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On the existence of solutions of a three steps crisis integro-differential equation

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Abstract

There are many natural phenomena including a crisis (such as a spate or contest) which could be described in three steps. We investigate the existence of solutions for a three step crisis integro-differential equation. We suppose that the second step is a point-wise defined singular fractional differential equation.

Keywords: Caputo derivative; Point-wise defined singular equation; Three steps crisis phenomena

1 Introduction

In most phenomena there appears usually a crisis. Our imagination as regards crises has effects on economy while there are distinct types of crisis-phenomena study in different fields of science such chemistry, social sciences, physics, mathematics, engineering and economy (see, for example, [1–7] and [8]). Considering the importance of modeling of crisis phenomena, some researchers are working and publishing in this area (see, for example, [9–13]). In 2016, Almeida, Bastos and Monteiro published a paper about modeling of some real phenomena by fractional differential equations [14]. As is well known, one of the best methods for mathematical describing this type phenomena is modeling of the problems as singular fractional integro-differential equations, which have been studied by researchers especially in recent decades (see, for example, [15–20] and [21]).

In 2010, Agarwal, O'Regan and Stanek investigated the existence of solutions for the problem $D^\alpha u(t) + f(t,u(t)) = 0$ with boundary conditions $u'(0) = \cdots = u^{(n-1)} = 0$ and $u(1) = \int_0^1 u(s)\,d\mu(s)$, where $n \geq 2$, $\alpha \in (n-1,n)$, $\mu(s)$ is a functional of bounded variation with $\int_0^1 d\mu(s) < 1$, and f may have a singularity at $t = 0$ [15]. They reviewed the existence of positive solutions for the system $D^\alpha u_i(t) + f_i(t,u_1(t),u_2(t)) = 0$ with boundary conditions $u_i(0) = u_i'(0) = 0$ and $u_i(1) = \int_0^1 u_i(t)\,d\eta(t)$ for $i = 1,2$, where $t \in (0,1)$, $\alpha \in (2,3]$, $\int_0^1 u_i(t)\,d\eta(t)$ denotes the Riemann–Stieltjes integral, $f_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ and D^α is the Riemann–Liouville fractional derivative of order α [16]. In 2013, Bai and Qui studied the singular problem $D^\alpha u + f(t,u,D^\gamma u,D^\mu u) + g(t,u,D^\gamma u,D^\mu u) = 0$ with boundary conditions $u(0) = u'(0) = u''(0) = u'''(0) = 0$, where $3 < \alpha < 4$, $0 < \gamma < 1$, $1 < \mu < 2$, D^α is the Caputo fractional derivative and f is a Caratheodory function on $[0,1] \times (0,\infty)^3$ [17]. Recently, the multi-singular point-wise defined fractional integro-differential equation $D^\mu x(t) + f(t,x(t),x'(t),D^\beta x(t),I^p x(t)) = 0$ with boundary conditions $x'(0) = x(\xi)$ and $x(1) = \int_0^\eta x(s)\,ds$ when $\mu \in [2,3)$ and $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)\,ds$ and $x^{(j)}(0) = 0$ for $j = 2,\ldots,[\mu]-1$ when

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$\mu \in [3,\infty)$ has been studied, where $0 \le t \le 1$, $x \in C^1[0,1]$, $\mu \in [2,\infty)$, $\beta, \xi, \eta \in (0,1)$, $p > 1$, D^μ is the Caputo fractional derivative of order μ and $f : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ is a function such that $f(t,\cdot,\cdot,\cdot,\cdot)$ is singular at some points $t \in [0,1]$ [19]. By using these ideas and providing a new method for modeling of crisis phenomena, we investigate the existence of solutions for the point-wise defined three steps crisis integro-differential equation

$$D^\alpha x(t) + f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi)x(\xi)\,d\xi, \phi\bigl(x(t)\bigr)\right) = 0 \tag{1}$$

with boundary conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \ge 2$, $\lambda, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that $\|\phi(x) - \phi(y)\| \le \theta_0\|x - y\| + \theta_1\|x' - y'\|$ for some nonnegative real numbers θ_0 and $\theta_1 \in [0,\infty)$ and all $x, y \in X$, D^α is the Caputo fractional derivative of order α, $f(t, x_1(t), \dots, x_5(t)) = f_1(t, x_1(t), \dots, x_5(t))$ for all $t \in [0,\lambda)$, $f(t, x_1(t), \dots, x_5(t)) = f_2(t, x_1(t), \dots, x_5(t))$ for all $t \in [\lambda, \mu]$ and $f(t, x_1(t), \dots, x_5(t)) = f(t, x_1(t), \dots, x_5(t))$ for all $t \in (\mu, 1]$, $f_1(t,\cdot,\cdot,\cdot,\cdot)$ and $f_3(t,\cdot,\cdot,\cdot,\cdot)$ are continuous on $[0,\lambda)$ and $(\mu, 1]$ and $f_2(t,\cdot,\cdot,\cdot,\cdot)$ is multi-singular [19].

2 Preliminaries

Recall that $D^\alpha x(t) + f(t) = 0$ is a point-wise defined equation on $[0,1]$ if there exists a set $E \subset [0,1]$ such that the measure of E^c is zero and the equation holds on E [19]. In this paper, we use $\|\cdot\|_1$ for the norm of $L^1[0,1]$, $\|\cdot\|$ for the sup norm of $Y = C[0,1]$ and $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for the norm of $X = C^1[0,1]$. As is well known, the Riemann–Liouville integral of order p with the lower limit $a \ge 0$ for a function $f : (a,\infty) \to \mathbb{R}$ is defined by $I^p_{a^+} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s)\,ds$, provided that the right-hand side is point-wise defined on (a,∞) [22]. We denote $I^p_{0^+} f(t)$ by $I^p f(t)$. Also, the Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}\,ds$, where $n = [\alpha] + 1$ and $f : (a,\infty) \to \mathbb{R}$ is a function [22]. Let Ψ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t > 0$ (see [23]). One can check that $\psi(t) < t$ for all $t > 0$. Let (X,d) be a metric space and $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ two maps. Then T is called an α-admissible map whenever $\alpha(x,y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$ [23]. The map T is called an α-admissible map whenever $\alpha(x,y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$ [23]. Let (X,d) be a metric space, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever $\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y))$ for all $x, y \in X$ [23]. To prove the existence of solutions, we need next results.

Lemma 2.1 ([23]) *Let (X,d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0,\infty)$ a map and $T : X \to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$, then T has a fixed point.*

Lemma 2.2 ([24]) *Let $n - 1 \le \alpha < n$ and $x \in C(0,1) \cap L^1(0,1)$. Then we have $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0, \dots, c_{n-1}$.*

Lemma 2.3 ([21]) *Let $\beta > 0$ and $\alpha > -1$. Then $\int_0^t (t-s)^{\alpha-1} s^\beta\,ds = B(\beta+1, \alpha)t^{\alpha+\beta}$, where $B(\beta,\alpha) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.*

Lemma 2.4 ([25]) *Let E be a Banach space, $P \subseteq E$ a cone and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\overline{\Omega_1} \subset \Omega_2$. Suppose that $F : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either*

(i_1) $\|F(x)\| \leq \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Fx\| \geq \|x\|$ *for all* $x \in P \cap \partial\Omega_2$, *or*
(i_2) $\|Fx\| \geq \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Fx\| \leq \|x\|$ *for all* $x \in P \cap \partial\Omega_2$
holds. Then F *has a fixed point in* $P \cap (\Omega_2 \backslash \Omega_1)$.

3 Main results

Now, we are ready for providing our results.

Lemma 3.1 *Let* $\alpha \geq 2$, $n = [\alpha] + 1$ *and* $f \in L^1[0,1]$. *A map* u *is a solution for the pointwise defined equation* $D^\alpha x(t) + f(t) = 0$ *with boundary conditions* $x'(1) = x(0) = x''(0) = \cdots = x^{n-1}(0) = 0$ *if and only if* $u(t) = \int_0^1 G(t,s)f(s)\,ds$ *for all* $t \in [0,1]$, *where* $G(t,s) = \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ *whenever* $0 \leq t \leq s \leq 1$ *and* $G(t,s) = \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ *whenever* $0 \leq s \leq t \leq 1$.

Proof Let E be a subset of $[0,1]$ such that $m(E^c) = 0$ and $D^\alpha x(t) + f(t) = 0$ for all $t \in E$. Here, m is the Lebesgue measure on $\mathbb{R}$. Note that E is dense in $[0,1]$. Let $f_0 \in C[0,1]$ be a function such that $f_0 = f$ on E. Then we have

$$\begin{aligned}
I^\alpha\big(f(t)\big) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\left(\int_{[0,t]\cap E}(t-s)^{\alpha-1}f(s)\,ds + \int_{[0,t]\cap E^c}(t-s)^{\alpha}f(s)\,ds\right) \\
&= \frac{1}{\Gamma(\alpha)}\int_{[0,t]\cap E}(t-s)^{\alpha-1}f_0(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\left(\int_{[0,t]\cap E}(t-s)^{\alpha-1}f_0(s)\,ds + \int_{[0,t]\cap E^c}(t-s)^{\alpha-1}f_0(s)\,ds\right) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_0(s)\,ds = I^\alpha\big(f_0(t)\big)
\end{aligned}$$

for all $t \in E$. Let $t \in E^c \backslash \{0\}$. Choose a sequence $\{t_n\}_{n\geq 1}$ in E such that $t_n \to t^-$. Then

$$\begin{aligned}
I^\alpha\big(f(t)\big) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \\
&= \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-s)^{\mu-1}f(s)\,ds = \lim_{n\to\infty} I^\alpha\big(f(t_n)\big) \\
&= \lim_{n\to\infty} I^\alpha\big(f_0(t_n)\big) = \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-s)^{\alpha-1}f_0(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \\
&= I^\alpha\big(f_0(t)\big).
\end{aligned}$$

For $t = 0 \in E^c$, we get $I^\alpha(f(t)) = I^\alpha(f_0(t)) = 0$ and so $I^\alpha(f(t)) = I^\alpha(f_0(t))$ for all $t \in [0,1]$. Thus, the equation $D^\alpha x(t) + f(t) = 0$ equivalents to $I^\alpha(D^\alpha x(t)) = I^\alpha(-f_0(t))$ on $[0,1]$. By using Lemma 2.2 and the boundary condition, we get $x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + c_1 t$ and so $x'(t) = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)\,ds + c_1$. Hence, $x'(1) = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)\,ds + c_1$. Since $x'(1) = 0$, $c_1 = \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)\,ds$ and so

