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Positive solutions for a class of nonlinear Hadamard fractional differential equations with a parameter

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Abstract

In this paper, we investigate a class of boundary value problem of nonlinear Hadamard fractional differential equations with a parameter. By means of the properties of the Green function and Guo–Krasnosel'skii fixed-point theorem on cones, the existence and nonexistence of positive solutions are obtained. Finally, some examples are presented to show the effectiveness of our main results.

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1 Introduction

Fractional differential equations have given rise to abroad attention of many researchers by the intensive development of the theory of fractional calculus itself. On the other hand, fractional differential equations can better describe many phenomena than ordinary differential equations in many diverse and widespread fields of science and engineering. For the development of fractional calculus and applications, we refer the reader to [1–10] and the references therein. For example, by employing the Avery–Henderson fixed-point theorem, Li [11] obtained the existence of positive solutions as considered for a fractional differential equation with p -Laplacian operator. In [12], existence and uniqueness results for a new class of boundary value problems of sequential fractional differential equations with nonlocal non-separated boundary conditions involving lower-order fractional derivatives were given by some standard fixed-point theorems. The existence and multiplicity of solutions or positive solutions for nonlinear boundary value problems involving fractional differential equations with kinds of boundary value conditions were studied by some well-known fixed-point theorems, the lower and upper solutions method and the monotone iterative technique; see [13, 14] and the references therein. For example, the authors of [15] investigated the solutions of fractional integrodifferential equations with boundary value conditions, respectively. In [16], the existence and multiplicity of positive solutions were obtained for nonlinear Caputo fractional differential equations with integral boundary conditions. Henderson and Luca investigated the positive solutions of nonlinear boundary value problems for systems of fractional differential equations in the book [17]. In [18], by applying the fixed-point theorem due to Leggett–Williams, the authors considered the

existence of positive solutions for a system of fractional multi-point boundary value problem with p -Laplacian operator.

In the past ten years, most of the work on the topic is based on Riemann–Liouville- and Caputo-type fractional differential equations. Recently, more and more scholars paid attention to the boundary value problems of nonlinear Hadamard fractional differential equations [19–24]. By applying some standard fixed-point theorems, Ahmad and Ntouyas [25, 26] studied the existence and uniqueness of solutions for Hadamard-type fractional differential equations for boundary value problems and systems with integral boundary conditions, respectively. Based on standard fixed-point theorems for multivalued maps, Ahmad *et al.* [27] investigated the existence of solutions for fractional boundary value problems involving Hadamard-type fractional differential inclusions and integral boundary conditions. Aljoudi *et al.* [28] studied a nonlocal boundary value problem of Hadamard-type coupled sequential fractional differential equations supplemented with coupled strip conditions. By discussing a continuity, integrable estimation, and the asymptotic property on Mittag–Leffler functions, Li and Wang [29] investigated the existence of solutions and finite-time stability for a class of nonlinear Hadamard fractional differential equations with constant coefficient. In [30, 31], the existence of positive solutions for nonlinear Hadamard fractional differential equations with four-point coupled and coupled integral boundary conditions were given by the Guo–Krasnosel’skii fixed-point theorems, respectively.

From the above review of the literature concerning fractional differential equations, most of the authors investigated only the existence of solutions or positive solutions for Hadamard fractional differential equations without considering the influence of a parameter. In this paper, we will study the influence of parameter intervals for Hadamard fractional differential equation boundary value problems. Motivated by the work mentioned above, we consider the following nonlinear Hadamard fractional differential equation with a parameter:

$$(D^\alpha x)(t) + \lambda a(t)f(x(t)) = 0, \quad x(1) = (\delta x)(1) = (\delta x)(e) = 0, \alpha \in (2, 3], t \in [1, e], \tag{1}$$

where λ is a positive parameter, D^α is the left-sided Hadamard fractional derivative order α , $(\delta x)(t) = t dx(t)/dt$, $a : (1, e) \rightarrow [0, \infty)$ and $f : [0, \infty) \rightarrow [0, \infty)$ are two continuous functions. The main aim of this paper is to investigate the above Hadamard fractional differential equation boundary value problem (1). With the help of the properties of the Green function and the Guo–Krasnosel’skii fixed-point theorem on cones, we establish the existence and nonexistence of positive solutions. At the end, we give some examples to illustrate the feasibility of our proposed theoretical results.

2 Preliminaries

For convenience of the reader, we present some necessary definitions and lemmas from Hadamard fractional calculus theory in this section.

