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# Robust stability analysis of quaternion-valued neural networks via LMI approach

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## Abstract

This paper is concerned with the issue of robust stability for quaternion-valued neural networks (QVNNs) with leakage, discrete and distributed delays by employing a linear matrix inequality (LMI) approach. Based on the homeomorphic mapping theorem, the quaternion matrix theorem and the Lyapunov theorem, some criteria are developed in the form of real-valued LMIs for guaranteeing the existence, uniqueness, and global robust stability of the equilibrium point of the delayed QVNNs. Two numerical examples are provided to demonstrate the effectiveness of the obtained results.

**Keywords:** Quaternion-valued neural networks; Delay effects; Global robust stability; Linear matrix inequality; Modulus inequality technique

## 1 Introduction

The quaternions are members of a noncommutative division algebra invented independently by Carl Friedrich Gauss in 1819 and William Rowan Hamilton in 1843 [1]. Quaternions provide a concise mathematical method for representing the automorphisms of three- and four-dimensional spaces. The representations by quaternions are more compact and quicker to compute than the representations by matrices [2]. For this reason, an increasing number of applications based on quaternions are found in various fields, such as computer graphics, quantum mechanics, attitude control, signal processing, and orbital mechanics [3–5]. For example, in attitude-control systems, it will lead to the problem of the so-called “gimbal lock” by using Euler angles. As an alternative approach, quaternions have the technical advantage not to suffer from the problem [6].

On the other hand, over the past three decades, neural networks (NNs) have been applied in various areas throughout science and engineering, such as signal processing, image processing, pattern recognition, associative memory and optimization [7–16]. Furthermore, real-valued NNs (RVNNs) and complex-valued NNs (CVNNs) have been extensively investigated and a great number of results have been reported [17–22]. Recently, quaternion-valued neural networks (QVNNs) have drawn a great deal of attention [23, 24]. Due to the simple representation of quaternions and the high efficiency in dealing with multidimensional data, QVNNs have demonstrated better performances than CVNNs and RVNNs in their wide applications [25–31]. For example, in the image compression

[25, 26], one color is synthesized by the three primary colors with a certain proportion, which needs three real- or complex-valued neurons to store. However, one quaternion-valued neuron is enough to represent one color via three channels  $i$ ,  $j$  and  $\kappa$  of QVNNs, which leads to a significant reduction in the dimension of the system and to a great increase in the computational efficiency. In some practical applications, it is required that the designed system has a unique equilibrium point which is globally stable. Therefore, the dynamics of QVNNs has been an active research topic [32–36]. In [32, 33], some  $\mu$ -stability criteria in the form of linear matrix inequalities (LMIs) were provided for QVNNs with time-varying delays. In [34], several sufficient conditions were presented to check the global exponential stability for QVNNs with time-varying delays. In [35], several sufficient criteria were derived to ensure the existence, uniqueness, and global robust stability of the equilibrium point for delayed QVNNs with parameter uncertainties. In [36], some algebraic conditions on the global dissipativity for QVNNs with time-varying delays were devised.

Moreover, when implementing neural networks, time delays are unavoidably encountered due to delay transmission line, partial element equivalent circuit, integration and communication. The existence of time delays in neural networks frequently will lead to undesirable complex dynamical behaviors [37–39]. As pointed out by Gopalsamy [40], time delays in the negative feedback terms will have a tendency to destabilize a system, which are known as leakage or forgetting delays. Moreover, many biological and artificial neural networks contain inherent discrete time delays in signal transmission, which may cause oscillation and instability. Furthermore, since the neural networks usually are of a spatial nature associated with the presence of an amount of parallel pathway of a variety of axon sizes and lengths, it is desirable to model them by introducing distributed delays. In [40, 41], the stability problem was investigated for RVNNs with the introduction of the leakage delays. In [42, 43], some dynamical behaviors of RVNNs with distributed delays were studied. In [44], the multistability issue of competitive RVNNs with discrete and distributed delays was investigated. In [45, 46], the authors considered the effects of leakage and discrete delays in CVNNs.

Strongly motivated by the above discussions, in the present paper we consider the robust stability problem of QVNNs with leakage, discrete and distributed delays. There are two main challenging problems of the current research. The first one is how to construct a proper Lyapunov–Krasovskii functional corresponding to the considered QVNNs. The second one is how to make sure that the obtained criteria are depend on the upper and lower bounds of system parameters. For the first one, we adopt quaternion self-conjugate and positive definite matrices to construct the Lyapunov–Krasovskii functional so that we can directly deal with the QVNNs rather than any decomposition. For the second one, we utilize modulus inequality technique to compute the derivative of the Lyapunov–Krasovskii functional so that the obtained criteria are not only real-valued but also related to the bounds of parameters.

**Notations** Throughout this paper,  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$  denote the real field, the complex field and the skew field of quaternions, respectively.  $\mathbb{R}^n$ ,  $\mathbb{C}^n$  and  $\mathbb{H}^n$  denote  $n$ -dimensional vectors with entries from  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$ , respectively.  $\mathbb{R}^{n \times m}$ ,  $\mathbb{C}^{n \times m}$  and  $\mathbb{H}^{n \times m}$  denote  $n \times m$  matrices with entries from  $\mathbb{R}$ ,  $\mathbb{C}$  and  $\mathbb{H}$ , respectively. Specially,  $\mathbb{R}_d^{n \times n}$  denotes  $n \times n$  real diagonal matrices. The notation  $\bar{A}$ ,  $A^T$  and  $A^*$  stand for the conjugate, the transpose

and the conjugate transpose, respectively, of the matrix  $A$ . For  $A = (a_{ij})_{n \times n} \in \mathbb{C}^{n \times n}$ , let  $\|A\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2}$  denote the norm of  $A$ . The notation  $X \geq Y$  (respectively,  $X > Y$ ) means that  $X - Y$  is positive semi-definite (respectively, positive definite). For a positive definite Hermitian matrix  $P \in \mathbb{C}^{n \times n}$ ,  $\lambda_{\max}(P)$  and  $\lambda_{\min}(P)$  are defined as the largest and the smallest eigenvalues of  $P$ , respectively. In the four-dimensional algebra  $\mathbb{H}$ , the four basis elements are denoted by  $1, i, j, \kappa$ , which obey the following multiplication table:  $i^2 = j^2 = \kappa^2 = -1, ij = -ji = \kappa, j\kappa = -\kappa j = i, \kappa i = -i\kappa = j$ , and  $1 \cdot a = a \cdot 1 = a$  for every quaternion  $a$ . For a quaternion  $a = a_0 + a_1i + a_2j + a_3\kappa \in \mathbb{H}$ , we call  $a_0, a_1, a_2$  and  $a_3$  the first, second, third and fourth parts of the quaternion, respectively. Let  $a^* = a_0 - a_1i - a_2j - a_3\kappa$  be the conjugate of  $a$ , and  $|a| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$  be the modulus of  $a$ . For  $q = (q_1, q_2, \dots, q_n)^T \in \mathbb{H}^n$ , let  $|q| = (|q_1|, |q_2|, \dots, |q_n|)^T$  be the modulus of  $q$ , and  $\|q\| = \sqrt{\sum_{i=1}^n |q_i|^2}$  be the norm of  $q$ . For  $a, b \in \mathbb{H}$ ,  $a \leq b$  denotes  $a_i \leq b_i, i = 0, 1, 2, 3$ , where  $a = a_0 + a_1i + a_2j + a_3\kappa$  and  $b = b_0 + b_1i + b_2j + b_3\kappa$ . For  $A, B \in \mathbb{H}^{n \times n}$ ,  $A \leq B$  denotes  $a_{ij} \leq b_{ij}, i, j = 1, 2, \dots, n$ , where  $A = (a_{ij})_{n \times n}$  and  $B = (b_{ij})_{n \times n}$ . In addition, the symbol  $\star$  always denotes the conjugate transpose of a suitable block in a Hermitian matrix.

### 2 Problem formulation and preliminaries

Consider the following QVNNs model with three kinds of time delays including leakage delay, discrete delay and distributed delay:

$$\dot{q}(t) = -Dq(t - \delta) + Af(q(t)) + Bf(q(t - \tau)) + C \int_{-\infty}^t k(t - s)f(q(s)) ds + J, \tag{1}$$

for  $t \geq 0$ , where  $q(t) = (q_1(t), q_2(t), \dots, q_n(t))^T \in \mathbb{H}^n$  is the state vector of the neural networks with  $n$  neurons at time  $t$ .  $D = \text{diag}(d_1, d_2, \dots, d_n) \in \mathbb{R}_d^{n \times n}$  where  $d_i > 0$  ( $i = 1, 2, \dots, n$ ) is the self-feedback connection weight matrix;  $A = (a_{ij})_{n \times n} \in \mathbb{H}^{n \times n}$ ,  $B = (b_{ij})_{n \times n} \in \mathbb{H}^{n \times n}$  and  $C = (c_{ij})_{n \times n} \in \mathbb{H}^{n \times n}$  are, respectively, the connection weight matrix, the discretely delayed connection weight matrix and the distributively delayed connection weight matrix.  $J = (J_1, J_2, \dots, J_n)^T \in \mathbb{H}^n$  is the external input vector.  $f(q(t)) = (f_1(q_1(t)), f_2(q_2(t)), \dots, f_n(q_n(t)))^T \in \mathbb{H}^n$  denotes the neuron activations.  $\delta > 0$  and  $\tau > 0$  are the leakage time delay and the discrete time delay, respectively.  $k(\cdot) : [0, +\infty) \rightarrow [0, +\infty)$  is the delay kernel, which satisfies  $\int_0^\infty k(s) ds = 1$ .