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + \frac{t}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)\,ds = \int_0^1 G(t,s)y(s)\,ds,$$

where $G(t,s) = \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ whenever $0 \le t \le s \le 1$ and $G(t,s) = \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ whenever $0 \le s \le t \le 1$. Also, an easy calculation shows that $u(t) = \int_0^1 G(t,s)f(s)\,ds$ is a solution for the equation with the boundary conditions. This completes the proof. □

Note that for the Green function $G(t,s)$ in the last result we have $G(t,s) \ge \frac{(\alpha-2)|t-s|^{\alpha-1}}{\Gamma(\alpha)} \ge 0$, $G(t,s) \le \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$, $\frac{\partial}{\partial t}G(t,s) \ge 0$ and $\frac{\partial}{\partial t}G(t,s) \le \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ for all $t,s \in [0,1]$. Also, G and $\frac{\partial}{\partial t}G$ are continuous with respect to t. Consider the space $X = C^1[0,1]$ with the norm $\|\cdot\|_*$, where $\|x\|_* = \max\{\|x\|, \|x'\|\}$ and $\|\cdot\|$ is the supremum norm on $C[0,1]$. Let $\lambda, \mu \in (0,1)$ with $\lambda < \mu$. Suppose that f_1 and f_3 are continuous functions (with respect to the first variable) on $[0,\lambda] \times X^5$ and $[\mu,1] \times X^5$, respectively, and f_2 is a function on $(\lambda,\mu) \times X^5$ which is singular at some points $t \in (\lambda,\mu)$. Let f be a map on $[0,1] \times X^5$ such that $f|_{[0,\lambda]\times X^5} = f_1$, $f|_{(\lambda,\mu)\times X^5} = f_2$ and $f|_{[\mu,0]\times X^5} = f_3$. We denote this case briefly by $[\lambda,\mu,f = (f_1,f_2,f_3)]$. Define the map $F: X \to X$ by $F_x(t) = \int_0^1 G(t,s)f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi(x(s)))\,ds$ for all $t \in [0,1]$. Note that the singular point-wise defined equation (1) has a solution $u_0 \in X$ if and only if u_0 a fixed point of the map F.

Theorem 3.2 *Let* $[\lambda,\mu,f = (f_1,f_2,,f_3)]$ *with* $f_1(s,0,0,0,0,0) = f_3(t,0,0,0,0,0) = 0$ *for all* $s \in [0,\lambda]$ *and* $t \in [\mu,1]$. *Assume that there exist two maps* $H : X^5 \to [0,\infty)$ *and* $\Phi : (\lambda,\mu) \to [0,\infty)$ *such that* $f_2(t,x_1,x_2,\ldots,x_5) \le \Phi(t)H(x_1,x_2,\ldots,x_5)$ *for all* $(x_1,\ldots,x_5) \in X^5$ *and* almost *all* $t \in (\lambda,\mu)$, *where* $H : X^5 \to [0,\infty)$ *is nondecreasing with respect to all its components*, $\int_\lambda^\mu (1-s)^{\alpha-1}\Phi(s)\,ds < \infty$ *and* $\lim_{z\to 0^+} \frac{H(z,z,z,z,z)}{z} = 0$. *Suppose that the map* q *defined by* $q(t) = \lim_{\max\{\|x_1\|,\ldots,\|x_5\|\}\to\infty} \frac{f_2(t,x_1x_2,\ldots,x_5)}{\max\{\|x_1\|,\ldots,\|x_5\|\}}$ *for almost all* $t \in (\lambda,\mu)$ *has the property that* $\frac{\alpha-2}{\Gamma(\alpha)}\int_\lambda^\mu (\mu-s)^{\alpha-2}q(s)\,ds > 1$. *Assume that there exist nonnegative real numbers* $l_1,\ldots,l_5$, $l'_1,\ldots,l'_5$ *and mappings* $a_1,\ldots,a_5 : (\lambda,\mu) \to [0,\infty)$ *and* $\Lambda_1,\ldots,\Lambda_5 : X^5 \to [0,\infty)$ *such that* $|f_1(t,x_1,\ldots,x_5) - f_1(t,y_1,\ldots,y_5)| \le \sum_{i=1}^5 l_i|x_i - y_i|$,

$$\bigl|f_2(t,x_1,\ldots,x_5) - f_2(t,y_1,\ldots,y_5)\bigr| \le \sum_{i=1}^5 a_i(t)\Lambda_i\bigl(|x_1-y_1|,\ldots,|x_5-y_5|\bigr)$$

and $|f_3(t,x_1,\ldots,x_5) - f_3(t,y_1,\ldots,y_5)| \le \sum_{i=1}^5 l'_i|x_i - y_i|$ *for all* t *and* $x_1,\ldots,x_5 \in X$. *If* $\lim_{z\to 0^+} \frac{\Lambda_i(z,z,z,z,z)}{z} = q_i < \infty$ *and* $[\frac{L(1-(1-\lambda)^{\alpha-1})}{\Gamma(\alpha)} + \frac{L'}{\Gamma(\alpha)}(1-\mu)^{\alpha-1}] < 1$ *for* $i = 1,\ldots,5$, *where* $m_0 = \int_0^1 |h(\xi)|\,d\xi$, $L = l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + m_0 l_4 + \theta_0 l_5 + \theta_1 l_5$ *and* $L' = l'_1 + l'_2 + \frac{l'_3}{\Gamma(2-\beta)} + m_0 l'_4 + \theta_0 l'_5 + \theta_1 l'_5$, *then the problem* (1) *has a solution*.

Proof Consider the closed cone $P = \{x \in X : x(t) \ge 0 \text{ and } x'(t) \ge 0 \text{ for all } t \in [0,1]\}$ in X. Let $\epsilon > 0$ be given, $\{x_n\}_{n\ge 1}$ a sequence in X with $x_n \to x$. Choose a natural number N such that $\|x_n - x\| < \epsilon$ for all $n \ge N$. Take $\epsilon > 0$ such that

$$\left[\frac{L(1-(1-\lambda)^{\alpha-1})}{\Gamma(\alpha)} + \frac{(q_i+\epsilon)\epsilon}{\Gamma(\alpha-1)}\sum_{i=1}^5 M_i(\lambda,\mu) + \frac{L'}{\Gamma(\alpha)}(1-\mu)^{\alpha-1}\right] < 1$$

for $i = 1,\ldots,5$, where $M_i(\lambda,\mu) = \int_\lambda^\mu (1-s)^{\alpha-2}a_i(s)\,ds$. Note that

$$\begin{aligned}\bigl|F_{x_n}(t) - F_x(t)\bigr| \le \int_0^\lambda G(t,s)\biggl|&f_1\biggl(s,x_n(s),x'_n(s),D^\beta x_n(s),\int_0^s h(\xi)x_n(\xi)\,d\xi,\phi\bigl(x_n(s)\bigr)\biggr)\\ &- f_1\biggl(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\bigl(x(s)\bigr)\biggr)\biggr|\,ds\end{aligned}$$

$$
\begin{aligned}
&+\int_\lambda^\mu G(t,s)\biggl|f_2\biggl(s,x_n(s),x_n'(s),D^\beta x_n(s),\int_0^s h(\xi)x_n(\xi)\,d\xi,\phi\bigl(x_n(s)\bigr)\biggr)\\
&-f_2\biggl(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\bigl(x(s)\bigr)\biggr)\biggr|\,ds\\
&+\int_\mu^1 G(t,s)\biggl|f_3\biggl(s,x_n(s),x_n'(s),D^\beta x_n(s),\int_0^s h(\xi)x_n(\xi)\,d\xi,\phi\bigl(x_n(s)\bigr)\biggr)\\
&-f_3\biggl(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\bigl(x(s)\bigr)\biggr)\biggr|\,ds\\
\le{}&\int_0^\lambda G(t,s)\biggl(l_1\bigl|x_n(s)-x(s)\bigr|+l_2\bigl|x_n'(s)-x'(s)\bigr|+l_3\bigl|D^\beta(x_n-x)(s)\bigr|\\
&+l_4\int_0^s\bigl|x_n(\xi)-x(\xi)\bigr|\,d\xi+l_5\bigl|\phi\bigl(x_n(s)-x(s)\bigr)\bigr|\biggr)\\
&+\int_\lambda^\mu G(t,s)(a_1(s)\Lambda_1\biggl(\bigl|x_n(s)-x(s)\bigr|,\bigl|x_n'(s)-x'(s)\bigr|,\\
&\bigl|D^\beta(x_n-x)(s)\bigr|,\int_0^s\bigl|x_n(\xi)-x(\xi)\bigr|d\xi,\bigl|\phi\bigl(x_n(s)-x(s)\bigr)\bigr|\biggr)\\
&+\cdots+a_5(s)\Lambda_5\biggl(\bigl|x_n(s)-x(s)\bigr|,\bigl|x_n'(s)-x'(s)\bigr|,\\
&\bigl|D^\beta(x_n-x)(s)\bigr|,\int_0^s\bigl|x_n(\xi)-x(\xi)\bigr|d\xi,\bigl|\phi\bigl(x_n(s)-x(s)\bigr)\bigr|\biggr)\\
&+\int_\mu^1 G(t,s)\biggl(l_1'\bigl|x_n(s)-x(s)\bigr|+l_2'\bigl|x_n'(s)-x'(s)\bigr|+l_3'\bigl|D^\beta(x_n-x)(s)\bigr|\\
&+l_4'\int_0^s\bigl|x_n(\xi)-x(\xi)\bigr|\,d\xi+l_5'\bigl|\phi\bigl(x_n(s)-x(s)\bigr)\bigr|\biggr)\\
\le{}&\int_0^\lambda G(t,s)\biggl(l_1\|x_n-x\|+l_2\bigl\|x_n'-x'\bigr\|+\frac{l_3}{\Gamma(2-\beta)}\bigl\|x_n'-x'\bigr\|\\
&+m_0l_4\|x_n-x\|+\theta_0l_5\|x_n-x\|+\theta_1l_5\bigl\|x_n'-x'\bigr\|\biggr)\\
&+\int_\lambda^\mu G(t,s)(a_1(s)\Lambda_1\biggl(\|x_n-x\|,\bigl\|x_n'-x'\bigr\|,\frac{1}{\Gamma(2-\beta)}\bigl\|x_n'-x'\bigr\|,\\
&m_0\|x_n-x\|,\theta_0l_5\|x_n-x\|+\theta_1l_5\bigl\|x_n'-x'\bigr\|\biggr)\\
&+\cdots+a_5(s)\Lambda_1\biggl(\|x_n-x\|,\bigl\|x_n'-x'\bigr\|,\frac{1}{\Gamma(2-\beta)}\bigl\|x_n'-x'\bigr\|,\\
&m_0\|x_n-x\|,\theta_0l_5\|x_n-x\|+\theta_1l_5\bigl\|x_n'-x'\bigr\|\biggr)\,ds\\
&+\int_\mu^1 G(t,s)\biggl(l_1'\|x_n-x\|+l_2'\bigl\|x_n'-x'\bigr\|+\frac{l_3'}{\Gamma(2-\beta)}\bigl\|x_n'-x'\bigr\|\\
&+m_0l_4'\|x_n-x\|+\theta_0l_5'\|x_n-x\|+\theta_1l_5'\bigl\|x_n'-x'\bigr\|\biggr)\,ds\\
\le{}&\biggl(l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m_0+l_5\theta_0+l_5\theta_1\biggr)\|x_n-x\|_*\int_0^\lambda G(t,s)\,ds
\end{aligned}
$$