Definition 2.1 ([32]) The left-sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ of the function $y(t)$ are defined by

$$(I^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s) \frac{ds}{s} \quad (1 \leq t \leq e),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([32]) The left-sided Hadamard fractional derivatives of order $\alpha \in [n - 1, n)$, $n \in \mathbb{Z}^+$, of the function $y(t)$ are defined by

$$(D^\alpha y)(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\ln \frac{t}{s} \right)^{n - \alpha - 1} y(s) \frac{ds}{s} \quad (1 \leq t \leq e),$$

where $\Gamma(\cdot)$ is the Gamma function.

Now we will give the Green function of linear equation and some properties of the Green function.

Lemma 2.1 Let $\alpha \in (0, 1]$ be fixed and $y \in C[1, e]$, then the linear boundary value problem

$$(D^\alpha x)(t) + y(t) = 0, \quad x(1) = (\delta x)(1) = (\delta x)(e) = 0, \alpha \in (2, 3], t \in [1, e], \tag{2}$$

has a unique solution which is given by the following integral representation of the solution:

$$x(t) = \int_1^e G(t, s)y(s) \frac{ds}{s}, \quad t \in [1, e], \tag{3}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1 - \ln s)^{\alpha - 2} (\ln t)^{\alpha - 1}, & 1 \leq t \leq s \leq e, \\ (1 - \ln s)^{\alpha - 2} (\ln t)^{\alpha - 1} - (\ln(t/s))^{\alpha - 1}, & 1 \leq s \leq t \leq e. \end{cases} \tag{4}$$

Proof As argued in [32], the solution of Hadamard differential equation (2) can be written the following equivalent integral equations:

$$x(t) = c_1 (\ln t)^{\alpha - 1} + c_2 (\ln t)^{\alpha - 2} + c_3 (\ln t)^{\alpha - 3} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha - 1} y(s) \frac{ds}{s}, \tag{5}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. From the boundary condition $x(1) = 0$, we have $c_3 = 0$. Furthermore, from (5), we can get

$$(\delta x)(t) = c_1 (\alpha - 1) (\ln t)^{\alpha - 2} + c_2 (\alpha - 2) (\ln t)^{\alpha - 3} - \frac{\alpha - 1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha - 2} y(s) \frac{ds}{s}. \tag{6}$$

From the boundary condition $(\delta x)(0) = (\delta x)(1) = 0$ and (6), we obtain $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \ln s)^{\alpha - 2} y(s) \frac{ds}{s}. \tag{7}$$

Substituting (7) and $c_2 = c_3 = 0$ into (5), we can observe

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^e (1 - \ln s)^{\alpha - 2} (\ln t)^{\alpha - 1} y(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha - 1} y(s) \frac{ds}{s} \\ &= \int_1^e G(t, s)y(s) \frac{ds}{s}, \end{aligned}$$

which implies (3). The proof is completed. □

Lemma 2.2 *Let $G(t, s)$ be defined as in (4) and $g(t) = (\ln t)^{\alpha-1}$. Then the following inequalities hold:*

$$G(t, s) \geq 0, \quad g(t)G(e, s) \leq G(t, s) \leq G(e, s), \quad \forall 1 \leq s, t \leq e.$$

Proof For $1 \leq t \leq s \leq e$, It is easy to see that $G(t, s) \geq 0$ and $g(t)G(e, s) = G(t, s) \leq G(e, s)$.

For $1 \leq s \leq t \leq e$, we have

$$\begin{aligned} \Gamma(\alpha)G(t, s) &= (1 - \ln s)^{\alpha-2}(\ln t)^{\alpha-1} - (\ln(t/s))^{\alpha-1} \\ &= (1 - \ln s)^{\alpha-2}(\ln t)^{\alpha-1} - (1 - \ln s/\ln t)^{\alpha-1}(\ln t)^{\alpha-1} \\ &\geq (1 - \ln s)^{\alpha-2}(\ln t)^{\alpha-1} - (1 - \ln s)^{\alpha-1}(\ln t)^{\alpha-1} = 0, \end{aligned}$$

which implies $G(t, s) \geq 0$. Let $h(t) = (1 - \ln s)^{\alpha-2}(\ln t)^{\alpha-1} - (\ln(t/s))^{\alpha-1}$ for $1 \leq t \leq e$. Then

$$\begin{aligned} dh(t)/dt &= (\alpha - 1)[(1 - \ln s)^{\alpha-2}(\ln t)^{\alpha-2} - (\ln(t/s))^{\alpha-2}] \\ &\geq (\alpha - 1)(\ln t)^{\alpha-2}[(1 - \ln s)^{\alpha-2} - (1 - \ln s)^{\alpha-2}] = 0, \end{aligned}$$