The following assumptions will be needed throughout the paper:

- (A<sub>1</sub>) The parameters  $D, A, B, C, J$  in QVNNs (1) are assumed to be in the following sets, respectively,

$$\begin{aligned} D_I &= \{D \in \mathbb{R}_d^{n \times n} : 0 < \check{D} \leq D \leq \hat{D}\}, \\ A_I &= \{A \in \mathbb{H}^{n \times n} : \check{A} \leq A \leq \hat{A}\}, \\ B_I &= \{B \in \mathbb{H}^{n \times n} : \check{B} \leq B \leq \hat{B}\}, \\ C_I &= \{C \in \mathbb{H}^{n \times n} : \check{C} \leq C \leq \hat{C}\}, \\ J_I &= \{J \in \mathbb{H}^n : \check{J} \leq J \leq \hat{J}\}, \end{aligned}$$

where  $\check{D}, \hat{D} \in \mathbb{R}_d^{n \times n}$ ,  $\check{A}, \hat{A}, \check{B}, \hat{B}, \check{C}, \hat{C} \in \mathbb{H}^{n \times n}$ , and  $\check{J}, \hat{J} \in \mathbb{H}^n$ . Moreover, let  $\check{A} = (\check{a}_{ij})_{n \times n}$ ,  $\hat{A} = (\hat{a}_{ij})_{n \times n}$ ,  $\check{B} = (\check{b}_{ij})_{n \times n}$ ,  $\hat{B} = (\hat{b}_{ij})_{n \times n}$ ,  $\check{C} = (\check{c}_{ij})_{n \times n}$ , and  $\hat{C} = (\hat{c}_{ij})_{n \times n}$ . Then we de-

fine  $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ ,  $\tilde{B} = (\tilde{b}_{ij})_{n \times n}$  and  $\tilde{C} = (\tilde{c}_{ij})_{n \times n}$ , where  $\tilde{a}_{ij} = \max\{|\check{a}_{ij}|, |\hat{a}_{ij}|\}$ ,  $\tilde{b}_{ij} = \max\{|\check{b}_{ij}|, |\hat{b}_{ij}|\}$  and  $\tilde{c}_{ij} = \max\{|\check{c}_{ij}|, |\hat{c}_{ij}|\}$ .

(A<sub>2</sub>) For  $i = 1, 2, \dots, n$ , the neuron activation function  $f_i$  is continuous and satisfies

$$|f_i(s_1) - f_i(s_2)| \leq \gamma_i |s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbb{H},$$

where  $\gamma_i$  is a real constant. Moreover, define  $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$ .

**Definition 1** The QVNNs defined by (1) with the parameter ranges defined by (A<sub>1</sub>) are globally asymptotically robust stable if the unique equilibrium point  $\check{q}$  of QVNNs (1) is globally asymptotically stable for all  $D \in D_I$ ,  $A \in A_I$ ,  $B \in B_I$ ,  $C \in C_I$  and  $J \in J_I$ .

**Lemma 1** ([35]) For any  $x, y \in \mathbb{H}^n$ , if  $P \in \mathbb{H}^{n \times n}$  is a positive definite Hermitian matrix, then

$$x^*y + y^*x \leq x^*Px + y^*P^{-1}y.$$

**Lemma 2** ([35]) If  $H(z) : \mathbb{H}^n \rightarrow \mathbb{H}^n$  is a continuous map and satisfies the following conditions:

- (i)  $H(z)$  is injective on  $\mathbb{H}^n$ ,
- (ii)  $\lim_{\|z\| \rightarrow \infty} \|H(z)\| = \infty$ ,

then  $H(z)$  is a homeomorphism of  $\mathbb{H}^n$  onto itself.

**Lemma 3** ([35]) For any positive definite constant Hermitian matrix  $W \in \mathbb{H}^{n \times n}$  and any scalar function  $\omega(s) : [a, b] \rightarrow \mathbb{H}^n$  with scalars  $a < b$  such that the integrations concerned are well defined,

$$\left( \int_a^b \omega(s) \, ds \right)^* W \left( \int_a^b \omega(s) \, ds \right) \leq (b - a) \int_a^b \omega^*(s) W \omega(s) \, ds.$$

In the following, we provide some modulus inequalities of quaternions, which play a major role in analyzing the problem in this paper.

**Lemma 4** Suppose  $A \in \mathbb{H}^{n \times n}$ ,  $\check{A} = (\check{a}_{ij})_{n \times n} \in \mathbb{H}^{n \times n}$ ,  $\hat{A} = (\hat{a}_{ij})_{n \times n} \in \mathbb{H}^{n \times n}$ , and  $\check{A} \preceq A \preceq \hat{A}$ . Then, for any  $x, y \in \mathbb{H}^n$ , the following inequalities hold:

$$x^*A^*Ax \leq |x|^T |A|^T |A| |x| \leq |x|^T \tilde{A}^T \tilde{A} |x|, \tag{2}$$

$$x^*A^*y + y^*Ax \leq |x|^T |A|^T |y| + |y|^T |A| |x| \leq |x|^T \tilde{A}^T |y| + |y|^T \tilde{A} |x|, \tag{3}$$

where  $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ ,  $\tilde{a}_{ij} = \max\{|\check{a}_{ij}|, |\hat{a}_{ij}|\}$ .

*Proof* It should be noted that  $|a + b| \leq |a| + |b|$  for any  $a, b \in \mathbb{H}$ . By the Cauchy–Schwarz inequality, the modulus inequalities (2) and (3) can be obtained. We omit the details because the proof is direct. □

*Remark 1* In Lemma 4, if  $A$  is a real positive diagonal matrix, then  $|A| = A$  and  $\tilde{A} = \hat{A}$ . Therefore, the modulus inequality (3) will reduce to

$$x^*A^*y + y^*Ax \leq |x|^T A^T |y| + |y|^T A |x| \leq |x|^T \hat{A}^T |y| + |y|^T \hat{A} |x|.$$

**Lemma 5** (Schur complement [47]) *A given real symmetric matrix*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where  $S_{11}^T = S_{11}$ ,  $S_{12}^T = S_{21}$  and  $S_{22}^T = S_{22}$ , satisfies  $S < 0$  if and only if any one of the following two conditions holds:

- (i)  $S_{22} < 0$  and  $S_{11} - S_{12}S_{22}^{-1}S_{21} < 0$ ,
- (ii)  $S_{11} < 0$  and  $S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$ .

### 3 Main results

In this section, we first analyze the existence and uniqueness of the equilibrium point of the delayed QVNNs under Assumptions (A<sub>1</sub>) and (A<sub>2</sub>). Then we investigate the global robust stability of the equilibrium point of the delayed QVNNs.

**Theorem 1** *Under Assumptions (A<sub>1</sub>) and (A<sub>2</sub>), QVNNs (1) have a unique equilibrium point, if there exist four real positive diagonal matrices U, V<sub>1</sub>, V<sub>2</sub> and V<sub>3</sub> such that the following LMI holds:*

$$\Sigma = \begin{pmatrix} \Sigma_{11} & U\tilde{A} & U\tilde{B} & U\tilde{C} \\ \star & -V_1 & 0 & 0 \\ \star & \star & -V_2 & 0 \\ \star & \star & \star & -V_3 \end{pmatrix} < 0, \tag{4}$$

where  $\Sigma_{11} = -\check{D}U - U\check{D} + \Gamma V_1\Gamma + \Gamma V_2\Gamma + \Gamma V_3\Gamma$ .

*Proof* We define the following continuous map  $\mathcal{H} : \mathbb{H}^n \rightarrow \mathbb{H}^n$  associated with system (1):

$$\mathcal{H}(q) = -Dz + Af(q) + Bf(q) + Cf(q) + J. \tag{5}$$

Then the proof is divided into two steps.