$$+\int_\lambda^\mu G(t,s)(\sum_{i=1}^5 a_i(s)\Lambda_i\big(l\|x_n-x\|_*, l\|x_n-x\|_*, l\|x_n-x\|_*,$$
$$l\|x_n-x\|_*, l\|x_n-x\|_*\big)\,ds$$
$$+\left(l_1'+l_2'+\frac{l_3'}{\Gamma(2-\beta)}+l_4'm_0+l_5'\theta_0+l_5'\theta_1\right)\|x_n-x\|_*\int_\mu^1 G(t,s)\,ds$$

for all $t\in[0,1]$, where $l=\max\{1,\frac{1}{\Gamma(2-\beta)},m_0,\theta_0+\theta_1\}$. For each $1\le i\le 5$ choose $0<\delta_i(\epsilon)<\epsilon^2$ such that $\frac{\Lambda_i(z,z,z,z,z)}{z}<q_i+\epsilon$ for all $z\in(0,\delta_i(\epsilon)]$. Thus, $\Lambda_i(z,z,z,z,z)<(q_i+\epsilon)z$ for all $z\in(0,\delta_i(\epsilon)]$ and $1\le i\le 5$. Put $\delta:=\min_{1\le i\le 5}\delta_i(\epsilon)$. Then we have

$$\Lambda_i(\delta,\delta,\delta,\delta,\delta)<(q_i+\epsilon)\delta<(q_i+\epsilon)\epsilon^2.$$

Let m_1 be a natural number such that $l\|x_n-x\|_*<\delta$ for all $n\ge m_1$. This implies that $\Lambda_i(l\|x_n-x\|_*,\dots,l\|x_n-x\|_*)<\Lambda_i(\delta,\delta,\delta,\delta,\delta)<(q_i+\epsilon)\epsilon^2$ for all $n\ge m_1$ and $i=1,\dots,5$. Thus,

$$\big|F_{x_n}(t)-F_x(t)\big|\le L\|x_n-x\|_*\int_0^\lambda G(t,s)\,ds$$
$$+(q_i+\epsilon)\epsilon^2\int_\lambda^\mu G(t,s)\sum_{i=1}^5 a_i(s)\,ds+L'\|x_n-x\|_*\int_\mu^1 G(t,s)\,ds$$

for all $n\ge\max\{N,m_1\}$. This implies that

$$\begin{aligned}&\big|F_{x_n}(t)-F_x(t)\big|\\&\quad\le\frac{L\epsilon t}{\Gamma(\alpha-1)}\int_0^\lambda(1-s)^{\alpha-2}\,ds\\&\qquad+\frac{(q_i+\epsilon)\epsilon^2 t}{\Gamma(\alpha-1)}\sum_{i=1}^5\int_\lambda^\mu(1-s)^{\alpha-2}a_i(s)\,ds+\frac{L'\epsilon t}{\Gamma(\alpha-1)}\int_\mu^1(1-s)^{\alpha-2}\,ds\\&\quad=\frac{L\epsilon t(1-(1-\lambda)^{\alpha-1})}{\Gamma(\alpha)}+\frac{(q_i+\epsilon)\epsilon^2 t}{\Gamma(\alpha-1)}\sum_{i=1}^5 M_i(\lambda,\mu)+\frac{L'\epsilon t}{\Gamma(\alpha)}(1-\mu)^{\alpha-1}\end{aligned}$$

for all $n\ge\max\{N,m_1\}$ and $t\in[0,1]$ and so

$$\|F_{x_n}-F_x\|\le\left[\frac{L(1-(1-\lambda)^{\alpha-1})}{\Gamma(\alpha)}+\frac{(q_i+\epsilon)\epsilon}{\Gamma(\alpha-1)}\sum_{i=1}^5 M_i(\lambda,\mu)+\frac{L'}{\Gamma(\alpha)}(1-\mu)^{\alpha-1}\right]\epsilon<\epsilon.$$

By using similar calculations, we get

$$\big|F'_{x_n}(t)-F'_x(t)\big|\le\int_0^\lambda\frac{\partial G}{\partial t}(t,s)\bigg|f_1\left(s,x_n(s),x_n'(s),D^\beta x_n(s),\int_0^s h(\xi)x_n(\xi)\,d\xi,\phi\big(x_n(s)\big)\right)$$
$$-f_1\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\right)\bigg|\,ds$$
$$+\int_\lambda^\mu\frac{\partial G}{\partial t}(t,s)\bigg|f_2\left(s,x_n(s),x_n'(s),D^\beta x_n(s),\int_0^s h(\xi)x_n(\xi)\,d\xi,\phi\big(x_n(s)\big)\right)$$

$$
\begin{aligned}
&-f_2\Big(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\Big|\,ds\\
&+\int_\mu^1 \frac{\partial G}{\partial t}(t,s)\Big|f_3\Big(s,x_n(s),x_n'(s),D^\beta x_n(s),\int_0^s h(\xi)x_n(\xi)\,d\xi,\phi\big(x_n(s)\big)\Big)\\
&-f_3\Big(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\Big|\,ds\\
\le{}&\int_0^\lambda \frac{\partial G}{\partial t}(t,s)\Big(l_1\|x_n-x\|+l_2\big\|x_n'-x'\big\|+\frac{l_3}{\Gamma(2-\beta)}\big\|x_n'-x'\big\|\\
&+m_0l_4\|x_n-x\|+\theta_0l_5\|x_n-x\|+\theta_1l_5\big\|x_n'-x'\big\|\Big)\,ds\\
&+\int_\lambda^\mu \frac{\partial G}{\partial t}(t,s)(a_1(s)\Lambda_1\Big(\|x_n-x\|,\big\|x_n'-x'\big\|,\frac{1}{\Gamma(2-\beta)}\big\|x_n'-x'\big\|,\\
&m_0\|x_n-x\|,\theta_0l_5\|x_n-x\|+\theta_1l_5\big\|x_n'-x'\big\|\Big)\\
&+\cdots+a_5(s)\Lambda_5\Big(\|x_n-x\|,\big\|x_n'-x'\big\|,\frac{1}{\Gamma(2-\beta)}\big\|x_n'-x'\big\|,\\
&m_0\|x_n-x\|,\theta_0l_5\|x_n-x\|+\theta_1l_5\big\|x_n'-x'\big\|\Big)\,ds\\
&+\int_\mu^1 \frac{\partial G}{\partial t}(t,s)\Big(l_1'\|x_n-x\|+l_2'\big\|x_n'-x'\big\|+\frac{l_3'}{\Gamma(2-\beta)}\big\|x_n'-x'\big\|\\
&+m_0l_4'\|x_n-x\|+\theta_0l_5'\|x_n-x\|+\theta_1l_5'\big\|x_n'-x'\big\|\Big)\,ds\\
\le{}&\bigg[\frac{L(1-(1-\lambda)^{\alpha-1})}{\Gamma(\alpha)}+\frac{(q_i+\epsilon)\epsilon}{\Gamma(\alpha-1)}\sum_{i=1}^5 M_i(\lambda,\mu)+\frac{L'}{\Gamma(\alpha)}(1-\mu)^{\alpha-1}\bigg]\epsilon
\end{aligned}
$$

for all $n\ge\max\{N,m_1\}$ and $t\in[0,1]$. Hence, $\|F'_{x_n}-F'_x\|\le\epsilon$ for sufficiently large n and so $\|F_{x_n}-F_x\|_*=\max\{\|F_{x_n}-F_x\|,\|F'_{x_n}-F'_x\|\}<\epsilon$ for sufficiently large n. This implies that $F_{x_n}\to F_x$ in X. Now, we prove that F maps bounded sets into bounded sets of X. Let M be a bounded set of X. Choose $r>0$ such that $\|x\|_*<r$ for all $x\in M$. Let $x\in M$. Then

$$
\begin{aligned}
\big|F_x(t)\big|\le{}&\bigg|\int_0^\lambda G(t,s)f_1\Big(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\,ds\\
&+\int_\lambda^\mu G(t,s)f_2\Big(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\,ds\\
&+\int_\mu^1 G(t,s)f_3\Big(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\,ds\bigg|\\
\le{}&\int_0^\lambda G(t,s)\Big|f_1\Big(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\\
&-f_1(s,0,0,0,0,0)\Big|\,ds\\
&+\int_0^\lambda G(t,s)\big|f_1(s,0,0,0,0,0)\big|\,ds\\
&+\int_\lambda^\mu G(t,s)\Phi(s)H\Big(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\,ds
\end{aligned}
$$

