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Wirtinger inequality using Bessel functions

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Abstract
This paper presents of some new Wirtinger-type integral inequalities by using Bessel functions. We establish one weighted Wirtinger inequality.

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1 Introduction

The Wirtinger inequality plays a very important role in the theory of approximation, the theory of Sobolev's spaces, the theory of function of several variables and functional analysis. In 1916. Wirtinger established an integral inequality.

Theorem 1.1 (Wirtinger inequality) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with period 2π and let $f' \in L^2$. Then, if $\int_0^{2\pi} f(x)\,dx = 0$ the following inequality holds:*

$$\int_0^{2\pi} f^2(x)\,dx \le \int_0^{2\pi} f'^2(x)\,dx,$$

with equality if and only if $f(x) = a\cos x + b\sin x$, where a and b are constants.

Theorem 1.2 *Let $f(x)$ be a smooth function with period 2π. Then, for all real t,*

$$\int_0^{2\pi} \bigl[f(x) - f(x+t)\bigr]^2\,dx \le 4\sin^2\frac{t}{2}\int_0^{2\pi} f'^2(x)\,dx. \tag{1}$$

Equality is attained if and only if $f(x) = a\cos x + b\sin x + c$, where a, b, c are real constants (for $t = 0$ equality holds always).

In [1], Beesack obtained the following generalization of the Wirtinger inequality: If $k > 1$, $f(x) \in C^1([0,\pi])$, $f(0) = 0$, then

$$\int_0^{\pi} \bigl(f'(x)\bigr)^{2k}\,dx \ge \frac{2k-1}{(k\sin\frac{\pi}{2k})^{2k}}\int_0^{\pi} f^{2k}(x)\,dx, \quad k \ge 1. \tag{2}$$

In [2], Hall proved the following theorem:

Theorem 1.3 *Suppose that $k \in N$, $f(x) \in C^2[0,\pi]$ and $f(0) = f(\pi) = 0$. Let $H(u)$ be an even function, increasing and strictly convex on R^+, and such that $H(0) = H'(0) = 0$; moreover, $uH''(u) \to 0$ as $u \to 0$. Then we have*

$$\int_0^\pi H\big(f'(x)/f(x)\big)f^{2k}(x) \geq (2k-1)\lambda \int_0^\pi f^{2k}(x)\,dx, \quad \lambda = \lambda(k,H), \tag{3}$$

where $\lambda = \lambda(k,H)$ is determined by the equation

$$\int_0^\infty \frac{G'(u)}{G(u)+(2k-1)\lambda}\frac{du}{u} = k\pi, \quad G(u) := uH'(u) - H(u). \tag{4}$$

For each non-negative constant p, the associated Bessel equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \big(x^2 - p^2\big)y = 0. \tag{5}$$

Since Bessel's differential equation is a second-order equation, there must be two linearly independent solutions, which are called Bessel functions. These functions play important roles in many areas of applied mathematics (see [3, 4]). Typically the general solution is given as

$$y = a_1 J_p(x) + a_2 Y_p(x),$$

where a_1 and a_2 are arbitrary constants.

Special functions $J_p(x)$ are Bessel functions of the first kind, which are finite at $x = 0$ for all real values of p, and $Y_p(x)$ are Bessel functions of the second kind, which are singular at $x = 0$.

The Bessel function of the first kind of order p can be determined using an infinite power series expansion as follows: $J_p(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)}(\frac{x}{2})^{2k+p}$. Since $\Gamma(k+1) = k!$, it follows that

$$J_p(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+p)!}\left(\frac{x}{2}\right)^{2k+p}. \tag{6}$$

For integer order p, functions J_p and J_{-p} are not linearly independent, $J_{-p} = (-1)^p J_p$. In contrast, for non-integer orders, J_p and J_{-p} are linearly independent.