First, we prove that  $\mathcal{H}(q)$  is an injective map on  $\mathbb{H}^n$ . Suppose that there exist  $q, \tilde{q} \in \mathbb{H}^n$  with  $q \neq \tilde{q}$ , such that  $\mathcal{H}(q) = \mathcal{H}(\tilde{q})$ , which implies that

$$\begin{aligned} 0 &= (q - \tilde{q})^*U[\mathcal{H}(q) - \mathcal{H}(\tilde{q})] + [\mathcal{H}(q) - \mathcal{H}(\tilde{q})]^*U(q - \tilde{q}) \\ &= -(q - \tilde{q})^*(UD + DU)(q - \tilde{q}) + (q - \tilde{q})^*UA(f(q) - f(\tilde{q})) \\ &\quad + (f(q) - f(\tilde{q}))^*A^*U(q - \tilde{q}) + (q - \tilde{q})^*UB(f(q) - f(\tilde{q})) \\ &\quad + (f(q) - f(\tilde{q}))^*B^*U(q - \tilde{q}) + (q - \tilde{q})^*UC(f(q) - f(\tilde{q})) \\ &\quad + (f(q) - f(\tilde{q}))^*C^*U(q - \tilde{q}) \\ &\leq -(q - \tilde{q})^*(UD + DU)(q - \tilde{q}) + (q - \tilde{q})^*U(A)V_1^{-1}A^*U(q - \tilde{q}) \\ &\quad + (f(q) - f(\tilde{q}))^*V_1(f(q) - f(\tilde{q})) + (q - \tilde{q})^*UBV_2^{-1}B^*U(q - \tilde{q}) \\ &\quad + (f(q) - f(\tilde{q}))^*V_2(f(q) - f(\tilde{q})) + (q - \tilde{q})^*UCV_3^{-1}C^*U(q - \tilde{q}) \\ &\quad + (f(q) - f(\tilde{q}))^*V_3(f(q) - f(\tilde{q})) \end{aligned}$$

$$\begin{aligned} &\leq |q - \tilde{q}|^* [-\check{D}U - U\check{D} + U\tilde{A}V_1^{-1}\tilde{A}^*U + U\tilde{B}V_2^{-1}\tilde{B}^*U \\ &\quad + U\tilde{C}V_3^{-1}\tilde{C}^*U] |q - \tilde{q}| + (f(q) - f(\tilde{q}))^* (V_1 + V_2 + V_3)(f(q) - f(\tilde{q})). \end{aligned} \tag{6}$$

Here, in the computation above, we have applied Lemmas 1 and 4. Since  $V_1, V_2,$  and  $V_3$  are real positive diagonal matrices, it follows from Assumption  $(A_2)$  that

$$\begin{aligned} &(f(q) - f(\tilde{q}))^* (V_1 + V_2 + V_3)(f(q) - f(\tilde{q})) \\ &\leq (q - \tilde{q})^* \Gamma (V_1 + V_2 + V_3) \Gamma (q - \tilde{q}) = |q - \tilde{q}|^* \Gamma (V_1 + V_2 + V_3) \Gamma |q - \tilde{q}|. \end{aligned} \tag{7}$$

We can get from (6) and (7)

$$0 \leq |q - \tilde{q}|^* \Xi |q - \tilde{q}|, \tag{8}$$

where  $\Xi = -\check{D}U - U\check{D} + \Gamma (V_1 + V_2 + V_3) \Gamma + U\tilde{A}V_1^{-1}\tilde{A}^*U + U\tilde{B}V_2^{-1}\tilde{B}^*U + U\tilde{C}V_3^{-1}\tilde{C}^*U$ . From Lemma 5 and the LMI (4), we see that  $\Xi < 0$ . Then  $q - \tilde{q} = 0$  from (8). Therefore,  $\mathcal{H}(q)$  is an injective map on  $\mathbb{H}^n$ . The first step is completed.

Second, we prove  $\|\mathcal{H}(q)\| \rightarrow \infty$  as  $\|q\| \rightarrow \infty$ . Let  $\tilde{\mathcal{H}}(q) = \mathcal{H}(q) - \mathcal{H}(0)$ . By Lemmas 1 and 4, we can compute that

$$\begin{aligned} q^* U \tilde{\mathcal{H}}(q) + \tilde{\mathcal{H}}(q)^* U q &= -q^* (UD + DU)q + q^* U(A + B + C)(f(q) - f(0)) \\ &\quad + (f(q) - f(0))^* (A^* + B^* + C^*) U q \\ &\leq -q^* (UD + DU)q + q^* U A V_1^{-1} A^* U q \\ &\quad + q^* U B V_2^{-1} B^* U q + q^* U C V_3^{-1} C^* U q \\ &\quad + (f(q) - f(0))^* (V_1 + V_2 + V_3)(f(q) - f(0)) \\ &\leq |q|^* \Xi |q| \leq -\lambda_{\min}(-\Xi) \|q\|^2. \end{aligned}$$

An application of the Cauchy–Schwarz inequality yields

$$\lambda_{\min}(-\Xi) \|q\|^2 \leq 2 \|q^*\| \|U\| \|\tilde{\mathcal{H}}(q)\|.$$

When  $q \neq 0$ , we have

$$\|\tilde{\mathcal{H}}(q)\| \geq \frac{\lambda_{\min}(-\Xi) \|q\|}{2 \|U\|}.$$

Therefore,  $\|\tilde{\mathcal{H}}(q)\| \rightarrow \infty$  as  $\|q\| \rightarrow \infty$ , which implies  $\|\mathcal{H}(q)\| \rightarrow \infty$  as  $\|q\| \rightarrow \infty$ . From Lemma 2 we know that  $\mathcal{H}(q)$  is a homeomorphism of  $\mathbb{H}^n$ . Thus, system (1) has a unique equilibrium point.  $\square$

In what follows, we further consider the global robust stability of the equilibrium point based on Theorem 1.

**Theorem 2** *Under Assumptions  $(A_1)$  and  $(A_2)$ , QVNNs (1) have a unique equilibrium point and the equilibrium point is globally robust stable, if there exist nine real positive*

diagonal matrices  $P_1, P_2, P_3, P_4, Q_1, Q_2, R, S_1$  and  $S_2$  such that the following LMI holds:

$$\Omega = \begin{pmatrix} \Omega_{11} & 0 & 0 & 0 & P_1\tilde{A} & P_1\tilde{B} & P_1\tilde{C} & \hat{D}P_1\hat{D} \\ \star & \Omega_{22} & \Omega_{23} & 0 & Q_1^T\tilde{A} & Q_1^T\tilde{B} & Q_1^T\tilde{C} & 0 \\ \star & \star & \Omega_{33} & 0 & Q_2^T\tilde{A} & Q_2^T\tilde{B} & Q_2^T\tilde{C} & 0 \\ \star & \star & \star & \Omega_{44} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -S_1 & 0 & 0 & \tilde{A}^T P_1 \hat{D} \\ \star & \star & \star & \star & \star & -S_2 & 0 & \tilde{B}^T P_1 \hat{D} \\ \star & \star & \star & \star & \star & \star & -R & \tilde{C}^T P_1 \hat{D} \\ \star & \star & \star & \star & \star & \star & \star & -P_3 \end{pmatrix} < 0, \tag{9}$$

where  $\Omega_{11} = -P_1\check{D} - \check{D}P_1 + P_2 + \delta^2 P_3 + P_4 + \Gamma R \Gamma + \Gamma S_1 \Gamma$ ,  $\Omega_{22} = -Q_1^T - Q_1$ ,  $\Omega_{23} = Q_1^T \hat{D} + Q_2$ ,  $\Omega_{33} = -P_2 - Q_2^T \check{D} - \check{D}Q_2$ ,  $\Omega_{44} = -P_4 + \Gamma S_2 \Gamma$ .

*Proof* The proof will be divided into two steps. First, we will proof the QVNNs (1) have a unique equilibrium point under LMI (9) based on Theorem 1. Second, we will prove the equilibrium point is globally robust stable by the Lyapunov theorem.

Step 1: For convenience, we let

$$\Omega_1 = \begin{pmatrix} \Omega_{11} & P_1\tilde{A} & P_1\tilde{B} & P_1\tilde{C} \\ \star & -S_1 & 0 & 0 \\ \star & \star & -S_2 & 0 \\ \star & \star & \star & -R \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} \Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \Omega_1,$$

where  $\Theta = \Omega_{44} - P_2 - \delta^2 P_3$ . It should be noted that  $\Omega_1$  is a principal sub-matrix of  $\Omega$  formed by rows 1, 5, 6, 7 and columns 1, 5, 6, 7. Therefore,  $\Omega_1 < 0$ . Since  $\Theta < 0$ , we see that

$$\Omega_2 < 0. \tag{10}$$

Then we let  $U = P_1 > 0$ ,  $V_1 = S_1 > 0$ ,  $V_2 = S_2 > 0$  and  $V_3 = R > 0$ , we can obtain

$$\Sigma = \Omega_2, \tag{11}$$

where  $\Sigma$  is defined in LMI (4) of Theorem 1. It follows from (10) and (11) that  $\Sigma < 0$ , which means LMI (4) holds. Thus, system (1) has a unique equilibrium point by Theorem 1.