$$\begin{aligned}
&+\int_\mu^1 G(t,s)\Big|f_3\Big(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\Big)\\
&-f_3(s,0,0,0,0,0)\Big|\,ds\\
&+\int_\mu^1 G(t,s)\big|f_3(s,0,0,0,0,0)\big|\,ds\\
\le{}&\int_0^\lambda G(t,s)\Big(l_1\|x\|+l_2\big\|x'\big\|+l_3\big\|D^\beta x\big\|+l_4\|x\|\int_0^s\big|h(\xi)\big|\,d\xi+l_5\phi\big(\|x\|\big)\Big)\,ds\\
&+H\big(l\|x\|_*,\dots,l\|x\|_*\big)\int_\lambda^\mu G(t,s)\Phi(s)\\
&+\int_\mu^1 G(t,s)\Big(l'_1\|x\|+l'_2\big\|x'\big\|+l'_3\big\|D^\beta x\big\|\\
&+l'_4\|x\|\int_0^s\big|h(\xi)\big|\,d\xi+l'_5\phi\big(\|x\|\big)\Big)\,ds\\
\le{}&\frac{t}{\Gamma(\alpha-1)}\int_0^\lambda(1-s)^{\alpha-2}\Big(l_1\|x\|+l_2\big\|x'\big\|+\frac{l_3}{\Gamma(2-\beta)}\big\|x'\big\|\\
&+l_4m_0\|x\|+l_5\theta_0\|x\|+l_5\theta_1\big\|x'\big\|\Big)\,ds\\
&+\frac{H(l\|x\|_*,\dots,l\|x\|_*)t}{\Gamma(\alpha-1)}\int_\lambda^\mu(1-s)^{\alpha-2}\Phi(s)\\
&+\frac{t}{\Gamma(\alpha-1)}\int_\mu^1(1-s)^{\alpha-2}\Big(l'_1\|x\|+l'_2\big\|x'\big\|,\frac{l'_3}{\Gamma(2-\beta)}\big\|x'\big\|+l'_4m_0\|x\|\\
&+l'_5\theta_0\|x\|+l'_5\theta_1\big\|x'\big\|\Big)\,ds\\
\le{}&\frac{tL}{\Gamma(\alpha-1)}\|x\|_*+\frac{H(l\|x\|_*,\dots,l\|x\|_*)t}{\Gamma(\alpha-1)}\int_\lambda^\mu(1-s)^{\alpha-2}\Phi(s)\,ds+\frac{tL}{\Gamma(\alpha-1)}\|x\|_*
\end{aligned}$$

and so $\|F_x\|\le\frac{L}{\Gamma(\alpha-1)}\|x\|_*+\frac{H(l\|x\|_*,\dots,l\|x\|_*)}{\Gamma(\alpha-1)}\int_\lambda^\mu(1-s)^{\alpha-2}\Phi(s)\,ds+\frac{L}{\Gamma(\alpha-1)}\|x\|_*$. By using similar calculations, we get $\|F'_x\|\le\frac{L}{\Gamma(\alpha-1)}\|x\|_*+\frac{H(l\|x\|_*,\dots,l\|x\|_*)}{\Gamma(\alpha-1)}\int_\lambda^\mu(1-s)^{\alpha-2}\Phi(s)\,ds+\frac{L}{\Gamma(\alpha-1)}\|x\|_*$. This implies that

$$\begin{aligned}
\|F_x\|_*&=\max\big\{\|F_x\|,\big\|F'_x\big\|\big\}\\
&\le\frac{L}{\Gamma(\alpha-1)}\|x\|_*+\frac{H(l\|x\|_*,\dots,l\|x\|_*)}{\Gamma(\alpha-1)}\int_\lambda^\mu(1-s)^{\alpha-2}\Phi(s)\,ds+\frac{L}{\Gamma(\alpha-1)}\|x\|_*\\
&<\infty.
\end{aligned}$$

This proves the claim. Since G and G' are continuous with respect to t, it is easy to check that $F_x(t_2)\to F_x(t_1)$ as $t_2\to t_1$. By using the Arzela–Ascoli theorem, we get $\overline{T(M)}$ is relatively compact and so $F:P\to P$ is completely continuous. Since $\lim_{z\to0^+}\frac{H(z,z,z,z,z)}{z}=0$, one concludes that $\lim_{\|x\|_*\to0^+}\frac{H(l\|x\|_*,\dots,l\|x\|_*)}{l\|x\|_*}=0$. Let $\epsilon>0$ be given. Choose $\delta=\delta(\epsilon)>0$ such that $\|x\|_*<\delta$ implies $\frac{H(l\|x\|_*,\dots,l\|x\|_*)}{l\|x\|_*}<\epsilon$ and so $H(l\|x\|_*,\dots,l\|x\|_*)<\epsilon l\|x\|_*$. Since $\frac{L(1-(1-\lambda)^{\alpha-1})+L'(1-\mu)^{\alpha-1}}{\Gamma(\alpha)}<1$, there exists $\epsilon_0>0$ such that

$$\frac{L(1-(1-\lambda)^{\alpha-1})+L'(1-\mu)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\epsilon_0 l\|\Phi\|^*}{\Gamma(\alpha-1)}<1,$$

where $\|\Phi\|^* = \int_\lambda^\mu (1-s)^{\alpha-2}\Phi(s)\,ds$. Let $\delta_0 = \delta(\epsilon_0)$. Define $\Omega_1 = \{x \in X \text{ s.t. } \|x\|_* < \delta\}$. Then

$$\begin{aligned}
\left|F_x(t)\right| &\le \int_0^\lambda G(t,s)\left|f_1\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\right)\right| ds \\
&\quad + \int_\lambda^\mu G(t,s)\left|f_2\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\right)\right| ds \\
&\quad + \int_\mu^1 G(t,s)\left|f_3\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\right)\right| ds \\
&\le \int_0^\lambda G(t,s)\bigg(l_1\|x\| + l_2\left\|x'\right\| + \frac{l_3}{\Gamma(2-\beta)}\left\|x'\right\| + l_4 m_0\|x\| \\
&\quad + l_5\theta_0\|x\| + l_5\theta_1\left\|x'\right\|\bigg) ds \\
&\quad + \int_\lambda^\mu G(t,s)\Phi(s)H\left(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\right) ds \\
&\quad + \int_\mu^1 G(t,s)\bigg(l'_1\|x\| + l'_2\left\|x'\right\| + \frac{l'_3}{\Gamma(2-\beta)}\left\|x'\right\| + l'_4 m_0\|x\| \\
&\quad + l'_5\theta_0\|x\| + l'_5\theta_1\left\|x'\right\|\bigg) ds \\
&\le \frac{tL}{\Gamma(\alpha-1)}\|x\|_* \int_0^\lambda (1-s)^{\alpha-2}\,ds \\
&\quad + \frac{tH(l\|x\|_*,l\|x\|_*,l\|x\|_*,l\|x\|_*,l\|x\|_*)}{\Gamma(\alpha-1)}\int_\lambda^\mu (1-s)^{\alpha-2}\Phi(s)\,ds \\
&\quad + \int_\mu^1 \frac{tL'}{\Gamma(\alpha-1)}\|x\|_* \int_0^\lambda (1-s)^{\alpha-2}\,ds
\end{aligned}$$

for all $x \in \Omega_1$ and $t \in [0,1]$. Hence,

$$\|F_x\| \le \left[\frac{L(1-(1-\lambda)^{\alpha-1}) + L'(1-\mu)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\epsilon_0 l\|\Phi\|^*}{\Gamma(\alpha-1)}\right]\|x\|_* \le \|x\|_*.$$

Similarly, we get $\|F'_x\| \le \|x\|_*$ and so $\|F_x\|_* \le \|x\|_*$. Since $\lim_{\max\|x_i\|\to\infty} \frac{f_2(t,x_1x_2,\dots,x_5)}{\max\|x_i\|} = q(t)$, there exists $R = R(\epsilon) > 0$ such that $\max\|x_i\| > R(\epsilon)$ implies that $\frac{f_2(t,x_1x_2,\dots,x_5)}{\max\|x_i\|} > q(t) - \epsilon$ and so $f_2(t,x_1x_2,\dots,x_5) > (\max\|x_i\|)(q(t)-\epsilon)$. Recall that

$$\frac{\alpha-2}{\Gamma(\alpha)}\int_\lambda^\mu (\mu-s)^{\alpha-1}\,ds - \frac{\epsilon_1(\alpha-2)(\mu-\lambda)^\alpha}{\Gamma(\alpha+1)} > 1.$$

Choose $R_1 = R(\epsilon_1) > 0$. Put $\Omega_2 = \{x \in X : \|x\|_* < R_1\}$. Then

$$\begin{aligned}
\|F_x\| &= \sup_{t\in[0,1]}\left|F_x(t)\right| \\
&\ge \left|F_x(\mu)\right| \\
&\ge \int_\lambda^\mu G(t,s)f_2\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\right) ds \\
&\ge \int_\lambda^\mu \frac{(\mu-s)^{\alpha-1}(\alpha-2)}{\Gamma(\alpha)}\big(q(s)-\epsilon_1\big)\max\left\{\|x\|,\left\|x'\right\|,\dots,\left\|\phi(x(s)\right\|\right\} ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \|x\|_* \int_\lambda^\mu \frac{(\mu-s)^{\alpha-1}(\alpha-2)}{\Gamma(\alpha)}\bigl(q(s)-\epsilon_1\bigr)\,ds \\
&= \|x\|_* \biggl[\int_\lambda^\mu \frac{(\mu-s)^{\alpha-1}(\alpha-2)}{\Gamma(\alpha)}q(s)\,ds - \epsilon_1 \int_\lambda^\mu \frac{(\mu-s)^{\alpha-1}(\alpha-2)}{\Gamma(\alpha)}\,ds\biggr] \\
&= \|x\|_* \biggl[\frac{\alpha-2}{\Gamma(\alpha)}\int_\lambda^\mu (\mu-s)^{\alpha-1}q(s)\,ds - \frac{\epsilon_1(\alpha-2)(\mu-\lambda)^\alpha}{\Gamma(\alpha+1)}\biggr] > \|x\|_*
\end{aligned}
$$

for all $x \in P \cap \partial\Omega_2$. Hence, $\|F_x\|_* \geq \|x\|_*$ on $P \cap \partial\Omega_2$. Now by using Lemma 2.4, $F : X \to X$ has a fixed point on $P \cap (\Omega_2 \backslash \Omega_1)$ which is a solution for the problem (1). □