which implies that $h(t)$ is the monotone nondecreasing function, i.e. $G(t, s) \leq G(e, s)$. On the other hand,

$$\begin{aligned} \frac{G(t, s)}{G(e, s)} &= \frac{(1 - \ln s)^{\alpha-2}(\ln t)^{\alpha-1} - (\ln(t/s))^{\alpha-1}}{(1 - \ln s)^{\alpha-2} - (1 - \ln s)^{\alpha-1}} \\ &\geq \frac{(1 - \ln s)^{\alpha-2}(\ln t)^{\alpha-1} - (\ln t - \ln t \ln s)^{\alpha-1}}{(1 - \ln s)^{\alpha-2} - (1 - \ln s)^{\alpha-1}} = (\ln t)^{\alpha-1} = g(t), \end{aligned}$$

which implies $g(t)G(e, s) \leq G(t, s)$. The proof of Lemma 2.2 is completed. □

Our main results are based on the following Guo–Krasnosel’skii fixed-point theorem on cones.

Lemma 2.3 ([33, 34]) *Let \mathcal{X} be a Banach space, and let $\mathcal{P} \subset \mathcal{X}$ be a cone in \mathcal{X} . Assume that Ω_1, Ω_2 are open subsets of \mathcal{X} with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : \mathcal{P} \rightarrow \mathcal{P}$ be a completely continuous operator such that, either*

$$(B_1) \quad \|Sw\| \leq \|w\|, \quad w \in \mathcal{P} \cap \partial\Omega_1, \text{ and } \|Sw\| \geq \|w\|, \quad w \in \mathcal{P} \cap \partial\Omega_2,$$

or

$$(B_2) \quad \|Sw\| \geq \|w\|, \quad w \in \mathcal{P} \cap \partial\Omega_1, \text{ and } \|Sw\| \leq \|w\|, \quad w \in \mathcal{P} \cap \partial\Omega_2.$$

Then S has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $\mathcal{E} = C[1, e]$ be the Banach space endowed with the norm $\|x\| = \sup_{t \in [1, e]} |x(t)|$. Define the cone $\mathcal{P} \subset \mathcal{E}$ by

$$\mathcal{P} = \{x \in \mathcal{E} : x(t) \geq g(t)\|x\|, t \in [1, e]\}, \quad g(t) = (\ln t)^{\alpha-1}.$$

Suppose that x is a solution of boundary value problem (1). Then from Lemma 2.1, we obtain

$$x(t) = \lambda \int_1^e G(t, s)a(s)f(x(s)) \frac{ds}{s}, \quad t \in [1, e].$$

We define an operator $S_\lambda : \mathcal{P} \rightarrow \mathcal{E}$ as follows:

$$(S_\lambda x)(t) = \lambda \int_1^e G(t,s)a(s)f(x(s)) \frac{ds}{s}, \quad t \in [1, e].$$

By Lemma 2.2, we have

$$\begin{aligned} \|S_\lambda x\| &\leq \lambda \int_1^e G(e,s)a(s)f(x(s)) \frac{ds}{s}, \\ (S_\lambda x)(t) &\geq \lambda \int_1^e g(t)G(e,s)a(s)f(x(s)) \frac{ds}{s} \geq g(t)\|S_\lambda x\|. \end{aligned}$$

Thus, $S_\lambda(\mathcal{P}) \subset \mathcal{P}$. Then we have the following lemma.

Lemma 2.4 $S_\lambda : \mathcal{P} \rightarrow \mathcal{P}$ is completely continuous.

Proof The operator $S_\lambda : \mathcal{P} \rightarrow \mathcal{P}$ is continuous in view of continuity of $G(t,s)$, $a(t)$ and $f(x(t))$. By means of the Arzela–Ascoli theorem, $S_\lambda : \mathcal{P} \rightarrow \mathcal{P}$ is completely continuous. □

3 Main results

In this section, we establish some sufficient conditions for the existence and nonexistence of positive solutions for boundary value problem (1).