Step 2: Let  $\check{q}$  be the unique equilibrium point of system (1). For convenience, we shift the equilibrium to the origin by letting  $\tilde{q}(t) = q(t) - \check{q}$ , and then system (1) can be transformed into

$$\dot{\tilde{q}}(t) = -D\tilde{q}(t - \delta) + A f(\tilde{q}(t)) + B f(\tilde{q}(t - \tau)) + C \int_{-\infty}^t k(t-s) f(\tilde{q}(s)) ds, \tag{12}$$

where  $f(\tilde{q}(t)) = f(q(t)) - f(\check{q})$  and  $f(\tilde{q}(t - \tau)) = f(q(t - \tau)) - f(\check{q})$ .

With some preparation above, consider the following Lyapunov–Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t),$$

where

$$V_1(t) = \left( \tilde{q}(t) - D \int_{t-\delta}^t \tilde{q}(s) \, ds \right)^* P_1 \left( \tilde{q}(t) - D \int_{t-\delta}^t \tilde{q}(s) \, ds \right), \tag{13}$$

$$V_2(t) = \int_{t-\delta}^t \tilde{q}^*(s) P_2 \tilde{q}(s) \, ds, \tag{14}$$

$$V_3(t) = \delta \int_0^\delta \int_{t-u}^t \tilde{q}^*(s) P_3 \tilde{q}(s) \, ds \, du, \tag{15}$$

$$V_4(t) = \int_{t-\tau}^t \tilde{q}^*(s) P_4 \tilde{q}(s) \, ds, \tag{16}$$

$$V_5(t) = \sum_{j=1}^n r_j \int_0^\infty k(s) \int_{t-s}^t f_j^*(\tilde{q}_j(t)) f_j(\tilde{q}_j(t)) \, dt \, ds, \tag{17}$$

where  $r_j$  are the main diagonal entries of  $R$ . That is,  $R = \text{diag}(r_1, r_2, \dots, r_n)$ .

Then the derivatives of  $V_1(t)$ ,  $V_2(t)$ ,  $V_3(t)$ ,  $V_4(t)$  and  $V_5(t)$  are calculated as follows:

$$\begin{aligned} \dot{V}_1(t) &= \left[ \tilde{q}(t) - D \int_{t-\delta}^t \tilde{q}(s) \, ds \right]^* P_1 \left[ \dot{\tilde{q}}(t) - D\dot{\tilde{q}}(t) + D\tilde{q}(t-\delta) \right] \\ &\quad + \left[ \dot{\tilde{q}}(t) - D\dot{\tilde{q}}(t) + D\tilde{q}(t-\delta) \right]^* P_1 \left[ \tilde{q}(t) - D \int_{t-\delta}^t \tilde{q}(s) \, ds \right] \\ &= \left[ \tilde{q}(t) - D \int_{t-\delta}^t \tilde{q}(s) \, ds \right]^* P_1 \left[ -D\dot{\tilde{q}}(t) + Af(\tilde{q}(t)) + Bf(\tilde{q}(t-\tau)) \right. \\ &\quad \left. + C \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right] + \left[ -D\dot{\tilde{q}}(t) + Af(\tilde{q}(t)) + Bf(\tilde{q}(t-\tau)) \right. \\ &\quad \left. + C \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right]^* P_1 \left[ \tilde{q}(t) - D \int_{t-\delta}^t \tilde{q}(s) \, ds \right] \\ &= -\tilde{q}^*(t)(P_1 D + DP_1)\tilde{q}(t) + \tilde{q}^*(t)P_1 Af(\tilde{q}(t)) + f^*(\tilde{q}(t))A^*P_1\tilde{q}(t) \\ &\quad + \tilde{q}^*(t)P_1 Bf(\tilde{q}(t-\tau)) + f^*(\tilde{q}(t-\tau))B^*P_1\tilde{q}(t) \\ &\quad + \tilde{q}^*(t)P_1 C \left[ \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right] + \left[ \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right]^* \\ &\quad \times C^*P_1\tilde{q}(t) + \tilde{q}^*(t)DP_1 D \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right] + \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right]^* DP_1 D\tilde{q}(t) \\ &\quad - f^*(\tilde{q}(t))A^*P_1 D \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right] - \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right]^* DP_1 Af(\tilde{q}(t)) \\ &\quad - f^*(\tilde{q}(t-\tau))B^*P_1 D \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right] \\ &\quad - \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right]^* DP_1 Bf(\tilde{q}(t-\tau)) \\ &\quad - \left[ \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right]^* C^*P_1 D \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right] \\ &\quad - \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right]^* DP_1 C \left[ \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right], \tag{18} \end{aligned}$$



$$\dot{V}_2(t) = \tilde{q}^*(t)P_2\tilde{q}(t) - \tilde{q}^*(t - \delta)P_2\tilde{q}(t - \delta), \tag{19}$$

$$\begin{aligned} \dot{V}_3(t) &= \delta^2 \tilde{q}^*(t)P_3\tilde{q}(t) - \delta \int_0^\delta \tilde{q}^*(t - u)P_3\tilde{q}(t - u) \, du \\ &= \delta^2 \tilde{q}^*(t)P_3\tilde{q}(t) - \delta \int_{t-\delta}^t \tilde{q}^*(s)P_3\tilde{q}(s) \, ds \\ &\leq \delta^2 \tilde{q}^*(t)P_3\tilde{q}(t) - \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right]^* P_3 \left[ \int_{t-\delta}^t \tilde{q}(s) \, ds \right], \end{aligned} \tag{20}$$

$$\dot{V}_4(t) = \tilde{q}^*(t)P_4\tilde{q}(t) - \tilde{q}^*(t - \tau)P_4\tilde{q}(t - \tau), \tag{21}$$

$$\begin{aligned} \dot{V}_5(t) &= \sum_{j=1}^n r_j \int_0^{+\infty} k(s)f_j^*(\tilde{q}(t))f_j(\tilde{q}(t)) \, ds \\ &\quad - \sum_{j=1}^n r_j \int_0^{+\infty} k(s)f_j^*(\tilde{q}(t - s))f_j(\tilde{q}(t - s)) \, ds \\ &= f^*(\tilde{q}(t))Rf(\tilde{q}(t)) \\ &\quad - \sum_{j=1}^n r_j \int_0^{+\infty} k(s) \int_0^{+\infty} k(s)f_j^*(\tilde{q}(t - s))f_j(\tilde{q}(t - s)) \, ds \\ &\leq \tilde{q}^*(t)\Gamma R\Gamma\tilde{q}(t) \\ &\quad - \sum_{j=1}^n r_j \int_0^{+\infty} k(s)f_j^*(\tilde{q}(t - s)) \, ds \int_0^{+\infty} k(s)f_j(\tilde{q}(t - s)) \, ds \\ &\leq \tilde{q}(t)\Gamma R\Gamma\tilde{q}(t) \\ &\quad - \left[ \int_{-\infty}^t k(t - s)f(q(s)) \, ds \right]^* R \left[ \int_{-\infty}^t k(t - s)f(q(s)) \, ds \right]. \end{aligned} \tag{22}$$

In deriving inequality (20), we have made use of Lemma 3. Since  $S_1$  and  $S_2$  are real positive diagonal matrices, it follows from Assumption  $(A_2)$  that

$$0 \leq \tilde{q}^*(t)\Gamma S_1\Gamma\tilde{q}(t) - f^*(\tilde{q}(t))S_1f(\tilde{q}(t)), \tag{23}$$

$$0 \leq \tilde{q}^*(t - \tau)\Gamma S_2\Gamma\tilde{q}(t - \tau) - f^*(\tilde{q}(t - \tau))S_2f(\tilde{q}(t - \tau)). \tag{24}$$

From system (12), it is obvious that

$$\begin{aligned} 0 &= [Q_1\dot{\tilde{q}}(t) + Q_2\tilde{q}(t - \delta)]^* \left[ -\dot{\tilde{q}}(t) - D\tilde{q}(t - \delta) + Af(\tilde{q}(t)) \right. \\ &\quad \left. + Bf(\tilde{q}(t - \tau)) + C \int_{-\infty}^t k(t - s)f(q(s)) \, ds \right] \\ &\quad + \left[ -\dot{\tilde{q}}(t) - D\tilde{q}(t - \delta) + Af(\tilde{q}(t)) + Bf(\tilde{q}(t - \tau)) \right. \\ &\quad \left. + C \int_{-\infty}^t k(t - s)f(q(s)) \, ds \right]^* [Q_1\dot{\tilde{q}}(t) + Q_2\tilde{q}(t - \delta)]. \end{aligned} \tag{25}$$