Example 3.1 Define the map d on $[0.1, 0.9]$ by $d(t) = \frac{1}{c(t)}$ whenever $t \in [0.1, 0.9] \cap \mathbb{Q}$ where $c(t) = 0$ on $[0.1, 0.9] \cap \mathbb{Q}$ and $d(t) = 10$ whenever $t \in [0.1, 0.9] \cap \mathbb{Q}^c$. Now, consider the point-wise defined fractional integro-differential equation $D^{\frac{7}{2}}x(t) + f(t, x(t), x'(t), D^{\frac{1}{2}}x(t), \int_0^t x(s)\,ds, D^{\frac{1}{3}}x(t)) = 0$, where

$$
f(t,x_1,x_2,x_3,x_4,x_5) = \begin{cases} t\sum_{i=1}^5 x_i & 0 \leq t < 0.1, \\ d(t)H(x_1,x_2,x_3,x_4,x_5) & 0.1 \leq t \leq 0.9, \\ (1-t)\sum_{i=1}^5 x_i & 0.9 < t \leq 1, \end{cases}
$$

and $H(x_1,x_2,x_3,x_4,x_5) = \sum_{i=1}^5 \frac{\|x_i\|^2}{1+\|x_i\|}$. Put $f_1(t,x_1,x_2,x_3,x_4,x_5) = t\sum_{i=1}^5 x_i$,

$$
f_2(t,x_1,x_2,x_3,x_4,x_5) = d(t)H(x_1,x_2,x_3,x_4,x_5),
$$

and $f_3(t,x_1,x_2,x_3,x_4,x_5) = (1-t)\sum_{i=1}^5 x_i$. Note that

$$
f_1(t,0,0,0,0,0) = f_3(t,0,0,0,0,0) = 0,
$$

$$
f_1(t,x_1,x_2,x_3,x_4,x_5) - f_1(t,y_1,y_2,y_3,y_4,y_5) \leq t\sum_{i=1}^5 \|x_i - y_i\| \leq 0.1\sum_{i=1}^5 \|x_i - y_i\|,
$$

$$
\begin{aligned}
&\bigl|f_2(t,x_1,x_2,x_3,x_4,x_5) - f_2(t,y_1,y_2,y_3,y_4,y_5)\bigr| \\
&\quad = d(t)\sum_{i=1}^5 \biggl|\frac{\|x_i\|^2}{1+\|x_i\|} - \frac{\|y_i\|^2}{1+\|y_i\|}\biggr| \\
&\quad = d(t)\sum_{i=1}^5 \biggl|\frac{\|x_i\|^2 - \|x_i\|^2\|y_i\| - \|y_i\|^2 - \|y_i\|^2\|x_i\|}{(1+\|x_i\|)(1+\|y_i\|)}\biggr| \\
&\quad = d(t)\sum_{i=1}^5 \biggl|\frac{\|x_i\|^2 - \|y_i\|^2 + \|x_i\|(\|x_i\|\|y_i\| - \|y_i\|^2)}{(1+\|x_i\|)(1+\|y_i\|)}\biggr| \\
&\quad = d(t)\sum_{i=1}^5 \biggl|\frac{(\|x_i\| - \|y_i\|)(\|x_i\| + \|y_i\|) + \|x_i\|(\|x_i\| - \|y_i\|)\|y_i\|}{(1+\|x_i\|)(1+\|y_i\|)}\biggr| \\
&\quad = d(t)\sum_{i=1}^5 \biggl|\frac{(\|x_i\| - \|y_i\|)(\|x_i\| + \|y_i\| + \|x_i\|\|y_i\|)}{\|x_i\| + \|y_i\| + \|x_i\|\|y_i\| + 1}\biggr| \\
&\quad \leq d(t)\sum_{i=1}^5 \biggl|\frac{(\|x_i\| - \|y_i\|)(\|x_i\| + \|y_i\| + \|x_i\|\|y_i\|)}{\|x_i\| + \|y_i\| + \|x_i\|\|y_i\|}\biggr|
\end{aligned}
$$

$$= d(t)\sum_{i=1}^{5}\bigl|\|x_i\| - \|y_i\|\bigr| \le d(t)\sum_{i=1}^{5}\|x_i - y_i\|$$

$$:= d(t)\sum_{i=1}^{5}\Lambda_i(x_1 - y_1,\dots,x_5 - y_5)$$

and $f_3(t,x_1,x_2,x_3,x_4,x_5) - f_3(t,y_1,y_2,y_3,y_4,y_5) \le 0.1\sum_{i=1}^{5}\|x_i - y_i\|$, where $\Lambda_i(x_1,\dots,x_5) = \|x_i\|$ for $i = 1,\dots,5$. Note that

$$L = \left(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + m_0 l_4 + \theta_0 l_5 + \theta_1 l_5\right)$$
$$= (0.1 + 0.1 + \frac{0.1}{\Gamma(2-\frac{1}{2})} + 0.1 + \frac{0.1}{\Gamma(2-\frac{1}{3})} < 0.4,$$
$$L' = \left(l_1' + l_2' + \frac{l_3'}{\Gamma(2-\beta)} + m_0 l_4' + \theta_0 l_5' + \theta_1 l_5'\right)$$
$$= (0.1 + 0.1 + \frac{0.1}{\Gamma(2-\frac{1}{2})} + 0.1 + \frac{0.1}{\Gamma(2-\frac{1}{3})} < 0.4,$$

and $\lim_{z\to 0^+}\frac{\Lambda_i(z,z,z,z,z)}{z} = 1 := q_i$ for $i = 1,\dots,5$. Then we have

$$\frac{L(1-(1-\lambda)^{\alpha-1})}{\Gamma(\alpha)} + \frac{L'}{\Gamma(\alpha)}(1-\mu)^{\alpha-1} < \frac{0.4(1-(1-0.1)^{\frac{5}{2}})}{\Gamma(\frac{7}{2})} + \frac{0.4(1-0.9)^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} < 1$$

and for almost all $t \in [0,1]$

$$q(t) := \lim_{\max\|x_i\|\to\infty}\frac{f_2(t,x_1x_2,\dots,x_5)}{\max\|x_i\|} = d(t)\lim_{\max\|x_i\|\to\infty}\frac{\sum_{i=1}^{5}\frac{\|x_i\|^2}{1+\|x_i\|}}{\max\|x_i\|}$$
$$\ge d(t)\lim_{\|x_r\|\to\infty}\frac{\|x_i\|^2}{\|x_r\|(1+\|x_r\|)} = d(t)\lim_{\|x_r\|\to\infty}\frac{\|x_r\|}{1+\|x_r\|} = d(t),$$

where $\|x_r\| = \max_{1\le i\le 5}\|x_i\|$. Thus, we obtain

$$\frac{\alpha-2}{\Gamma(\alpha)}\int_{\lambda}^{\mu}(\mu-s)^{\alpha-2}q(s)\,ds \ge \frac{\frac{3}{2}}{\Gamma(\frac{7}{2})}\int_{0.1}^{0.9}10(0.9-s)^{\frac{3}{2}}\,ds > 1.$$

Now, by using Theorem 3.2, the problem has a solution.

Theorem 3.3 *Let* $[\lambda,\mu,f = (f_1,f_2,,f_3)]$ *with* $f_1(s,0,0,0,0,0) = f_3(t,0,0,0,0,0) = 0$ *for all* $s \in [0,\lambda]$ *and* $t \in [\mu,1]$. *Assume that there exist nonnegative functions* $a \in L^1[0,\lambda]$, $c \in L^1[\mu,1]$ *and* $b_1,\dots,b_5 : [\lambda,\mu] \to \mathbb{R}$ *with* $\hat{b}_i := (1-t)^{\alpha-2}b_i(t) \in L^1[\lambda,\mu]$ $(i = 1,\dots,5)$ *such that* $|f_1(t,x_1,\dots,x_5) - f_1(t,y_1,\dots,y_5)| \le a(t)\sum_{i=1}^{5}\|x_i - y_i\|$,

$$\bigl|f_2(t,x_1,\dots,x_5) - f_2(t,y_1,\dots,y_5)\bigr| \le \sum_{i=1}^{5}b_i(t)\|x_i - y_i\|,$$

and $|f_3(t,x_1,\dots,x_5) - f_3(t,y_1,\dots,y_5)| \le c(t)\sum_{i=1}^{5}\|x_i - y_i\|$ *for all* $x_1,\dots,x_5,y_1,\dots,y_5 \in X$ *and almost all* $t \in [0,1]$. *Suppose that there exist a natural number* n_0 *and nonnegative functions* $\phi_1,\dots,\phi_{n_0}$ *with* $\hat{\phi}_i := (1-t)^{\alpha-2}\phi_i(t) \in L^1[\lambda,\mu]$ *and nonnegative and nondecreasing* *with respect to all components maps* $\Lambda_1, \ldots, \Lambda_{n_0} : X^5 \to [0, \infty)$ *with* $\lim_{z\to 0^+} \frac{\Lambda_i(z,z,z,z,z)}{z} = 0$ *such that* $|f_2(t, x_1, \ldots, x_5)| \leq \sum_{i=1}^{n_0} \phi_i \Lambda_i(x_1, \ldots, x_5)$ *for all* $(x_1, \ldots, x_5) \in X$ *and almost all* $t \in [\lambda, \mu]$. *If* $(2 + \frac{1}{\Gamma(2-\beta)} + m_0 + \theta_0 + \theta_1)(\|a\|_{[0,\lambda]} + \sum_{i=1}^{5} \|\hat{b}_i\| + (1-\mu)^{\alpha-2}\|c\|_{[1,\mu]}) < \Gamma(\alpha - 1)$, *then the problem* (1) *has a solution.*

Proof First we show that F is a continuous map on X. Let $x_1, x_2 \in X$ and $t \in [0, 1]$. Then