For convenience, we denote

$$\begin{aligned} F_0 &= \limsup_{x \rightarrow 0^+} \frac{f(x)}{x}, & F_\infty &= \limsup_{x \rightarrow +\infty} \frac{f(x)}{x}, \\ f_0 &= \liminf_{x \rightarrow 0^+} \frac{f(x)}{x}, & f_\infty &= \liminf_{x \rightarrow +\infty} \frac{f(x)}{x}, \\ C_1 &= \int_1^e G(e,s)a(s) \frac{ds}{s}, & C_2 &= \int_1^e g(s)G(e,s)a(s) \frac{ds}{s}. \end{aligned}$$

Theorem 3.1 *If there exists $l \in (1, e)$ such that $g(l)f_\infty C_2 > F_0 C_1$ holds, for each*

$$\lambda \in ((g(l)f_\infty C_2)^{-1}, (F_0 C_1)^{-1}), \tag{8}$$

then boundary value problem (1) has at least one positive solution. Here we impose $(g(l)f_\infty C_2)^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0 C_1)^{-1} = +\infty$ if $F_0 = 0$.

Proof Let λ satisfy (8) and $\varepsilon > 0$ be such that

$$(g(l)(f_\infty - \varepsilon)C_2)^{-1} \leq \lambda \leq ((F_0 + \varepsilon)C_1)^{-1}. \tag{9}$$

By the definition of F_0 , we see that there exists $r_1 > 0$ such that

$$f(x) \leq (F_0 + \varepsilon)x, \quad \text{for } 0 < x \leq r_1. \tag{10}$$

So if $x \in \mathcal{P}$ with $\|x\| = r_1$, then, by (9) and (10), we have

$$\begin{aligned} \|S_\lambda x\| &\leq \lambda \int_1^e G(e,s)a(s)f(x(s)) \frac{ds}{s} \\ &\leq \lambda \int_1^e G(e,s)a(s)(F_0 + \varepsilon)r_1 \frac{ds}{s} = \lambda(F_0 + \varepsilon)r_1 C_1 \leq r_1 = \|x\|. \end{aligned}$$

Hence, if we choose $\Omega_1 = \{x \in \mathcal{E} : \|x\| < r_1\}$, then we get

$$\|S_\lambda x\| \leq \|x\|, \quad \text{for } x \in \mathcal{P} \cap \partial\Omega_1. \tag{11}$$

Let $r_3 > 0$ be such that

$$f(x) \geq (f_\infty - \varepsilon)x, \quad \text{for } x \geq r_3. \tag{12}$$

If $x \in \mathcal{P}$ with $\|x\| = r_2 = \max\{2r_1, r_3\}$, then from (9) and (12), we obtain

$$\begin{aligned} \|S_\lambda x\| &\geq (S_\lambda x)(l) = \lambda \int_1^e G(e,s)a(s)f(x(s)) \frac{ds}{s} \geq \lambda \int_1^e g(l)G(e,s)a(s)f(x(s)) \frac{ds}{s} \\ &\geq \lambda \int_1^e g(l)G(e,s)a(s)(f_\infty - \varepsilon)x(s) \frac{ds}{s} \geq \lambda \int_1^e g(l)g(s)G(e,s)a(s)(f_\infty - \varepsilon)\|x\| \frac{ds}{s} \\ &= \lambda g(l)C_2(f_\infty - \varepsilon)\|x\| \geq \|x\|. \end{aligned}$$

Thus, if we set $\Omega_2 = \{x \in \mathcal{E} : \|x\| < r_2\}$, then we get

$$\|S_\lambda x\| \geq \|x\|, \quad \text{for } x \in \mathcal{P} \cap \partial\Omega_2. \tag{13}$$

Now, from (11), (13), and Lemma 2.3, we guarantee that S_λ has a fixed point $x \in \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$ with $r_1 \leq \|x\| \leq r_2$, and clearly x is a positive solution of boundary value problem (1). This completes the proof of Theorem 3.1. □

Theorem 3.2 *If there exists $l \in (1, e)$ such that $g(l)f_\infty C_2 > F_0 C_1$ holds, for each*

$$\lambda \in ((g(l)f_0 C_2)^{-1}, (F_\infty C_1)^{-1}), \tag{14}$$

then boundary value problem (1) has at least one positive solution. Here we impose $(g(l)f_0 C_2)^{-1} = 0$ if $f_0 = +\infty$ and $(F_\infty C_1)^{-1} = +\infty$ if $F_\infty = 0$.

