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Robust exponential stability results for uncertain infinite delay differential systems with random impulsive moments

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Abstract

In this paper, we establish some criteria on robust exponential stability by using the formula for the variation of parameters and estimating the Cauchy matrix. More importantly, the robust stability criteria do not require the stability of the corresponding continuous system, and so they can be more widely applied to stabilize the unstable continuous system with time delays and uncertainties by using random impulsive control. Further, we give some numerical examples to illustrate the theoretical results.

Keywords: random impulses; exponential stability; robustness; infinite delay differential systems

1 Introduction

Uncertainties happen frequently in various engineering, biological, economical, etc. systems. Delay differential systems usually encounter the uncertainties because of system parameters, modeling error or some factors. The uncertainties that affect a time delay system fall into two different categories. They can be classified into delay dependent and delay independent criteria. Since the delay dependent criteria make use of information on the length of delays, they are less conservative than the delay independent ones. It is well known that uncertainties often result in instability. Therefore, the robust stability is performed for generally bounded and uncertain domains. The robust stability has received considerable attention in recent years. Yang and Xu in [1] presented robust stability for an uncertain impulsive control system with time-varying delay. Li in [2] established robust exponential stability for impulsive systems with state-dependent delays, and several interesting results were established in [3–11].

Impulsive differential systems and impulsive control systems have attracted increasing interest in recent years. Such systems arise in many fields of science and engineering, see [2, 12–20]. When impulse time is random, the solutions of the differential system behave as a stochastic process. There are several research works in the literature on random impulsive differential systems. Wu et al. in [21] studied the existence and uniqueness of solutions to random impulsive differential systems. In [22] Anguraj and Vinodkumar proved the existence, uniqueness and stability results of random impulsive semilinear differential systems. Ravi Agarwal et al.[23] proved exponential stability for differential equations with random impulses at random times. For further study, refer to [21, 22, 24–30] and the references therein. So far there has been no paper reported dealing with uncertain random impulsive delay differential systems. Therefore, it is necessary to investigate the stability of uncertain random impulsive delay differential systems.

The paper is organized as follows. In Section 2, we recall briefly the notations, definitions, lemmas and preliminaries which are used throughout this paper. In Section 3, we prove robust exponential stability of uncertain random impulsive linear and nonlinear infinite delay differential systems by using the method of variation of parameters. Finally, in Section 4, we give some examples to illustrate our result.

2 Preliminaries

Let $\Re^n$ be the n-dimensional Euclidean space and Ω be a nonempty set. Assume that $\{\tau_k\}_{k=1}^{\infty}$ is a sequence of independent exponentially distributed random variables with parameter λ, and each random variable τ_k is defined from Ω to $D_k \overset{\text{def.}}{=} (0, d_k)$ for $k = 1, 2, \dots,$ where $0 < d_k < +\infty$. Let us denote by $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote by $\mathcal{F}_t$ the σ-algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. E stands for the mathematical expectation operator with respect to the given probability measure P.

For $x \in \Re^n$ and $A \in \Re^{n\times n}$, the norm is defined as follows:

$$\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2}, \qquad \|A\| = \sqrt{\lambda_{\max}\bigl(A^T A\bigr)}, \qquad \mu(A) = \frac{1}{2}\lambda_{\max}\bigl(A + A^T\bigr),$$

where $\lambda_{\max}(\cdot)$ is the largest eigenvalue of the matrix.

Consider the following nonlinear uncertain random impulsive control system with infinite delays:

$$\begin{aligned}
&\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)x^{\mu_1}\bigl(t - \tau(t)\bigr) \\
&\qquad + (C + \Delta C)\int_0^{\infty} h(\eta)x^{\mu_2}(t - \eta)\,d\eta, \quad t \neq \xi_k, t \geq t_0, \\
&x\bigl(\xi_k^+\bigr) = b_i(\tau_i)x\bigl(\xi_k^-\bigr), \quad k = 1, 2, \dots, \\
&x(s) = \varphi(s), \quad s \leq t_0,
\end{aligned} \tag{1}$$