$$\begin{aligned}
\bigl|F_{x_1}(t) - F_{x_2}(t)\bigr| \leq{}& \int_0^\lambda G(t,s)\biggl|f_1\biggl(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\,d\xi, \phi\bigl(x_1(s)\bigr)\biggr) \\
&- f_1\biggl(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\,d\xi, \phi\bigl(x_2(s)\bigr)\biggr)\biggr|\,ds \\
&+ \int_\lambda^\mu G(t,s)\biggl|f_2\biggl(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\,d\xi, \phi\bigl(x_1(s)\bigr)\biggr)\,ds \\
&- f_2\biggl(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\,d\xi, \phi\bigl(x_2(s)\bigr)\biggr)\biggr|\,ds \\
&+ \int_\mu^1 G(t,s)\biggl|f_3\biggl(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\,d\xi, \phi\bigl(x_1(s)\bigr)\biggr)\,ds \\
&- f_3\biggl(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\,d\xi, \phi\bigl(x_2(s)\bigr)\biggr)\biggr|\,ds \\
\leq{}& \int_0^\lambda \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} a(s)\biggl(\bigl|x_1(s) - x_2(s)\bigr| + \bigl|x_1'(s) - x_2'(s)\bigr| + \bigl|D^\beta(x_1 - x_2)(s)\bigr| \\
&+ \int_0^s \bigl|x_1(\xi) - x_2(\xi)\bigr|\,d\xi + \bigl|\phi\bigl(x_1(s) - x_2(s)\bigr)\bigr|\biggr)\,ds \\
&+ \int_\lambda^\mu \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\biggl(b_1(s)\bigl|x_1(s) - x_2(s)\bigr| + b_2(s)\bigl|x_1'(s) - x_2'(s)\bigr| \\
&+ b_3(s)\bigl|D^\beta(x_1 - x_2)(s)\bigr| \\
&+ b_4(s)\int_0^s \bigl|x_1(\xi) - x_2(\xi)\bigr|\,d\xi + b_5(s)\bigl|\phi\bigl(x_1(s) - x_2(s)\bigr)\bigr|\biggr)\,ds \\
&+ \int_\mu^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} c(s)\biggl(\bigl|x_1(s) - x_2(s)\bigr| \\
&+ \bigl|x_1'(s) - x_2'(s)\bigr| + \bigl|D^\beta(x_1 - x_2)(s)\bigr| \\
&+ \int_0^s \bigl|x_1(\xi) - x_2(\xi)\bigr|\,d\xi + \bigl|\phi\bigl(x_1(s) - x_2(s)\bigr)\bigr|\biggr)\,ds \\
\leq{}& \frac{t}{\Gamma(\alpha-1)}\int_0^\lambda (1-s)^{\alpha-2} a(s)\biggl(\|x_1 - x_2\| + \bigl\|x_1' - x_2'\bigr\| + \frac{\|x_1' - x_2'\|}{\Gamma(2-\beta)} \\
&+ m_0\|x_1 - x_2\| + \theta_0\|x_1 - x_2\| + \theta_1\bigl\|x_1' - x_2'\bigr\|\biggr)\,ds \\
&+ \frac{t}{\Gamma(\alpha-1)}\int_\lambda^\mu (1-s)^{\alpha-2}\biggl(b_1(s)\|x_1 - x_2\| + b_2(s)\bigl\|x_1' - x_2'\bigr\| \\
&+ b_3(s)\frac{\|x_1' - x_2'\|}{\Gamma(2-\beta)} + b_4(s)m_0\|x_1 - x_2\| \\
&+ b_5(s)\bigl(\theta_0\|x_1 - x_2\| + \theta_1\bigl\|x_1' - x_2'\bigr\|\bigr)\biggr)\,ds
\end{aligned}$$

$$+\frac{t}{\Gamma(\alpha-1)}\int_{\mu}^{1}(1-s)^{\alpha-2}c(s)\bigg(\|x_1-x_2\|+\|x_1'-x_2'\|+\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)}$$

$$+m_0\|x_1-x_2\|+\theta_0\|x_1-x_2\|+\theta_1\|x_1'-x_2'\|\bigg)\,ds$$

$$\leq\frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)}\|x_1-x_2\|_*\int_0^{\lambda}(1-s)^{\alpha-2}a(s)\,ds$$

$$+\frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)}\|x_1-x_2\|_*\sum_{i=1}^{5}\int_{\lambda}^{\mu}(1-s)^{\alpha-2}b_i(s)\,ds$$

$$+\frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)}\|x_1-x_2\|_*\int_{\mu}^{1}(1-s)^{\alpha-2}c(s)\,ds$$

$$\leq\frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)}\|x_1-x_2\|_*\bigg[\int_0^{\lambda}a(s)\,ds$$

$$+\sum_{i=1}^{5}\int_{\lambda}^{\mu}(1-s)^{\alpha-2}b_i(s)\,ds+\int_{\mu}^{1}c(s)\,ds\bigg]$$

and so

$$\|F_{x_1}-F_{x_2}\|\leq\frac{(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)}\bigg[\|a\|_{[0,\lambda]}$$

$$+\sum_{i=1}^{5}\|\hat{b}_i\|_{[\lambda,\mu]}+\|c\|_{[\mu,1]}\bigg]\|x_1-x_2\|_*.$$

By using similar calculations, we get

$$\big|F_{x_1}'(t)-F_{x_2}'(t)\big|$$

$$\leq\int_0^{\lambda}\frac{\partial G}{\partial t}(t,s)\bigg|f_1\bigg(s,x_1(s),x_1'(s),D^{\beta}x_1(s),\int_0^s h(\xi)x_1(\xi)\,d\xi,\phi\big(x_1(s)\big)\bigg)$$

$$-f_1\bigg(s,x_2(s),x_2'(s),D^{\beta}x_2(s),\int_0^s h(\xi)x_2(\xi)\,d\xi,\phi\big(x_2(s)\big)\bigg)\bigg|\,ds$$

$$+\int_{\lambda}^{\mu}\frac{\partial G}{\partial t}(t,s)\bigg|f_2\bigg(s,x_1(s),x_1'(s),D^{\beta}x_1(s),\int_0^s h(\xi)x_1(\xi)\,d\xi,\phi\big(x_1(s)\big)\bigg)\,ds$$

$$-f_2\bigg(s,x_2(s),x_2'(s),D^{\beta}x_2(s),\int_0^s h(\xi)x_2(\xi)\,d\xi,\phi\big(x_2(s)\big)\bigg)\bigg|\,ds$$

$$+\int_{\mu}^{1}\frac{\partial G}{\partial t}(t,s)\bigg|f_3\bigg(s,x_1(s),x_1'(s),D^{\beta}x_1(s),\int_0^s h(\xi)x_1(\xi)\,d\xi,\phi\big(x_1(s)\big)\bigg)\,ds$$

$$-f_3\bigg(s,x_2(s),x_2'(s),D^{\beta}x_2(s),\int_0^s h(\xi)x_2(\xi)\,d\xi,\phi\big(x_2(s)\big)\bigg)\bigg|\,ds$$

$$\leq\int_0^{\lambda}\frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}a(s)\bigg(\big|x_1(s)-x_2(s)\big|+\big|x_1'(s)-x_2'(s)\big|+\big|D^{\beta}(x_1-x_2)(s)\big|$$

$$+\int_0^s\big|x_1(\xi)-x_2(\xi)\big|\,d\xi+\big|\phi\big(x_1(s)-x_2(s)\big)\big|\bigg)\,ds$$

$$
\begin{aligned}
&+ \int_\lambda^\mu \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \Big(b_1(s)\big|x_1(s)-x_2(s)\big| + b_2(s)\big|x_1'(s)-x_2'(s)\big| \\
&+ b_3(s)\big|D^\beta(x_1-x_2)(s)\big| \\
&+ b_4(s)\int_0^s \big|x_1(\xi)-x_2(\xi)\big|\,d\xi + b_5(s)\big|\phi\big(x_1(s)-x_2(s)\big)\big| \Big)\,ds \\
&+ \int_\mu^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} c(s) \Big(\big|x_1(s)-x_2(s)\big| + \big|x_1'(s)-x_2'(s)\big| + \big|D^\beta(x_1-x_2)(s)\big| \\
&+ \int_0^s \big|x_1(\xi)-x_2(\xi)\big|\,d\xi + \big|\phi\big(x_1(s)-x_2(s)\big)\big| \Big)\,ds \\
\le{}& \frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)} \|x_1-x_2\|_* \int_0^\lambda (1-s)^{\alpha-2} a(s)\,ds \\
&+ \frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)} \|x_1-x_2\|_* \sum_{i=1}^5 \int_\lambda^\mu (1-s)^{\alpha-2} b_i(s)\,ds \\
&+ \frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)} \|x_1-x_2\|_* \int_\mu^1 (1-s)^{\alpha-2} c(s)\,ds \\
\le{}& \frac{t(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)} \|x_1-x_2\|_* \Bigg[\int_0^\lambda a(s)\,ds \\
&+ \sum_{i=1}^5 \int_\lambda^\mu (1-s)^{\alpha-2} b_i(s)\,ds + \int_\mu^1 c(s)\,ds \Bigg]
\end{aligned}
$$

and so

$$
\begin{aligned}
\big\|F_{x_1}' - F_{x_2}'\big\| \le{}& \frac{(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)} \\
&\times \Bigg[\|a\|_{[0,\lambda]} + \sum_{i=1}^5 \|\hat{b}_i\|_{[\lambda,\mu]} + \|c\|_{[\mu,1]} \Bigg] \|x_1-x_2\|_*.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|F_{x_1} - F_{x_2}\|_* \le{}& \frac{(2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1)}{\Gamma(\alpha-1)} \\
&\times \Bigg[\|a\|_{[0,\lambda]} + \sum_{i=1}^5 \|\hat{b}_i\|_{[\lambda,\mu]} + \|c\|_{[\mu,1]} \Bigg] \|x_1-x_2\|_*
\end{aligned}
$$

and so $F_{x_1} \to F_{x_2}$ in X as $x_2 \to x_1$. Thus, F is continuous on X. We have $\lim_{z\to 0^+} \frac{\Lambda_i(z,z,z,z,z)}{z} = 0$, $\lim_{z\to 0^+} \frac{\Lambda_i(lz,lz,lz,lz,lz)}{z} = 0$, where $l = \max\{1, \frac{1}{\Gamma(2-\beta)}, m_0, \theta_0+\theta_1\}$. Let $\epsilon > 0$ be given. Choose $\delta_i := \delta_i(\epsilon) > 0$ such that $0 < z \le \delta_i$ implies that $\lim_{z\to 0^+} \frac{\Lambda_i(lz,lz,lz,lz,lz)}{z} < \epsilon$ for $1 \le i \le n_0$. Hence, $\Lambda_i(lz,lz,lz,lz,lz) < \epsilon z$ for $0 < z \le \delta_i$ and so $\Lambda_i(lz,lz,lz,lz,lz) < \epsilon z$ for all $1 \le i \le n_0$ and $z \in (0.\delta]$, where $\delta := \delta(\epsilon) = \min_{1\le i\le n_0}\{\delta_i\}$. Since

$$
\left(2 + \frac{1}{\Gamma(2-\beta)} + m_0 + \theta_0 + \theta_1\right)\big(\|a\|_{[0,\lambda]} + (1-\mu)^{\alpha-2}\|c\|_{[1,\mu]}\big) < \Gamma(\alpha-1),
$$

there exists $\epsilon_0 > 0$ such that

$$\left(2 + \frac{1}{\Gamma(2-\beta)} + m_0 + \theta_0 + \theta_1\right)\left(\|a\|_{[0,\lambda]} + (1-\mu)^{\alpha-2}\|c\|_{[1,\mu]}\right) + \epsilon_0 \sum_{i=1}^{n_0} \|\hat{\phi}_i\|_{[\lambda,\mu]} < \Gamma(\alpha-1).$$