It follows from equalities or inequalities (18)–(25) and Lemma 4 that

$$\begin{aligned}
 \dot{V}(t) &\leq \tilde{q}^*(t)(-P_1D - DP_1 + P_2 + \delta^2P_3 + P_4 + \Gamma R\Gamma + \Gamma S_1\Gamma)\tilde{q}(t) \\
 &\quad + \tilde{q}^*(t)P_1Af(\tilde{q}(t)) + f^*(\tilde{q}(t))A^*P_1\tilde{q}(t) + \tilde{q}^*(t)P_1Bf(\tilde{q}(t - \tau)) \\
 &\quad + f^*(\tilde{q}(t - \tau))B^*P_1\tilde{q}(t) + \tilde{q}^*(t)P_1C\left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right] \\
 &\quad + \left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right]^*C^*P_1\tilde{q}(t) + \tilde{q}^*(t)DP_1D\left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right] \\
 &\quad + \left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right]^*DP_1D\tilde{q}(t) + \dot{\tilde{q}}^*(t)(-Q_1^* - Q_1)\dot{\tilde{q}}(t) \\
 &\quad + \dot{\tilde{q}}^*(t)(-Q_1^*D - Q_2)\tilde{q}(t - \delta) + \tilde{q}^*(t - \delta)(-DQ_1 - Q_2^*)\dot{\tilde{q}}(t) \\
 &\quad + \dot{\tilde{q}}^*(t)Q_1^*Af(\tilde{q}(t)) + f^*(\tilde{q}(t))A^*Q_1\dot{\tilde{q}}(t) + \dot{\tilde{q}}^*(t)Q_1^*Bf(\tilde{q}(t - \tau)) \\
 &\quad + f^*(\tilde{q}(t - \tau))B^*Q_1\dot{\tilde{q}}(t) + \dot{\tilde{q}}^*(t)Q_1^*C\left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right] \\
 &\quad + \left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right]^*C^*Q_1\dot{\tilde{q}}(t) \\
 &\quad + \tilde{q}^*(t - \delta)(-P_2 - Q_2^*D - DQ_2)\tilde{q}(t - \delta) \\
 &\quad + \tilde{q}^*(t - \delta)Q_2^*Af(\tilde{q}(t)) + f^*(\tilde{q}(t))A^*Q_2\tilde{q}(t - \delta) \\
 &\quad + \tilde{q}^*(t - \delta)Q_2^*Bf(\tilde{q}(t - \tau)) + f^*(\tilde{q}(t - \tau))B^*Q_2\tilde{q}(t - \delta) \\
 &\quad + \tilde{q}^*(t - \delta)Q_2^*C\left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right] \\
 &\quad + \left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right]^*CQ_2\tilde{q}(t - \delta) \\
 &\quad + \tilde{q}^*(t - \tau)(-P_4 + \Gamma S_2\Gamma)\tilde{q}(t - \tau) - f^*(\tilde{q}(t))S_1f(\tilde{q}(t)) \\
 &\quad - f^*(\tilde{q}(t))A^*P_1D\left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right] - \left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right]^*DP_1Af(\tilde{q}(t)) \\
 &\quad - f^*(\tilde{q}(t - \tau))S_2f(\tilde{q}(t - \tau)) - f^*(\tilde{q}(t - \tau))B^*P_1D\left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right] \\
 &\quad - \left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right]^*DP_1Bf(\tilde{q}(t - \tau)) - \left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right]^*R \\
 &\quad \times \left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right] - \left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right]^*P_3\left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right] \\
 &\quad - \left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right]^*C^*P_1D\left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right] \\
 &\quad - \left[\int_{t-\delta}^t \tilde{q}(s)\,ds\right]^*DP_1C\left[\int_{-\infty}^t k(t-s)f(q(s))\,ds\right] \\
 &\leq |\tilde{q}(t)|^T(-P_1\check{D} - \check{D}P_1 + P_2 + \delta^2P_3 + P_4 + \Gamma R\Gamma + \Gamma S_1\Gamma)|\tilde{q}(t)| \\
 &\quad + |\tilde{q}(t)|^T P_1\tilde{A}|f(\tilde{q}(t))| + |f(\tilde{q}(t))|^T \tilde{A}^T P_1|\tilde{q}(t)| + |\tilde{q}(t)|^T P_1\tilde{B}|f(\tilde{q}(t - \tau))| \\
 &\quad + |f(\tilde{q}(t - \tau))|^T \tilde{B}^T P_1|\tilde{q}(t)| + |\tilde{q}(t)|^T P_1\tilde{C}\left|\int_{-\infty}^t k(t-s)f(q(s))\,ds\right|
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right|^T \tilde{C}^T P_1 |\tilde{q}(t)| + |\tilde{q}(t)|^T \hat{D} P_1 \hat{D} \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right| \\
 & + \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right|^T \hat{D} P_1 \hat{D} |\tilde{q}(t)| + |\dot{\tilde{q}}(t)|^T (-Q_1^T - Q_1) |\dot{\tilde{q}}(t)| \\
 & + |\dot{\tilde{q}}(t)|^T (Q_1^T \hat{D} + Q_2) |\tilde{q}(t-\delta)| + |\tilde{q}(t-\delta)|^T (\hat{D} Q_1 + Q_2^T) |\dot{\tilde{q}}(t)| \\
 & + |\dot{\tilde{q}}(t)|^T Q_1^T \tilde{A} |f(\tilde{q}(t))| + |f(\tilde{q}(t))|^T \tilde{A}^T Q_1 |\dot{\tilde{q}}(t)| + |\dot{\tilde{q}}(t)|^T Q_1^T \tilde{B} |f(\tilde{q}(t-\tau))| \\
 & + |f(\tilde{q}(t-\tau))|^T \tilde{B}^T Q_1 |\dot{\tilde{q}}(t)| + |\dot{\tilde{q}}(t)|^T Q_1^T \tilde{C} \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right| \\
 & + \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right|^T \tilde{C}^T Q_1 |\dot{\tilde{q}}(t)| \\
 & + |\tilde{q}(t-\delta)|^T (-P_2 - Q_2^T \check{D} - \check{D} Q_2) |\tilde{q}(t-\delta)| \\
 & + |\tilde{q}(t-\delta)|^T Q_2^T \tilde{A} |f(\tilde{q}(t))| + |f(\tilde{q}(t))|^T \tilde{A}^T Q_2 |\tilde{q}(t-\delta)| \\
 & + |\tilde{q}(t-\delta)|^T Q_2^T \tilde{B} |f(\tilde{q}(t-\tau))| + |f(\tilde{q}(t-\tau))|^T \tilde{B}^T Q_2 |\tilde{q}(t-\delta)| \\
 & + |\tilde{q}(t-\delta)|^T Q_2^T \tilde{C} \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right| \\
 & + \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right|^T \tilde{C} Q_2 |\tilde{q}(t-\delta)| \\
 & + |\tilde{q}(t-\tau)|^T (-P_4 + \Gamma S_2 \Gamma) |\tilde{q}(t-\tau)| - |f(\tilde{q}(t))|^T S_1 |f(\tilde{q}(t))| \\
 & + |f(\tilde{q}(t))|^T \tilde{A}^T P_1 \hat{D} \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right| + \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right|^T \hat{D} P_1 \tilde{A} |f(\tilde{q}(t))| \\
 & - |f(\tilde{q}(t-\tau))|^T S_2 |f(\tilde{q}(t-\tau))| + |f(\tilde{q}(t-\tau))|^T \tilde{B}^T P_1 \hat{D} \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right| \\
 & + \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right|^T \hat{D} P_1 \tilde{B} |f(\tilde{q}(t-\tau))| - \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right|^T R \\
 & \times \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right| - \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right|^T P_3 \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right| \\
 & + \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right|^T \tilde{C}^T P_1 \hat{D} \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right| \\
 & + \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right|^T \hat{D} P_1 \tilde{C} \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right| \\
 & \leq \alpha^T \Omega \alpha, \tag{26}
 \end{aligned}$$

where

$$\begin{aligned}
 \alpha = & \left( |\tilde{q}^T(t)|, |\dot{\tilde{q}}^T(t)|, |\tilde{q}^T(t-\delta)|, |\tilde{q}^T(t-\tau)|, |f^T(\tilde{q}(t))|, \right. \\
 & \left. |f^T(\tilde{q}(t-\tau))|, \left| \int_{-\infty}^t k(t-s)f(q(s)) \, ds \right|^T, \left| \int_{t-\delta}^t \tilde{q}(s) \, ds \right|^T \right)^T.
 \end{aligned}$$

Therefore, we conclude that  $\dot{V}(t)$  is negative definite because of LMI (9). Moreover, it is obvious that  $V(t)$  is radially unbounded. Then the equilibrium point of system (1) is globally asymptotically stable by standard Lyapunov theorem.  $\square$

*Remark 2* It should be noted that RVNNs and CVNNs are special cases of QVNNs. So the results of the paper can also be applied to RVNNs and CVNNs in the form of (1).

#### 4 Numerical examples

In this section, two numerical examples will illustrate the effectiveness of the proposed results.