where $\mu_1 \geq 1$, $\mu_2 \geq 1$. Let $\mathcal{PC}((-\infty, t_0], \Re^n) = \{\varphi : (-\infty, t_0] \to \Re^n, \varphi(t)$ is piecewise continuous$\}$, and for $\varphi \in \mathcal{PC}((-\infty, t_0], \Re^n)$, the norm is defined as $E\|\varphi\|^2 = \sup_{t\leq t_0} E\|\varphi(t)\|^2$; $A, B, C \in \Re^{n\times n}$ are matrices; $\tau(t)$ is the time-varying delay function with $0 \leq \tau(t) \leq \tau$, τ is a given positive constant; ΔA, ΔB, ΔC are the uncertain matrices, which vary within the range of $\|\Delta A\| \leq a$, $\|\Delta B\| \leq b$, $\|\Delta C\| \leq c$, where a, b, c are known nonnegative constants; $h(s) \in C(\Re^+, \Re)$ satisfies $\int_0^{+\infty} |h(s)|e^{\mu_2 \eta s}\,ds < \infty$, where $\eta > 0$ is a given constant; $b_k : D_k \to \Re^{n\times n}$ is a matrix-valued function for each $k = 1, 2, \dots$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \dots,$ here $t_0 \in \Re$ is an arbitrary real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \lim_{k\to\infty} \xi_k = \infty$; $x(\xi_k^-) = \lim_{t\uparrow\xi_k} x(t)$ according to their paths with the norm $E\|x\|^2 = \sup_{t_0\leq s\leq t} E\|x(s)\|^2$ for each t satisfying $t \geq t_0$.

When $\mu_1, \mu_2 = 1$, system (1) becomes the linear uncertain random impulsive control system with infinite delays of the form

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)x\big(t - \tau(t)\big) \\ &\quad + (C + \Delta C)\int_0^{+\infty} h(\eta)x(t-\eta)\,d\eta, \quad t \neq \xi_k, t \geq t_0, \\ x\big(\xi_k^+\big) &= b_i(\tau_i)x\big(\xi_k^-\big), \quad k = 1, 2, \ldots, \\ x(s) &= \varphi(s), \quad s \leq t_0. \end{aligned} \tag{2}$$

If $h(\eta) = 0$, then system (2) becomes the linear uncertain random impulsive control system with infinite delays.

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)x\big(t - \tau(t)\big), \quad t \neq \xi_k, t \geq t_0, \\ x\big(\xi_k^+\big) &= b_i(\tau_i)x\big(\xi_k^-\big), \quad k = 1, 2, \ldots, \\ x(s) &= \varphi(s), \quad s \leq t_0. \end{aligned} \tag{3}$$

In particular, $\Delta A, \Delta B = 0$ then system (3) becomes the random impulsive control system with infinite delays.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx\big(t - \tau(t)\big), \quad t \neq \xi_k, t \geq t_0, \\ x\big(\xi_k^+\big) &= b_i(\tau_i)x\big(\xi_k^-\big), \quad k = 1, 2, \ldots, \\ x(s) &= \varphi(s), \quad s \leq t_0. \end{aligned} \tag{4}$$

We always assume that the solution $x(t)$ of (1) is continuous on the right and limitable on the left. Now, we introduce the following lemma and hypotheses used in our discussion.

Lemma 1 ([23]) *The probability that there will be exactly k impulses until the time t, $t \geq t_0$, where impulse moments ξ_k, $k = 1, 2, \ldots$, follow exponential distribution with parameter λ, is given by the equality $P(I_{[\xi_k,\xi_{k+1})}(t)) = \frac{\lambda^k (t-t_0)^k}{k!} e^{-\lambda(t-t_0)}$, where the events $I_{[\xi_k,\xi_{k+1})}(t) = \{\omega \in \Omega : \xi_k(\omega) < t < \xi_{k+1}(\omega)\}$, $k = 1, 2, \ldots$.*

Remark 1 From [23], the expected value of solution $x(t)$ for the random impulsive differential equations is given as

$$E\big[\|x(t)\|\big] = \sum_{k=0}^{\infty} E\big[\|x(t)\| | I_{[\xi_k,\xi_{k+1})}(t)\big] P\big(I_{[\xi_k,\xi_{k+1})}(t)\big),$$

where the impulse moments ξ_k, $k = 1, 2, \ldots$, follow exponential distribution with parameter λ.