Let $r = \delta(\epsilon_0)$. Then $\Lambda_i(lz, lz, lz, lz, lz) < \epsilon_0 z$ for all $1 \le i \le n_0$ and for $z \in (0, r]$. Put $C = \{x \in X : \|x\|_* < r\}$. Define the map $\alpha : X^2 \to [0, \infty)$ by $\alpha(x, y) = 1$ whenever $x, y \in C$ and $\alpha(x, y) = 0$ otherwise. We show that F is α-admissible. Let $x, y \in X$ be such that $\alpha(x, y) \ge 1$. Then $x, y \in C$, $\|x\|_* < r$ and $\|y\|_* < r$. Let $t \in [0, 1]$. Then we have

$$\begin{aligned}
\left|F_x(t)\right| \le{}& \int_0^\lambda G(t,s)\left|f_1\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\bigl(x(s)\bigr)\right)\right| ds \\
&+ \int_\lambda^\mu G(t,s)\left|f_2\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\bigl(x(s)\bigr)\right)\right| ds \\
&+ \int_\mu^1 G(t,s)\left|f_3\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\bigl(x(s)\bigr)\right)\right| ds \\
\le{}& \frac{t}{\Gamma(\alpha-1)} \int_0^\lambda (1-s)^{\alpha-2}\left|f_1\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\bigl(x(s)\bigr)\right)\right. \\
&\left. - f_1(s, 0, 0, 0, 0, 0)\right| ds + \frac{t}{\Gamma(\alpha-1)} \int_0^\lambda (1-s)^{\alpha-2}\left|f_1(s, 0, 0, 0, 0, 0)\right| ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\lambda^\mu (1-s)^{\alpha-2} \sum_{i=1}^{n_0} \phi_i(s)\Lambda_i\left(x(s), x'(s), D^\beta x(s),\right. \\
&\left.\int_0^s h(\xi)x(\xi)\,d\xi, \phi\bigl(x(s)\bigr)\right) ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\mu^1 (1-s)^{\alpha-2}\left|f_3\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\bigl(x(s)\bigr)\right)\right. \\
&\left. - f_3(s, 0, 0, 0, 0, 0)\right| ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_0^\lambda (1-s)^{\alpha-2}\left|f_3(s, 0, 0, 0, 0, 0)\right| ds \\
\le{}& \frac{t}{\Gamma(\alpha-1)} \int_0^\lambda (1-s)^{\alpha-2} a(s)\left(\|x\| + \left\|x'\right\| + \frac{\|x'\|}{\Gamma(2-\beta)}\right. \\
&\left. + m_0\|x\| + \theta_0\|x\| + \theta_1\|x\|\right) ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\lambda^\mu (1-s)^{\alpha-2} \sum_{i=1}^{n_0} \phi_i(s)\Lambda_i\left(\|x\| + \left\|x'\right\| + \frac{\|x'\|}{\Gamma(2-\beta)}\right. \\
&\left. + m_0\|x\| + \theta_0\|x\| + \theta_1\|x\|\right) ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\mu^1 (1-s)^{\alpha-2} c(s)\left(\|x\| + \left\|x'\right\| + \frac{\|x'\|}{\Gamma(2-\beta)}\right. \\
&\left. + m_0\|x\| + \theta_0\|x\| + \theta_1\|x\|\right) ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{t}{\Gamma(\alpha-1)}\Bigg(\bigg[2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1\bigg]\|x\|_*\int_0^\lambda \sup(1-s)^{\alpha-2}a(s)\,ds \\
&\quad+\sum_{i=1}^{n_0}\int_\lambda^\mu (1-s)^{\alpha-2}\phi_i(s)\Lambda_i\big(l\|x\|_*,l\|x\|_*,l\|x\|_*,l\|x\|_*,l\|x\|_*\big)\,ds \\
&\quad+\bigg[2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1\bigg]\|x\|_*\int_\mu^1 \sup(1-s)^{\alpha-2}c(s)\,ds\Bigg) \\
&\le \frac{1}{\Gamma(\alpha-1)}\Bigg(\bigg[2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1\bigg]\big(\|a\|_{[0,\lambda]}+(1-\mu)^{\alpha-2}\|c\|_{[\mu,1]}\big)r \\
&\quad+\sum_{i=1}^{n_0}\Lambda_i(lr,lr,lr,lr,lr)\int_\lambda^\mu \hat{\phi}_i(s)\,ds\Bigg) \\
&= \frac{1}{\Gamma(\alpha-1)}\Bigg(\bigg[2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1\bigg]\big(\|a\|_{[0,\lambda]}+(1-\mu)^{\alpha-2}\|c\|_{[\mu,1]}\big)r \\
&\quad+\sum_{i=1}^{n_0}\|\hat{\phi}_i\|\Lambda_i(lr,lr,lr,lr,lr)\Bigg) \\
&\le \frac{1}{\Gamma(\alpha-1)}\Bigg(\bigg[2+\frac{1}{\Gamma(2-\beta)}+m_0+\theta_0+\theta_1\bigg]\big(\|a\|_{[0,\lambda]}+(1-\mu)^{\alpha-2}\|c\|_{[\mu,1]}\big) \\
&\quad+\epsilon_0\sum_{i=1}^{n_0}\|\hat{\phi}_i\|\Bigg)r \\
&< \frac{1}{\Gamma(\alpha-1)}\Gamma(\alpha-1)r=r,
\end{aligned}
$$

and so $\|F_x\| < r$. Similarly one can prove that $\|F_x'\| < r$ and so $\|F_x\|_* = \max\{\|F_x\|, \|F_x'\|\} < r$. Hence, $F_x \in C$ and by same reason $F_y \in C$. This implies that $\alpha(F_x, F_y) \ge 1$ and so F is α-admissible. Also, $\alpha(x_0, F_{x_0}) \ge 1$ for all $x_0 \in C$ (note that C is nonempty). Let $x, y \in X$ and $t \in [0,1]$. Then we have

$$
\begin{aligned}
\big|F_x(t)-F_y(t)\big| &\le \int_0^\lambda G(t,s)\bigg|f_1\bigg(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\bigg) \\
&\quad-f_1\bigg(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)\,d\xi,\phi\big(y(s)\big)\bigg)\bigg|\,ds \\
&\quad+\int_\lambda^\mu G(t,s)\bigg|f_2\bigg(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\bigg)\,ds \\
&\quad-f_2\bigg(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)\,d\xi,\phi\big(y(s)\big)\bigg)\bigg|\,ds \\
&\quad+\int_\mu^1 G(t,s)\bigg|f_3\bigg(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi,\phi\big(x(s)\big)\bigg)\,ds \\
&\quad-f_3(s,y)),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)\,d\xi,\phi\big(y(s)\big))\bigg|\,ds \\
&\le \frac{t}{\Gamma(\alpha-1)}\int_0^\lambda (1-s)^{\alpha-2}a(s)\Big(\|x-y\|+\big\|x'-y'\big\|+\big\|D^\beta(x-y)\big\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^s \big|h(\xi)\big| \|x-y\|\,d\xi + \phi\big(\|x-y\|\big)\bigg)\,ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\lambda^\mu (1-s)^{\alpha-2} \bigg(b_1(s)\|x-y\| + b_2(s)\big\|x'-y'\big\| \\
&+ b_3(s)\big(D^\beta \|x-y\|\big) + b_4(s) \int_0^s \big|h(\xi)\big| \|x-y\|\,d\xi + b_5(s)\phi\big(\|x-y\|\big)\bigg)\,ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\mu^1 (1-s)^{\alpha-2} c(s) \bigg(\|x-y\| + \big\|x'-y'\big\| + \big\|D^\beta(x-y)\big\| \\
&+ \int_0^s \big|h(\xi)\big| \|x-y\|\,d\xi + \phi\big(\|x-y\|\big)\bigg)\,ds \\
\le{}& \frac{t}{\Gamma(\alpha-1)} \int_0^\lambda (1-s)^{\alpha-2} a(s) \bigg(\|x-y\| + \big\|x'-y'\big\| + \frac{\|x'-y'\|}{\Gamma(2-\beta)} \\
&+ m_0\|x-y\| + \theta_0\|x-y\| + \theta_1\big\|x'-y'\big\|\bigg)\,ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\lambda^\mu (1-s)^{\alpha-2} \bigg(b_1(s)\|x-y\| + b_2(s)\big\|x'-y'\big\| \\
&+ b_3(s)\frac{\|x'-y'\|}{\Gamma(2-\beta)} + b_4(s)m_0\|x-y\| \\
&+ b_5(s)\big(\theta_0\|x-y\| + \theta_1\big\|x'-y'\big\|\big)\bigg)\,ds \\
&+ \frac{t}{\Gamma(\alpha-1)} \int_\mu^1 (1-s)^{\alpha-2} c(s) \bigg(\|x-y\| + \big\|x'-y'\big\| + \frac{\|x'-y'\|}{\Gamma(2-\beta)} \\
&+ m_0\|x-y\| + \theta_0\|x-y\| + \theta_1\big\|x'-y'\big\|\bigg)\,ds \\
\le{}& \frac{t(2 + \frac{1}{\Gamma(2-\beta)} + m_0 + \theta_0 + \theta_1)}{\Gamma(\alpha-1)} \|x-y\|_* \bigg[\int_0^\lambda \sup(1-s)^{\alpha-2} a(s)\,ds \\
&+ \sum_{i=1}^5 \int_\lambda^\mu (1-s)^{\alpha-2} b_i(s)\,ds + \int_\mu^1 \sup(1-s)^{\alpha-2} c(s)\,ds \bigg] \\
\le{}& \frac{1}{\Gamma(\alpha-1)} \bigg(2 + \frac{1}{\Gamma(2-\beta)} + m_0 + \theta_0 + \theta_1 \bigg) \\
&\times \bigg(\|a\|_{[0,\lambda]} + \sum_{i=1}^5 \|\hat{b}_i\| + (1-\mu)^{\alpha-2}\|c\|_{[1,\mu]} \bigg) \|x-y\|_* \\
:={}& \psi\big(\|x-y\|_*\big).
\end{aligned}
$$

Similarly, one can show that $\|F'_x - F'_y\| \le \psi(\|x-y\|_*)$ and so $\alpha(x,y)\|F_x - F_y\|_* \le \psi(d(x,y))$ for all $x, y \in X$. We have

$$
\frac{1}{\Gamma(\alpha-1)} \bigg(2 + \frac{1}{\Gamma(2-\beta)} + m_0 + \theta_0 + \theta_1 \bigg) \bigg(\|a\|_{[0,\lambda]} + \sum_{i=1}^5 \|\hat{b}_i\| + (1-\mu)^{\alpha-2}\|c\|_{[1,\mu]} \bigg) < 1,
$$

$\psi \in \Psi$. By using Lemma 2.2, F has a fixed point which is a solution for the problem (1).