The most important Bessel functions are $J_0(x)$ and $J_1(x)$. For $p = -\frac{1}{2}$ and $p = \frac{1}{2}$, this functions expansion as follows:

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x, \tag{7}$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \tag{8}$$

2 Main results

Theorem 2.1 *Let $f' \in L^{2k}$ on $[0,\pi]$, with $f(0) = f(\pi) = 0$. Then the following inequality holds*:

$$\int_0^\pi f^{2k}(x)\,dx \leq \frac{1}{2k-1}\left(\frac{\pi}{2}\right)^{2k}\left(\int_0^{\frac{\pi}{2}} J_0\left(\frac{\pi}{2k}\cos t\right)\cos t\,dt\right)^{2k}\int_0^\pi f'^{2k}(x)\,dx. \tag{9}$$

Proof Since $J_n(z) = (\frac{z}{2})^n \sum_{r=0}^{+\infty} (-1)^r \frac{(\frac{z}{2})^{2r}}{r!(n+r)!}$, it follows that $J_0(\frac{\pi}{2k}\cos t) = \sum_{r=0}^{+\infty} (-1)^r \frac{(\frac{\pi}{2k}\frac{\cos t}{2})^{2r}}{(r!)^2}$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} J_0\left(\frac{\pi}{2k}\cos t\right)\cos t\,dt &= \int_0^{\frac{\pi}{2}} \sum_{r=0}^{+\infty} (-1)^r \frac{(\frac{\pi}{2k}\frac{\cos t}{2})^{2r}}{(r!)^2}\cos t\,dt \\ &= \sum_{r=0}^{+\infty} (-1)^r \frac{\pi^{2r}}{(2k)^{2r}2^{2r}(r!)^2}\int_0^{\frac{\pi}{2}} \cos^{2r+1} t\,dt.\end{aligned}$$

Using the integration by parts formula on the integral $I_{2r+1} = \int_0^{\frac{\pi}{2}} \cos^{2r+1} t\,dt$ and the fact that $\int_0^{\pi/2} \cos t\,dt = 1$, we obtained the recurrence relation $I_{2r+1} = \frac{2r}{2r+1} I_{2r-1}$, which implies $I_{2r+1} = \frac{(2r)!!}{(2r+1)!!} = \frac{((2r)!!)^2}{(2r+1)!} = \frac{(2^r r!)^2}{(2r+1)!}$.

The above equality becomes

$$\begin{aligned}\int_0^{\frac{\pi}{2}} J_0\left(\frac{\pi}{2k}\cos t\right)\cos t\,dt &= \sum_{r=0}^{+\infty} (-1)^r \frac{\pi^{2r}}{(2k)^{2r}2^{2r}(r!)^2}\frac{(2^r r!)^2}{(2r+1)!} \\ &= \frac{2k}{\pi}\sum_{r=0}^{+\infty} (-1)^r \frac{(\frac{\pi}{2k})^{2r+1}}{(2r+1)!} = \frac{2k}{\pi}\sin\frac{\pi}{2k},\end{aligned}$$

which implies

$$\sin^{2k}\frac{\pi}{2k} = \left(\frac{\pi}{2k}\right)^{2k}\left(\int_0^{\frac{\pi}{2}} J_0\left(\frac{\pi}{2k}\cos t\right)\cos t\,dt\right)^{2k}.$$

By (2) it follows that

$$\begin{aligned}&\int_0^{\pi} f^{2k}(x)\,dx \\ &\quad\le \frac{1}{2k-1}k^{2k}\frac{\pi^{2k}}{4^k k^{2k}}\left(\int_0^{\frac{\pi}{2}} J_0\left(\frac{\pi}{2k}\cos t\right)\cos t\,dt\right)^{2k}\int_0^{\pi} f'^{2k}(x)\,dx, \\ &\int_0^{\pi} f^{2k}(x)\,dx \\ &\quad\le \frac{1}{2k-1}\left(\frac{\pi}{2}\right)^{2k}\left(\int_0^{\frac{\pi}{2}} J_0\left(\frac{\pi}{2k}\cos t\right)\cos t\,dt\right)^{2k}\int_0^{\pi} f'^{2k}(x)\,dx.\end{aligned}$$