Proof Let λ satisfy (14) and let $\varepsilon > 0$ be such that

$$(g(l)(f_0 - \varepsilon)C_2)^{-1} \leq \lambda \leq ((F_\infty + \varepsilon)C_1)^{-1}. \tag{15}$$

From the definition of f_0 , we see that there exists $r_1 > 0$ such that

$$f(x) \geq (f_0 - \varepsilon)x, \quad \text{for } 0 < x \leq r_1.$$

Further, if $x \in \mathcal{P}$ with $\|x\| = r_1$, then similar to the second part of Theorem 3.1, we can obtain $\|S_\lambda x\| \geq \|x\|$. Thus, if we choose $\Omega_1 = \{x \in \mathcal{E} : \|x\| < r_1\}$, then

$$\|S_\lambda x\| \geq \|x\|, \quad \text{for } x \in P \cap \partial\Omega_1. \tag{16}$$

Next, we may choose $R_1 > 0$ such that

$$f(x) \leq (F_\infty + \varepsilon)x, \quad \text{for } x \geq R_1. \tag{17}$$

We consider the following two cases.

Case 1. Suppose f is bounded. Then there exists some $M > 0$, such that

$$f(x) \leq M, \quad \text{for } x \in (0, +\infty).$$

We define $r_3 = \max\{2r_1, \lambda MC_1\}$, and $x \in \mathcal{P}$ with $\|x\| = r_3$, then

$$\|S_\lambda x\| \leq \lambda \int_1^e G(e, s)a(s)f(x(s)) \frac{ds}{s} \leq \lambda M \int_1^e G(e, s)a(s) \frac{ds}{s} = \lambda MC_1 \leq r_3 = \|x\|.$$

Hence,

$$\|S_\lambda x\| \leq \|x\|, \quad \text{for } x \in \mathcal{P}_{r_3} = \{x \in \mathcal{P} : \|x\| \leq r_3\}.$$

Case 2. Suppose f is unbounded. Then there exists some $r_4 > \max\{2r_1, R_1\}$ such that

$$f(x) \leq f(r_4), \quad \text{for } 0 < x \leq r_4.$$

Let $x \in \mathcal{P}$ with $\|x\| = r_4$. Then, by (15) and (17), we have

$$\begin{aligned} \|S_\lambda x\| &\leq \lambda \int_1^e G(e, s)a(s)f(x(s)) \frac{ds}{s} \leq \lambda \int_1^e G(e, s)a(s)(F_\infty + \varepsilon)\|x\| \frac{ds}{s} \\ &= \lambda C_1(F_\infty + \varepsilon)\|x\| \leq \|x\|. \end{aligned}$$

Thus,

$$\|S_\lambda x\| \leq \|x\|, \quad \text{for } x \in \mathcal{P}_{r_4} = \{x \in \mathcal{P} : \|x\| \leq r_4\}.$$

In both Cases 1 and 2, if we set $\Omega_2 = \{x \in \mathcal{P} : \|x\| < r_2 = \max\{r_3, r_4\}\}$, then

$$\|S_\lambda x\| \leq \|x\|, \quad \text{for } x \in \mathcal{P} \cap \partial\Omega_2. \tag{18}$$

It follows from (16), (18) and Lemma 2.3 that S_λ has a fixed point $x \in \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$ with $r_1 \leq \|x\| \leq r_2$. It is clear that x is a positive solution of boundary value problem (1). The proof is complete. \square

Theorem 3.3 *If there exist $l \in (1, e)$ and $r_2 > r_1 > 0$ such that $g(l) > r_1/r_2$ and f satisfy*

$$\min_{g(l)r_1 \leq x \leq r_1} f(x) \geq \frac{r_1}{\lambda g(l)C_1}, \quad \max_{0 \leq x \leq r_2} f(x) \leq \frac{r_2}{\lambda C_1}.$$

Then boundary value problem (1) has a positive solution u with $r_1 \leq \|u\| \leq r_2$.

Proof Choose $\Omega_1 = \{x \in \mathcal{E} : \|x\| < r_1\}$; then, for $x \in \mathcal{P} \cap \partial\Omega_1$, we have

$$\begin{aligned} \|S_\lambda x\| &\geq (S_\lambda x)(l) = \lambda \int_1^e G(e,s)a(s)f(x(s)) \frac{ds}{s} \geq \lambda \int_1^e g(l)G(e,s)a(s)f(x(s)) \frac{ds}{s} \\ &\geq \lambda g(l)C_1 \frac{r_1}{\lambda g(l)C_1} = r_1 = \|x\|. \end{aligned}$$

On the other hand, choose $\Omega_2 = \{x \in \mathcal{E} : \|x\| < r_2\}$; then for $x \in \mathcal{P} \cap \partial\Omega_2$, we have

$$\begin{aligned} \|S_\lambda x\| &\leq \lambda \int_1^e G(e,s)a(s)f(x(s)) \frac{ds}{s} \leq \lambda \int_1^e G(e,s)a(s) \max_{0 \leq x \leq r_2} f(x(s)) \frac{ds}{s} \\ &\leq \lambda C_1 \frac{r_2}{\lambda C_1} = r_2 = \|x\|. \end{aligned}$$

From Lemma 2.3, boundary value problem (1) has a positive solution x with $r_1 \leq \|x\| \leq r_2$. The proof of Theorem 3.3 is completed. \square

For the remainder of this section, we will need the following assumption.