□

Example 3.2 Consider the problem $D^{\frac{9}{2}}x(t) + f(t, x(t), x'(t), D^{\frac{1}{2}}x(t), \int_0^t x(\xi)\,d\xi, I^{\frac{1}{3}}x(t)) = 0$, where

$$f(t,x_1,\ldots,x_5) = \begin{cases} f_1(t,x_1,\ldots,x_5) := \sin t(\sum_{i=1}^5 \|x_i\|) & t \in [0,0.2), \\ f_2(t,x_1,\ldots,x_5) := \frac{0.2}{p(t)} \sum_{i=1}^5 \frac{\|x_i\|^2}{1+\|x_i\|} & t \in [0.2,0.7], \\ f_3(t,x_1,\ldots,x_5) := t(\sum_{i=1}^5 \|x_i\|) & t \in [0.7,1], \end{cases}$$

and $p(t) = 0$ whenever $t \in [0.2, 0.07] \cap \mathcal{Q}$ and $p(t) = \sqrt{t}$ whenever $t \in [0.2, 0.07] \cap \mathcal{Q}^c$. Put $a(t) = \sin t$, $b_1(t) = \cdots = b_5(t) = \frac{1}{p(t)}$ and $c(t) = t$ for all t. Note that

$$\bigl|f_1(t,x_1,\ldots,x_5) - f_1(t,y_1,\ldots,y_5)\bigr| = \sin t \left|\sum_{i=1}^5 \|x_i\| - \|y_i\|\right| \le \sin t \sum_{i=1}^5 \|x_i - y_i\|,$$

$$\begin{aligned} &\bigl|f_2(t,x_1,\ldots,x_5) - f_2(t,y_1,\ldots,y_5)\bigr| \\ &\quad= \frac{0.2}{p(t)} \left|\sum_{i=1}^5 \frac{\|x_i\|^2}{1+\|x_i\|} - \frac{\|y_i\|^2}{1+\|y_i\|}\right| \\ &\quad= \frac{0.2}{p(t)} \left|\sum_{i=1}^5 \frac{\|x_i\|^2 + \|x_i\|^2\|y_i\| - \|x_i\|\|y_i\|^2 - \|y_i\|^2}{(1+\|x_i\|)(1+\|y_i\|)}\right| \\ &\quad= \frac{0.2}{p(t)} \left|\sum_{i=1}^5 \frac{(\|x_i\|+\|y_i\|)(\|x_i\|-\|y_i\|) + \|x_i\|(\|x_i\|-\|y_i\|)\|y_i\|}{(1+\|x_i\|)(1+\|y_i\|)}\right| \\ &\quad= \frac{0.2}{p(t)} \left|\sum_{i=1}^5 \frac{(\|x_i\|-\|y_i\|)(\|x_i\|+\|y_i\|+\|x_i\|\|y_i\|)}{(1+\|x_i\|+\|y_i\|+\|x_i\|\|y_i\|}\right| \\ &\quad\le \frac{0.2}{p(t)} \left|\sum_{i=1}^5 \|x_i\| - \|y_i\|\right| \le \frac{0.2}{p(t)} \sum_{i=1}^5 \|x_i - y_i\|. \end{aligned}$$

Define $\Lambda_i(x_1,\ldots,x_5) = \frac{\|x_i\|^2}{1+\|x_i\|}$ for $i = 1,\ldots,5$. Then $\lim_{z\to 0^+} \frac{\Lambda_i(z,z,z,z,z)}{z} = 0$ for all i. Put $b_i(t) = \phi_i(t) = \frac{0.2}{p(t)}$ for all i, $n_0 = 5$ and $\beta = \frac{1}{2}$. Since $|\int_0^t x(\xi)\,d\xi| \le t\|x\| \le \|x\|$, put $m_0 = 1$. Since $|I^{\frac{1}{3}}x(t)| = |\frac{1}{\Gamma(\frac{1}{3})}\int_0^t (t-s)^{\frac{1}{3}-1}x(s)\,ds| \le \frac{1}{\Gamma(\frac{1}{3})}\int_0^t |(t-s)^{\frac{1}{3}-1}x(s)|\,ds \le \frac{\|x\|}{\Gamma(\frac{1}{3})}\int_0^t \frac{ds}{(t-s)^{\frac{2}{3}}} \le \frac{\|x\|}{\Gamma(\frac{1}{2})}$, we put $\theta_0 = \frac{1}{\Gamma(\frac{1}{3})}$ and $\theta_1 = 0$. Note that $\|a\|_{[0,\lambda]} = \int_0^{0.2} \sin t\,dt \le 0.02$, $\|\hat{b}_i\|_{[\lambda,\mu]} = \int_{0.2}^{0.7} \frac{0.2}{\sqrt{t}}\,dt \le 0.08$, $\|c\|_{[\mu,1]} = \int_{0.7}^1 t\,dt = 0.045$ and

$$\begin{aligned} &\left(2 + \frac{1}{\Gamma(2-\beta)} + m_0 + \theta_0 + \theta_0 + \theta_1\right)\left(\|a\|_{[0,\lambda]} + \sum_{i=1}^5 \|\hat{b}_i\| + (1-\mu)^{\alpha-2}\|c\|_{[\mu,1]}\right) \\ &\quad\le \left(2 + \frac{1}{\Gamma(\frac{3}{2})} + 1 + \frac{1}{\Gamma(\frac{1}{2})}\right)\left(0.02 + \sum_{i=1}^5 0.8 + (1-0.7)^{\frac{7}{2}} 0.045\right) \\ &\quad\le \left(3 + \frac{2}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}}\right)(0.421) < \Gamma\left(\frac{7}{2}\right) = \Gamma(\alpha - 1). \end{aligned}$$

Now by using Theorem 3.3, the problem has a solution.

4 Conclusions

Most natural phenomena include crisis and it is important we could model this type phenomena. Researchers are going to use fractional integro-differential equations for modeling of crisis phenomena. In this work, we investigate the existence of solutions for a three steps crisis integro-differential equation by considering this assumption that the second step is a point-wise defined singular fractional differential equation, while the first and third parts have natural treatments.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by the third author. All authors read and approved the final manuscript.

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References

1. Berezowski, M.: Crisis phenomenon in a chemical reactor with recycle. Chem. Eng. Sci. **101**, 451–453 (2013)
2. Cheraghlou, A.M.: The aftermath of financial crises: a look on human and social wellbeing. World Dev. **87**, 88–106 (2016)
3. Ivanov, I., Kabaivanov, S., Bogdanova, B.: Stock market recovery from the 2008 financial crisis: the differences across Europe. Res. Int. Bus. Finance **37**, 360–374 (2016)
4. Naseradinmousavi, P., Nataraj, C.: Transient chaos and crisis phenomena in butterfly valves driven by solenoid actuators. Commun. Nonlinear Sci. Numer. Simul. **17**, 4336–4345 (2012)
5. Novelli, E.M., Gladwin, M.T.: Crises in sickle cell disease. Chest **149**, 1082–1093 (2016)
6. Surtaev, A., Pavlenko, A.: Observation of boiling heat transfer and crisis phenomena in falling water film at transient heating. Int. J. Heat Mass Transf. **74**, 342–352 (2014)
7. Surtaev, A.S., Pavlenko, A.N., Kuznetsov, D.V., Kalita, V.I., Komlev, D.I., Ivannikov, A.Y., Radyuk, A.A.: Heat transfer and crisis phenomena at pool boiling of liquid nitrogen on the surfaces with capillary-porous coatings. Int. J. Heat Mass Transf. **108**, 146–155 (2017)
8. Zhao, L., Li, W., Cai, X.: Structure and dynamics of stock market in times of crisis. Phys. Lett. A **380**, 654–666 (2016)
9. Alfaro, M., Coville, J.: Propagation phenomena in monostable integro-differential equations: acceleration or not? J. Differ. Equ. **263**(9), 5727–5758 (2017)
10. Calleja, R.C., Humphries, A.R., Krauskopf, B.: Resonance phenomena in a scalar delay differential equation with two state-dependent delays. SIAM J. Appl. Dyn. Syst. **16**(3), 1474–1513 (2017)
11. Chian, A.C.L., Rempel, E.L., Macau, E.E., Rosa, R.R., Christiansen, F.: High-dimensional interior crisis in the Kuramoto–Sivashinsky equation. Phys. Rev. E **65**(3), 035203 (2002)
12. Franaszek, M., Nabaglo, A.: General case of crisis-induced intermittency in the Duffing equation. Phys. Lett. A **178**(1–2), 85–91 (1993)
13. Gsponer, A., Hurni, J.P.: Lanczos's equation as a way out of the spin 3/2 crisis? Higher spins, QCD and beyond. Hadron. J. **26**(3–4), 327–350 (2003)
14. Almeida, R., Bastos, B.R.O., Monteiro, M.T.T.: Modeling some real phenomena by fractional differential equations. Math. Methods Appl. Sci. **39**(16), 4846–4855 (2016)
15. Agarwal, R.P., O'Regan, D., Stanek, S.: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57–68 (2010)
16. Agarwal, R.P., O'Regan, D., Stanek, S.: Positive solutions for mixed problems of singular fractional differential equations. Math. Nachr. **285**(1), 27–41 (2012)
17. Bai, Z., Qui, T.: Existence of positive solution for singular fractional differential equation. Appl. Math. Comput. **215**, 2761–2767 (2009)
18. Rezapour, S., Shabibi, M.: A singular fractional fractional differential equation with Riemann–Liouville integral boundary condition. J. Adv. Math. Stud. **8**(1), 80–88 (2015)
19. Shabibi, M., Postolache, M., Rezapour, S., Vaezpour, S.M.: Investigation of a multi-singular pointwise defined fractional integro-differential equation. J. Math. Anal. **7**(5), 61–77 (2016)
20. Stanek, S.: The existence of positive solutions of singular fractional boundary value problems. Comput. Math. Appl. **62**, 1379–1388 (2011)
21. Tatar, N.: An impulsive nonlinear singular version of the Gronwall–Bihari inequality. J. Inequal. Appl. **2006**, Article ID 84561 (2006)

22. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
23. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal. **75**, 2154–2165 (2012)
24. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integral and Derivative: Theory and Applications. Gordon & Breach, New York (1993)
25. Krasnoselskii, M.A.: Positive Solutions of Operator Equations. Noordhoff, Groningen (1964)