*Example 1* Suppose that the parameters of QVNNs (1) are given as follows:

$$\begin{aligned}\check{D} &= \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}, & \hat{D} &= \begin{pmatrix} 0.32 & 0 \\ 0 & 0.32 \end{pmatrix}, \\ \check{A} &= (\check{a}_{ij})_{2 \times 2}, & \hat{A} &= (\hat{a}_{ij})_{2 \times 2}, \\ \check{B} &= (\check{b}_{ij})_{2 \times 2}, & \hat{B} &= (\hat{b}_{ij})_{2 \times 2}, \\ \check{C} &= (\check{c}_{ij})_{2 \times 2}, & \hat{C} &= (\hat{c}_{ij})_{2 \times 2}, \\ \Gamma &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, & \delta &= 0.5, & \tau &= 1,\end{aligned}$$

where

$$\begin{aligned}\check{a}_{11} &= -0.1 - 0.1t - 0.1J - 0.1\kappa, \\ \check{a}_{12} &= -0.1 - 0.1t - 0.1J - 0.1\kappa, \\ \check{a}_{21} &= -0.15 - 0.15t - 0.15J - 0.15\kappa, \\ \check{a}_{22} &= -0.1 - 0.1t - 0.1J - 0.1\kappa, \\ \hat{a}_{11} &= 0.1 + 0.1t + 0.1J + 0.1\kappa, \\ \hat{a}_{12} &= 0.1 + 0.1t + 0.1J + 0.1\kappa, \\ \hat{a}_{21} &= 0.15 + 0.15t + 0.15J + 0.15\kappa, \\ \hat{a}_{22} &= 0.1 + 0.1t + 0.1J + 0.1\kappa, \\ \check{b}_{11} &= -0.1 - 0.1t - 0.1J - 0.1\kappa, \\ \check{b}_{12} &= -0.05 - 0.05t - 0.05J - 0.05\kappa, \\ \check{b}_{21} &= -0.1 - 0.1t - 0.1J - 0.1\kappa, \\ \check{b}_{22} &= -0.1 - 0.1t - 0.1J - 0.1\kappa, \\ \hat{b}_{11} &= 0.1 + 0.1t + 0.1J + 0.1\kappa, \\ \hat{b}_{12} &= 0.05 + 0.05t + 0.05J + 0.05\kappa, \\ \hat{b}_{21} &= 0.1 + 0.1t + 0.1J + 0.1\kappa, \\ \hat{b}_{22} &= 0.1 + 0.1t + 0.1J + 0.1\kappa, \\ \check{c}_{11} &= -0.05 - 0.05t - 0.05J - 0.05\kappa, \\ \check{c}_{12} &= -0.075 - 0.075t - 0.075J - 0.075\kappa,\end{aligned}$$

$$\begin{aligned} \check{c}_{21} &= -0.1 - 0.1t - 0.1J - 0.1\kappa, \\ \check{c}_{22} &= -0.05 - 0.05t - 0.05J - 0.05\kappa, \\ \hat{c}_{11} &= 0.05 + 0.05t + 0.05J + 0.05\kappa, \\ \hat{c}_{12} &= 0.075 + 0.075t + 0.075J + 0.075\kappa, \\ \hat{c}_{21} &= 0.1 + 0.1t + 0.1J + 0.1\kappa, \\ \hat{c}_{22} &= 0.05 + 0.05t + 0.05J + 0.05\kappa. \end{aligned}$$

According to the matrices  $\check{A}, \hat{A}, \check{B}, \hat{B}, \check{C}$  and  $\hat{C}$ , we get the following matrices:

$$\tilde{A} = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0.2 & 0.1 \\ 0.2 & 0.2 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 0.1 & 0.15 \\ 0.2 & 0.1 \end{pmatrix}.$$

Then, by using YALMIP with solver of SDPT3 in MATLAB, we obtain the feasible solutions of LMI (9) in Theorem 2 as follows:

$$\begin{aligned} P_1 &= \begin{pmatrix} 36.4456 & 0 \\ 0 & 23.2727 \end{pmatrix}, & P_2 &= \begin{pmatrix} 0.0807 & 0 \\ 0 & 0.0625 \end{pmatrix}, \\ P_3 &= \begin{pmatrix} 12.6787 & 0 \\ 0 & 7.8742 \end{pmatrix}, & P_4 &= \begin{pmatrix} 3.0234 & 0 \\ 0 & 1.7847 \end{pmatrix}, \\ Q_1 &= \begin{pmatrix} 0.3209 & 0 \\ 0 & 0.2297 \end{pmatrix}, & Q_2 &= \begin{pmatrix} 0.0793 & 0 \\ 0 & 0.0647 \end{pmatrix}, \\ R &= \begin{pmatrix} 54.7847 & 0 \\ 0 & 38.6931 \end{pmatrix}, & S_1 &= \begin{pmatrix} 91.8249 & 0 \\ 0 & 60.4696 \end{pmatrix}, \\ S_2 &= \begin{pmatrix} 75.0553 & 0 \\ 0 & 43.9460 \end{pmatrix}. \end{aligned}$$

Therefore, according to Theorem 2, the QVNNs (1) have a unique equilibrium point which is globally robust stable.

In what follows, we consider a special model in this example and give simulation results for the sake of verification of the proposed results. We choose the following fixed network parameters:

$$\begin{aligned} D &= \begin{pmatrix} 0.3 & 0 \\ 0 & 0.32 \end{pmatrix}, & A &= (a_{ij})_{2 \times 2}, & B &= (b_{ij})_{2 \times 2}, \\ C &= (c_{ij})_{2 \times 2}, & J &= \begin{pmatrix} 0.1 - 0.1t - 0.2J + 0.05\kappa \\ -0.2 + 0.1t + 0.05J - 0.1\kappa \end{pmatrix}, \end{aligned} \tag{27}$$

where

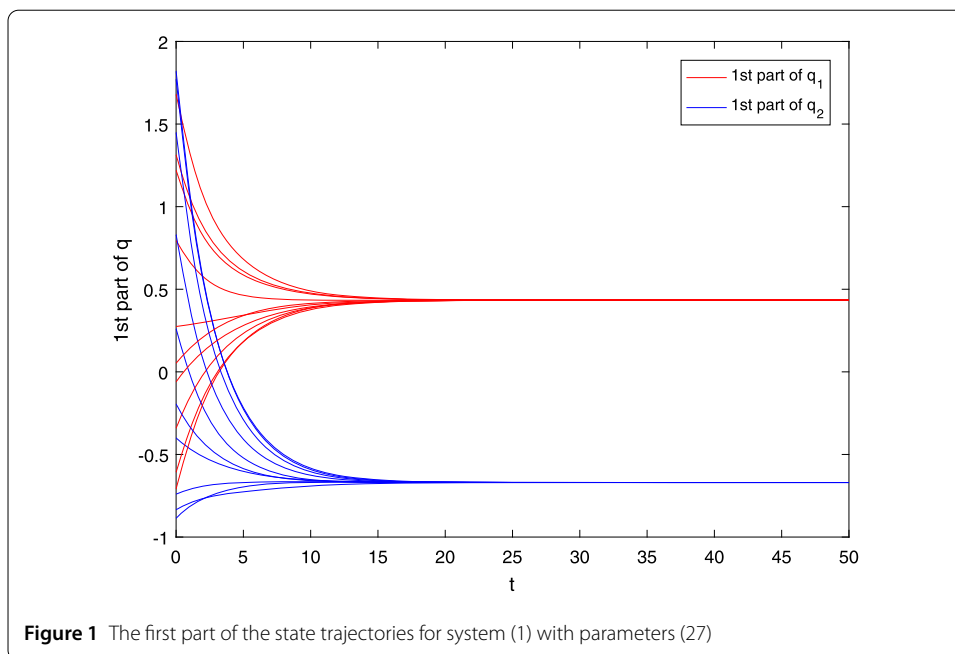
$$\begin{aligned} a_{11} &= 0.1 - 0.1t + 0.08J - 0.1\kappa, \\ a_{12} &= -0.1 + 0.1t - 0.1J + 0.05\kappa, \end{aligned}$$

$$\begin{aligned}
 a_{21} &= 0.15 + 0.1t - 0.15J - 0.1\kappa, \\
 a_{22} &= 0.08 - 0.08t + 0.1J + 0.1\kappa, \\
 b_{11} &= 0.1 - 0.06t + 0.1J + 0.09\kappa, \\
 b_{12} &= 0.05 + 0.05t - 0.05J - 0.03\kappa, \\
 b_{21} &= -0.1 + 0.1t - 0.08J - 0.05\kappa, \\
 b_{22} &= 0.1 - 0.1t + 0.1J + 0.08\kappa, \\
 c_{11} &= 0.05 - 0.03t + 0.05J - 0.04\kappa, \\
 c_{12} &= -0.07 + 0.05t - 0.06J + 0.05\kappa, \\
 c_{21} &= 0.1 + 0.1t - 0.1J - 0.08\kappa, \\
 c_{22} &= 0.05 - 0.05t + 0.05J + 0.03\kappa.
 \end{aligned}$$