Assumption 1 $(\min_{x \in [g(l)r,r]} f(x))/r > 0$, where $l \in (1, e)$.

For convenience, we denote

$$\lambda_1 = \sup_{r>0} \frac{r}{C_1 \max_{x \in [0,r]} f(x)}, \quad \lambda_2 = \inf_{r>0} \frac{r}{C_1 \min_{x \in [g(l)r,r]} f(x)}. \tag{19}$$

In view of the continuity of $f(x)$ and Assumption 1, we have $0 < \lambda_1 \leq +\infty$ and $0 \leq \lambda_2 < +\infty$.

Theorem 3.4 *Suppose Assumption 1 holds. If $f_0 = f_\infty = +\infty$, then boundary value problem (1) has at least two positive solutions for each $\lambda \in (0, \lambda_1)$.*

Proof Define $\varphi(r) = r/C_1 \max_{x \in [0,r]} f(x)$. By the continuity of $f(x)$ and $f_0 = f_\infty = +\infty$, we see that $\varphi(r) : (0, +\infty) \rightarrow (0, +\infty)$ is continuous and $\lim_{r \rightarrow 0} \varphi(r) = \lim_{r \rightarrow +\infty} \varphi(r) = 0$. From (19), there exists $r_0 \in (0, +\infty)$ such that $\varphi(r_0) = \sup_{r>0} \varphi(r) = \lambda_1$. then, for $\lambda \in (0, \lambda_1)$, there exist constants c_1, c_2 ($0 < c_1 < r_0 < c_2 < +\infty$) with $\varphi(c_1) = \varphi(c_2) = \lambda$. Thus, we have

$$f(x) \leq \frac{c_1}{\lambda C_1}, \quad \text{for } x \in [0, c_1], \tag{20}$$

and

$$f(x) \leq \frac{c_2}{\lambda C_1}, \quad \text{for } x \in [0, c_2]. \tag{21}$$

On the other hand, applying the conditions $f_0 = f_\infty = +\infty$, there exist constants d_1, d_2 ($0 < d_1 < c_1 < r_0 < c_2 < d_2 < +\infty$) with $f(x)/x \geq 1/(g^2(l)\lambda C_1)$, for $x \in (0, d_1) \cup (g(l)d_2, +\infty)$. Then we get

$$\min_{g(l)d_1 \leq x \leq d_1} f(x) \leq \frac{d_1}{\lambda g(l)C_1} \tag{22}$$

and

$$\min_{g(l)d_2 \leq x \leq d_2} f(x) \leq \frac{d_2}{\lambda g(l)C_1}. \tag{23}$$

By (20) and (22), (21) and (23), combining with Theorem 3.3 and Lemma 2.3, we can complete the proof. □

Corollary 3.1 *Suppose Assumption 1 holds. If $f_0 = +\infty$ or $f_\infty = +\infty$, then boundary value problem (1) has at least one positive solution for each $\lambda \in (0, \lambda_1)$.*

Theorem 3.5 *Suppose Assumption 1 holds. If $f_0 = f_\infty = 0$, then boundary value problem (1) has at least two positive solutions for each $\lambda \in (\lambda_2, +\infty)$.*

Proof Define $\psi(r) = r/(C_1 \min_{x \in [g(l)r, r]} f(x))$. By the continuity of $f(x)$ and $f_0 = f_\infty = 0$, we can easily see that $\psi(r) : (0, +\infty) \rightarrow (0, +\infty)$ is continuous and $\lim_{r \rightarrow 0} \psi(r) = \lim_{r \rightarrow +\infty} \psi(r) = +\infty$. From (19), there exists $r_0 \in (0, +\infty)$ such that $\psi(r_0) = \sup_{r>0} \psi(r) = \lambda_2$, then, for $\lambda \in (0, \lambda_1)$, there exist constants d_1, d_2 ($0 < d_1 < r_0 < d_2 < +\infty$) with $\psi(d_1) = \psi(d_2) = \lambda$. Therefore,

$$f(x) \leq \frac{d_1}{\lambda g(l)C_1}, \quad \text{for } x \in [g(l)d_1, d_1], \tag{24}$$

and

$$f(x) \leq \frac{d_2}{\lambda g(l)C_1}, \quad \text{for } x \in [g(l)d_2, d_2]. \tag{25}$$