□

Theorem 2.2 *If $f' \in L^{2k}$ is absolutely continuous on $[0,\pi]$, with $f(0) = f(\pi) = 0$ then*

$$\int_0^{\pi} f^{2k}(x)\,dx \le \frac{\pi^{2k}}{2k+1}C(k)\int_0^{\pi} f'^2(x)f^{2(k-1)}(x)\,dx, \tag{10}$$

where $C(k) := \frac{\int_0^{\frac{\pi}{2}} x^k J_{1/2}^{2k}(x)\,dx}{\int_0^{\frac{\pi}{2}} x^{k+1} J_{1/2}^{2k+1}(x)\,dx}$.

Proof Starting with the right side of (10), we obtain

$$\begin{aligned}&\frac{\pi^{2k}}{2k+1}C(k)\int_0^{\pi} f'^2(x)f^{2(k-1)}(x)\,dx \\ &\quad= \frac{\pi^{2k}}{2k+1}\frac{\int_0^{\frac{\pi}{2}} x^k J_{1/2}^{2k}(x)\,dx}{\int_0^{\frac{\pi}{2}} x^{k+1} J_{1/2}^{2k+1}(x)\,dx}\int_0^{\pi} f'^2(x)f^{2(k-1)}(x)\,dx\end{aligned}$$

$$= \frac{\pi^{2k} \int_0^{\frac{\pi}{2}} x^k (\sqrt{\frac{2}{\pi x}})^{2k} \sin^{2k} x\,dx}{(2k+1)\int_0^{\frac{\pi}{2}} x^{k+1} (\sqrt{\frac{2}{\pi x}})^{2k+1} \sin^{2k+1} x\,dx} \int_0^{\pi} f'^2(x) f^{2(k-1)}(x)\,dx$$

$$= \frac{\pi^{2k}}{2k+1}\sqrt{\frac{\pi}{2}} \frac{\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx}{\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx} \int_0^{\pi} f'^2(x) f^{2(k-1)}(x)\,dx.$$

Since $\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\,dx = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q}{2}+1)}$, for $p = 2k$ and $q = 0$, we get $\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2k+1}{2})}{2\Gamma(k+1)}$; For $p = 2k+1$ and $q = 0$, we get $\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx = \frac{\Gamma(\frac{1}{2})\Gamma(k+1)}{2\Gamma(\frac{2k+3}{2})}$.

By integrating by parts, we obtain $\Gamma(\frac{2k+1}{2}) = \frac{(2k)!}{2^k k!}\sqrt{\pi}$, $\Gamma(\frac{2k+3}{2}) = \frac{(2k+1)!}{2^{2k+1}k!}\sqrt{\pi}$, and since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(k+1) = k!$, it follows

$$\frac{\pi^{2k}}{2k+1}\sqrt{\frac{\pi}{2}} \frac{\frac{\pi}{2}\frac{(2k)!}{2^{2k}(k!)^2}}{\frac{(k!)^2}{(2k+1)!}} \int_0^{\pi} f'^2(x) f^{2(k-1)}(x)\,dx$$

$$= \frac{1}{2k+1}\left(\sqrt{\frac{\pi}{2}}\right)^{2k+\frac{3}{2}} \frac{((2k)!)^2(2k+1)}{(2^k k!)^2(k!)^2} \int_0^{\pi} f'^2(x) f^{2(k-1)}(x)\,dx$$

$$= \frac{1}{2k+1}\left(\sqrt{\frac{\pi}{2}}\right)^{2k+\frac{3}{2}} \left(\frac{(2k-1)!}{k(k!)}\right)^2 k^2 \int_0^{\pi} f'^2(x) f^{2(k-1)}(x)\,dx.$$