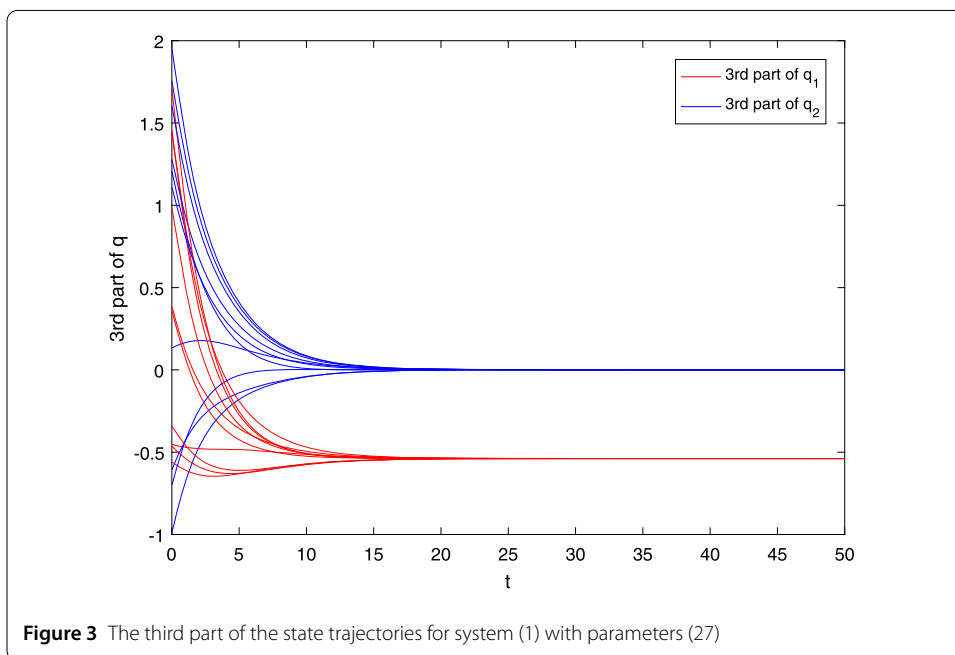
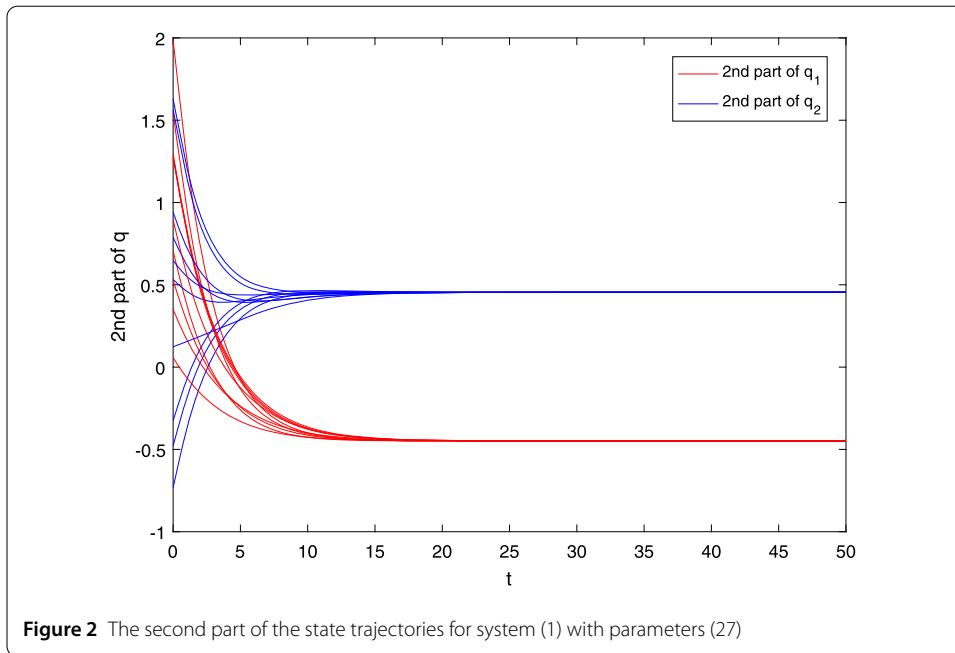
Besides, we choose the following functions as the activations and the delay kernel:

$$\begin{aligned}
 f_1(u) = f_2(u) &= 0.1(|u + 1| - |u - 1|), \quad \forall u \in \mathbb{H}, \\
 k(s) &= e^{-s}, \quad \forall s \in [0, +\infty).
 \end{aligned}$$

Based on these fixed parameters, we perform numerical simulation of the system by employing the fourth-order Runge–Kutta methods. Figures 1, 2, 3 and 4 show that the four parts of the states of the considered system, where the initial conditions are chosen by 10 random constant vectors. It can be seen from these figures that each neuron state converges to the stable equilibrium point, which is  $(0.4339 - 0.4488t - 0.5384J + 0.1301\kappa, -0.6699 + 0.4545t + 0.0010J - 0.4327\kappa)^T$ .

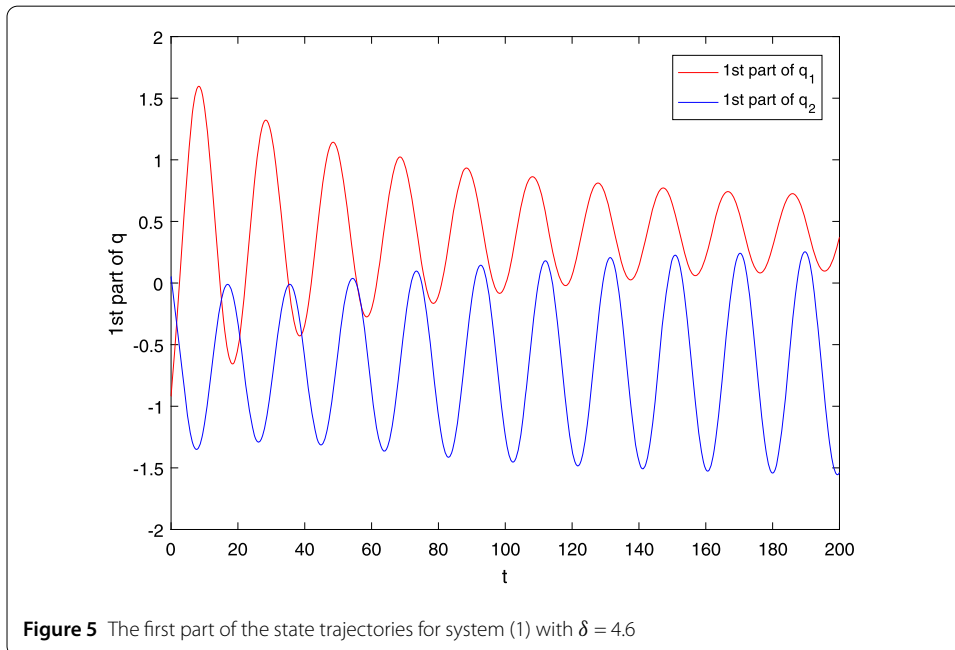
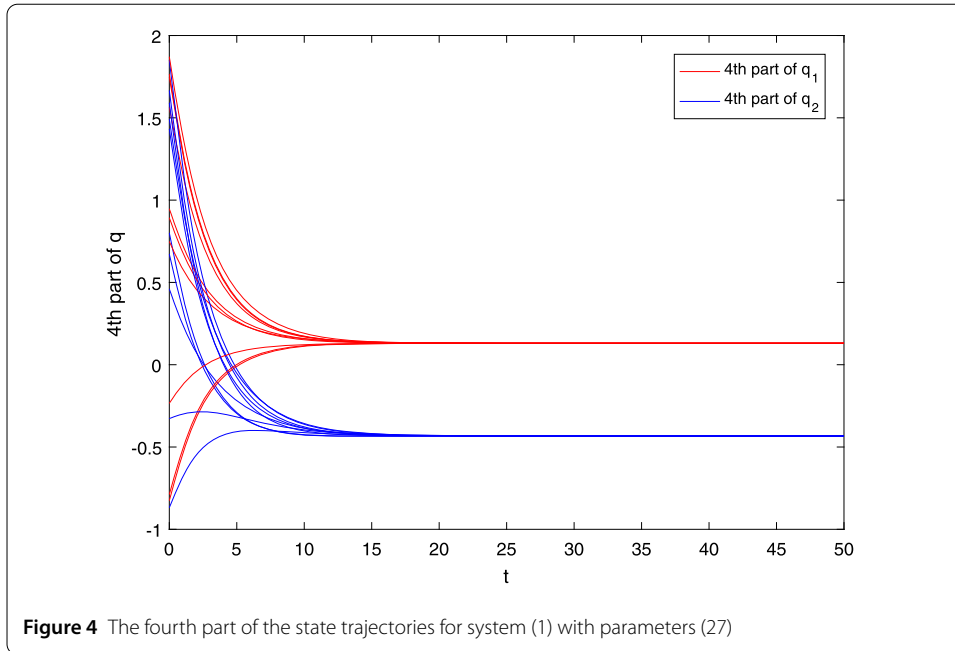


**Figure 1** The first part of the state trajectories for system (1) with parameters (27)



*Remark 3* Although the NNs (1) are quaternion-valued, the stability criteria are expressed in the form of LMIs (4) and (9), which are real-valued. In Example 1, we see that these LMIs can be checked directly by the mathematical software MATLAB.