On the other hand, applying the conditions $f_0 = 0$, there exist constants c_1 ($0 < c_1 < d_1$) with $f(x)/x \leq 1/(\lambda C_1)$, for $x \in (0, c_1)$. Then

$$\max_{0 \leq x \leq c_1} f(x) \leq \frac{c_1}{\lambda C_1}. \tag{26}$$

In view of $f_\infty = 0$, there exists a constant $c_2 \in (d_2, +\infty)$ such that $f(x)/x \leq 1/(\lambda C_1)$, for $x \in (c_2, +\infty)$. Let $M = \max_{0 \leq x \leq c_2} f(x)$, $c_2 \geq \lambda C_1 M$. It is easily seen that

$$\max_{0 \leq x \leq c_2} f(x) \leq \frac{c_2}{\lambda C_1}. \tag{27}$$

From (24) and (26), (25) and (27), combining with Theorem 3.3 and Lemma 2.3, we can complete the proof. □

Corollary 3.2 *Suppose Assumption 1 holds. If $f_0 = 0$ or $f_\infty = 0$, then boundary value problem (1) has at least one positive solution for each $\lambda \in (\lambda_2, +\infty)$.*

By the above theorems, we can obtain the following results.

Corollary 3.3 *Suppose Assumption 1 holds. If $f_0 = +\infty, f_\infty = d$ or $f_\infty = +\infty, f_0 = d$, then boundary value problem (1) has at least one positive solution for each $\lambda \in (0, (dC_1)^{-1})$.*

Corollary 3.4 *Suppose Assumption 1 holds. If $f_0 = 0, f_\infty = d$ or $f_\infty = 0, f_0 = d$, then boundary value problem (1) has at least one positive solution for each $\lambda \in ((g(l)dC_2)^{-1}, +\infty)$.*

Theorem 3.6 *Suppose Assumption 1 holds. If $F_0 < +\infty$ and $F_\infty < +\infty$, then there exists a $\lambda_0 > 0$ such that, for all $0 < \lambda < \lambda_0$, boundary value problem (1) has no positive solution.*

Proof Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist positive numbers m_1, m_2, r_1 and r_2 such that $r_1 < r_2$ and

$$f(x) \leq m_1x, \quad \text{for } x \in [0, r_1]; \quad f(x) \leq m_2x, \quad \text{for } x \in [r_2, +\infty).$$

Let $m = \max\{m_1, m_2, \max_{x \in [r_1, r_2]} \{f(x)/x\}\}$. Then we have

$$f(x) \leq mx, \quad \text{for } x \in [0, +\infty).$$

Assume $x(t)$ is a positive solution of boundary value problem (1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := (mC_1)^{-1}$. Since $S_\lambda x(t) = x(t)$ for $t \in [1, e]$,

$$\|x\| = \|S_\lambda x\| \leq \lambda \int_1^e G(e, s)a(s)f(x(s)) \frac{ds}{s} \leq m\lambda \|x\| \int_1^e G(e, s)a(s) \frac{ds}{s} < \|x\|,$$

which is a contradiction. Therefore, boundary value problem (1) has no positive solution. The proof is complete. □

Theorem 3.7 *Suppose Assumption 1 holds. If $f_0 > 0$ and $f_\infty > 0$, then there exists a $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, boundary value problem (1) has no positive solution.*

Proof Since $f_0 > 0$ and $f_\infty > 0$, there exist positive numbers n_1, n_2, r_1 and r_2 such that $r_1 < r_2$ and

$$f(u) \geq n_1x, \quad \text{for } x \in [0, r_1]; \quad f(x) \geq n_2x, \quad \text{for } x \in [r_2, +\infty).$$

Let $n = \min\{n_1, n_2, \min_{x \in [r_1, r_2]} \{f(x)/x\}\}$. Then we have

$$f(x) \geq nx, \quad \text{for } x \in [0, +\infty).$$

Assume $x(t)$ is a positive solution of boundary value problem (1). We will show that this leads to a contradiction for $\lambda > \lambda_0 := (g(l)nC_2)^{-1}$. Since $S_\lambda x(t) = x(t)$ for $t \in [1, e]$,

$$\|x\| = \|S_\lambda x\| \geq \lambda \int_1^e g(l)G(e, s)a(s)f(x(s)) \frac{ds}{s} > \|x\|,$$

which is a contradiction. Therefore, boundary value problem (1) has no positive solution. The proof of Theorem 3.7 is completed. □

4 Some examples

In this section, we will present some examples to illustrate the main results.