If in (4) we put $H(u) = u^2$, $G(u) = u^2$, then (6) gives $\lambda = \frac{1}{k^2(2k-1)}$, so (3) becomes

$$\int_0^{\pi} f'^2(x) f^{2(k-1)}(x)\,dx \geq \frac{1}{k^2}\int_0^{\pi} f^{2k}(x)\,dx,$$

which implies

$$\frac{\pi^{2k}}{2k+1} C(k) \int_0^{\pi} f'^2(x) f^{2(k-1)}(x)\,dx \geq \left(\sqrt{\frac{\pi}{2}}\right)^{2k+\frac{3}{2}} \left(\frac{(2k-1)!}{k(k!)}\right)^2 \int_0^{\pi} f^{2k}(x)\,dx.$$

Since $(\sqrt{\frac{\pi}{2}})^{2k+\frac{3}{2}} > 1$ and $(\frac{(2k-1)!}{k(k!)})^2 > 1$, inequality (10) is established. □

Theorem 2.3 *Let $f(x)$ be a smooth function with period 2π. Then, for all real t,*

$$\int_0^{2\pi} \left[f(x) - f(x+t)\right]^2 dx \leq t\pi J_{1/2}^2\left(\frac{t}{2}\right)\int_0^{2\pi} f'^2(x)\,dx. \tag{11}$$

Equality is attained if and only if $f(x) = A\cos x + B\sin x + C$, where A, B, C are real constants (for $t = 0$ equality holds always).

Proof From the equation $J_n^2(t) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} J_{2n}(2t\cos x)\,dx$, for $n = \frac{1}{2}$, the right side of (11) becomes

$$2t\int_0^{\frac{\pi}{2}} J_1(t\cos x)\,dx \int_0^{2\pi} f'^2(x)\,dx$$

$$= 2t\int_0^{\frac{\pi}{2}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(n+1)!}\left(\frac{t\cos x}{2}\right)^{2n+1} dx \int_0^{2\pi} f'^2(x)\,dx$$

$$
\begin{aligned}
&= 2t\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{t}{2}\right)^{2n+1}\int_0^{\frac{\pi}{2}}\cos^{2n+1}x\,dx\int_0^{2\pi}f'^2(x)\,dx \\
&= 2t\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(n+1)!}\frac{t^{2n+1}}{2\cdot 2^{2n}}\frac{2^{2n}n!n!}{(2n+1)!}\int_0^{2\pi}f'^2(x)\,dx \\
&= t\sum_{n=0}^{+\infty}\frac{(-1)^n n! t^{2n+1}}{(n+1)n!(2n+1)!}\int_0^{2\pi}f'^2(x)\,dx \\
&= t\sum_{n=0}^{+\infty}\frac{(-1)^n t^{2n+1}}{(n+1)(2n+1)!}\int_0^{2\pi}f'^2(x)\,dx \\
&= t\left[\frac{t}{1!}-\frac{t^3}{2\cdot 3!}+\frac{t^5}{3\cdot 5!}-\frac{t^7}{4\cdot 7!}+\cdots\right]\int_0^{2\pi}f'^2(x)\,dx \\
&= t\left[\frac{2t^2}{2!t}-\frac{4t^4}{2\cdot 4!t}+\frac{6t^6}{3\cdot 6!t}-\frac{8t^8}{4\cdot 8!t}+\frac{10t^{10}}{5\cdot 10!t}-\cdots\right]\int_0^{2\pi}f'^2(x)\,dx \\
&= 2t\left[\frac{t^2}{2!t}-\frac{t^4}{4!t}+\frac{t^6}{6!t}-\frac{t^8}{8!t}+\frac{t^{10}}{10!t}-\cdots\right]\int_0^{2\pi}f'^2(x)\,dx \\
&= 2t\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{t^{2n}}{t(2n)!}\int_0^{2\pi}f'^2(x)\,dx \\
&= t\left(1-\sum_{n=0}^{+\infty}(-1)^n\frac{t^{2n}}{(2n)!}\right)\int_0^{2\pi}f'^2(x)\,dx \\
&= 2(1-\cos t)\int_0^{2\pi}f'^2(x)\,dx = 4\sin^2\frac{t}{2}\int_0^{2\pi}f'^2(x)\,dx.
\end{aligned}
$$

Equation (1) implies the desired inequality (11). □

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Authors' contributions
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