*Remark 4* In [35], the authors considered the robust stability of QVNNs with both leakage and discrete delays but without distributed delay. The criteria obtained in [35] cannot be applied to check the robust stability of the system in Example 1, since the system has distributed delay.

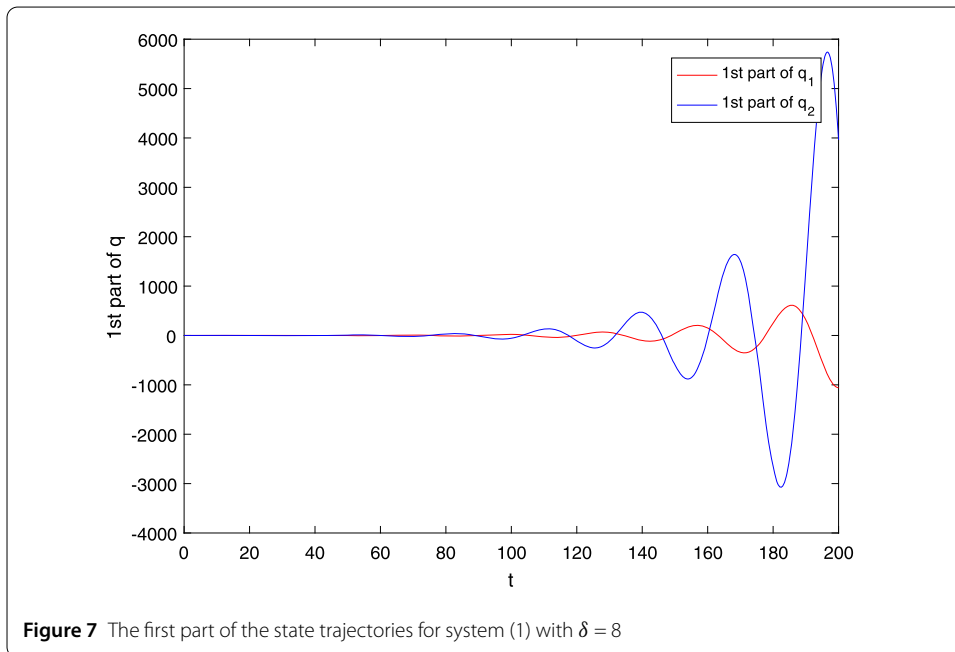
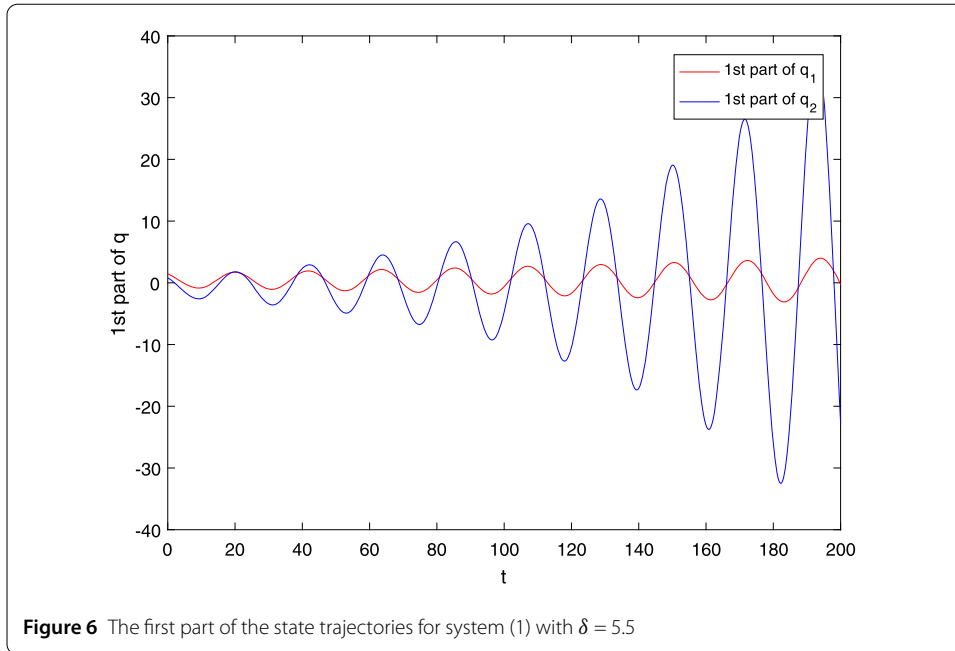


*Remark 5* For investigating the effects of time delays on the system, we set leakage delay  $\delta = 4.6, 5.5,$  and  $8$  in Example 1. Then we find that the equilibrium point of the system is unstable. Figures 5, 6 and 7 show that the first part of the states of the system do not converge to the equilibrium point.

*Example 2* Suppose that the parameters of system (1) are given as follows:

$$\check{D} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.7 \end{pmatrix},$$





$$\check{A} = \begin{pmatrix} -0.4 - 0.3t & -0.32 - 0.24t \\ -0.3 - 0.4t & -0.54 - 0.72t \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0.3 + 0.4t & 0.32 + 0.24t \\ 0.3 + 0.4t & 0.72 + 0.54t \end{pmatrix},$$

$$\check{B} = \begin{pmatrix} -0.24 - 0.32t & -0.18 - 0.24t \\ -0.3 - 0.4t & -0.32 - 0.24t \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0.32 + 0.24t & 0.18 + 0.24t \\ 0.4 + 0.3t & 0.24 + 0.32t \end{pmatrix},$$

$$\check{C} = \hat{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad \delta = 0, \quad \tau = 2.$$

According to the matrices  $\check{A}$ ,  $\hat{A}$ ,  $\check{B}$ ,  $\hat{B}$ ,  $\check{C}$  and  $\hat{C}$ , we get the following matrices:

$$\tilde{A} = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.9 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, by using YALMIP with solver of SDPT3 in MATLAB, we obtain the feasible solutions of LMI (9) in Theorem 2 as follows:

$$\begin{aligned} P_1 &= \begin{pmatrix} 10.0939 & 0 \\ 0 & 7.0398 \end{pmatrix}, & P_2 &= \begin{pmatrix} 0.0058 & 0 \\ 0 & 0.0077 \end{pmatrix}, \\ P_3 &= \begin{pmatrix} 974.6 & 0 \\ 0 & 1190.1 \end{pmatrix}, & P_4 &= \begin{pmatrix} 1.4122 & 0 \\ 0 & 1.2880 \end{pmatrix}, \\ Q_1 &= \begin{pmatrix} 0.0120 & 0 \\ 0 & 0.0075 \end{pmatrix}, & Q_2 &= \begin{pmatrix} 0.0053 & 0 \\ 0 & 0.0040 \end{pmatrix}, \\ R &= \begin{pmatrix} 0.0609 & 0 \\ 0 & 0.0848 \end{pmatrix}, & S_1 &= \begin{pmatrix} 40.2801 & 0 \\ 0 & 55.7370 \end{pmatrix}, \\ S_2 &= \begin{pmatrix} 35.2199 & 0 \\ 0 & 32.0899 \end{pmatrix}. \end{aligned}$$

Therefore, according to Theorem 2, system (1) has a unique equilibrium point which is globally robust stable.

*Remark 6* In Example 2, the system parameters are complex-valued. So the system with these parameters can be viewed as CVNNs. Moreover, since  $\delta = 0$  and  $\check{C} = \hat{C} = 0$ , the CVNNs have no leakage delay nor distributed delay. Then we try to apply the criteria in [22] to check the robust stability of the CVNNs. Via YALMIP with solver of SDPT3 in MATLAB, we cannot find feasible solutions of LMIs in [22]. Therefore, the results obtained in [22] cannot be applied to check the robust stability of the CVNNs.

## 5 Conclusion

In this paper, the robust stability problem of parametric uncertain QVNNs with both leakage, discrete and distributed delays has been investigated. Based on Homeomorphic mapping theorem and Lyapunov theorem, some criteria are obtained to check the existence, uniqueness, and global robust stability of the equilibrium point of the delayed QVNNs. Owing to using LMI approach and the modulus inequality technique, the presentation of the obtained criteria is in the form of real-valued LMIs, which can be solved by the mathematical software MATLAB directly and feasibly. In addition, two numerical examples are provided to substantiate the effectiveness of the proposed LMI conditions. It should be noted that the activation functions are continuous in this paper. Considering that the discontinuous neural network is one of the important dynamic systems, therefore, our further works will research the stability problem of QVNNs with discontinuous activations.

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**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

XC conceived, designed and performed the experiments. XC, LL and ZL wrote the paper. All authors read and approved the final manuscript.

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