Example 4.1 Consider the Hadamard fractional boundary value problem

$$(D^{2.5}x)(t) + \lambda \ln tx^\theta(t) = 0, \quad x(0) = (\delta x)(0) = (\delta x)(1) = 0, \quad 1 < t < e, \theta > 1. \tag{28}$$

Since $\alpha = 2.5$ and $a(t) = \ln t$, we have

$$C_1 = \int_1^e G(e, s)a(s) \frac{ds}{s} \leq \frac{1}{\Gamma(2.5)} \int_1^e \ln s(1 - \ln s)^{0.5} \frac{ds}{s} \approx 0.2006,$$

$$C_2 = \int_1^e g(s)G(e, s)a(s) \frac{ds}{s} \geq \frac{1}{\Gamma(2.5)} \int_1^e (\ln s)^{2.5}((1 - \ln s)^{0.5} - (1 - \ln s)^{1.5}) \frac{ds}{s} \approx 0.1146.$$

Let $f(x) = x^\theta, \theta > 1$. Then from [35], we have $F_0 = 0$ and $f_\infty = +\infty$. Choose $l = e^{0.5}$. Then $g(e^{0.5}) = 0.5^{1.5} \approx 0.3536$. So $g(l)C_2f_\infty > F_0C_1$ holds. Thus, by Theorem 3.1, the boundary value problem (28) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 4.2 Consider the Hadamard fractional boundary value problem

$$(D^{2.5}x)(t) + \lambda \ln tx^\vartheta(t) = 0, \quad x(0) = (\delta x)(0) = (\delta x)(1) = 0, \quad 1 < t < e, 0 < \vartheta < 1. \tag{29}$$

Since $\alpha = 2.5$ and $a(t) = \ln t$, we have $C_1 \leq 0.2006$ and $C_2 \geq 0.1146$. Let $f(x) = x^\vartheta, 0 < \vartheta < 1$. Then from [35], we have $F_\infty = 0$ and $f_0 = +\infty$. Choose $l = e^{0.5}$. Then $g(e^{0.5}) = 0.5^{1.5} \approx 0.3536$. So $g(l)C_2f_0 > F_\infty C_1$ holds. Thus, by Theorem 3.2, the boundary value problem (29) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 4.3 Consider the Hadamard fractional boundary value problem

$$(D^{2.5}x)(t) + \lambda \ln t \frac{(200x^2(t) + x(t))(2 + \sin x(t))}{x(t) + 1} = 0,$$

$$x(0) = (\delta x)(0) = (\delta x)(1) = 0, \quad 1 < t < e. \tag{30}$$

Since $\alpha = 2.5$ and $a(t) = \ln t$, we have $C_1 \leq 0.2006$ and $C_2 \geq 0.1146$. Let $f(x) = (200x^2 + x)(2 + \sin x)/(x + 1)$. Then from [35], we have $F_0 = f_0 = 2, F_\infty = 600, f_\infty = 200$, and $2x < f(x) < 600x$.

- (i) Choose $l = e^{0.5}$. Then $g(e^{0.5}) = 0.5^{1.5} \approx 0.3536$. So $g(l)C_2f_\infty > F_0C_1$ holds. Thus, by Theorem 3.1, the boundary value problem (30) has a positive solution for each $\lambda \in (0.1234, 2.4925)$.
- (ii) By Theorem 3.6, the boundary value problem (30) has no positive solution for all $\lambda \in (0, 0.0083)$.
- (iii) By Theorem 3.7, the boundary value problem (30) has no positive solution for all $\lambda \in (12.3388, +\infty)$.

5 Conclusions

By means of the properties of the Green function and the Guo–Krasnosel’skii fixed-point theorem on cones, we have investigated the existence and nonexistence of positive solutions for a class of boundary value problems of nonlinear Hadamard fractional differential equations with a parameter. Three examples are given to show the effectiveness of the obtained results. Furthermore, by using similarly the method in this paper, we can also obtain

the existence and nonexistence of positive solutions for nonlinear Hadamard fractional boundary value problems as follows: $(D^\alpha x)(t) + \lambda a(t)f(x(t)) = 0$, $x(1) = x(e) = (\delta x)(1) = 0$, $\alpha \in (2, 3]$, $t \in [1, e]$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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