Definition 1 Assume $x(t) = x(t, t_0, \varphi)$ to be the solution of (1) through (t_0, φ). Then the zero solution of (1) is said to be globally exponentially mean square stable if, for any initial data $x_{t_0} = \varphi$, there exist two positive numbers $\gamma > 0$, $M \geq 1$ such that

$$E\|x(t)\|^2 \leq ME\|\varphi\|^2 e^{-\gamma(t-t_0)}, \quad t \geq t_0.$$

Remark 2 The uncertain random impulsive dynamical system (1) is called robust exponentially mean square stable if the zero solution $x = 0$ of the system is globally exponentially mean square stable for any $\|\Delta A\| \leq a$, $\|\Delta B\| \leq b$, $\|\Delta C\| \leq c$, where a, b, c are known nonnegative constants.

Hypothesis ($\mathbf{H}_1$) The condition $E\{\max_{i,k}\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\}\}$ is uniformly bounded. That is, there is a constant $\alpha > 0$ such that

$$E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\left\|b_j(\tau_j)\right\|\right\}\right\} \leq \alpha \quad \text{for all } \tau_j \in D_j, j = 1, 2, \ldots.$$

3 Main results

We need the following lemma to prove the main results.

Lemma 2 *Let $\phi(t, t_0)$ be the Cauchy matrix of the linear system*:

$$\begin{aligned} &\dot{x}(t) = Ax(t), \quad t \neq \xi_k, t \geq t_0 \\ &x(\xi_k^+) = b_k(\tau_k)x(\xi_k^-), \\ &x(s) = \varphi(s), \quad s \leq t_0. \end{aligned} \tag{5}$$

Then it satisfies $\|\phi(t, t_0)\| \leq e^{[\mu(A)-\lambda(1-\alpha)](t-t_0)}$, $t \geq t_0$.

Proof For any $x_0 \in \Re^n$, let $x(t) = x(t, t_0, x_0)$ be a solution through (t_0, x_0). Calculating the upper right derivative $D^+\|x(t)\|$ along the solution $x(t)$ of equation (5), we have

$$D^+\left\|x(t)\right\| \leq \mu(A)\left\|x(t)\right\|, \quad t \neq \xi_k, t \geq t_0$$

and

$$\begin{aligned} &\left\|x(t)\right\|^2 \leq \left[\left\|\prod_{j=1}^{k} b_j(\tau_j)\right\| e^{\mu(A)(t-t_0)}\left\|x(t_0)\right\| I_{[\xi_k,\xi_{k+1}]}(t)\right]^2, \quad t \in [\xi_k, \xi_{k+1}], t \geq t_0, \\ &E\left\|x(t)\right\|^2 \leq \sum_{k=0}^{\infty}\prod_{j=1}^{k}\left\|b_j(\tau_j)\right\|^2 e^{\mu(A)(t-t_0)} E\left\|x(t_0)\right\|^2 P\left(I_{[\xi_k,\xi_{k+1}]}(t)\right), \quad t \geq t_0, \\ &E\left\|x(t)\right\|^2 \leq e^{\mu(A)(t-t_0)} E\left\|x(t_0)\right\|^2 \sum_{i=0}^{\infty}\frac{\alpha\lambda^i(t-t_0)^i}{i!} e^{-\lambda(t-t_0)}, \quad t \geq t_0 \\ &\qquad\qquad \leq e^{\mu(A)(t-t_0)} E\left\|x(t_0)\right\|^2 e^{-(1-\alpha)\lambda(t-t_0)}, \\ &E\left\|x(t)\right\|^2 \leq e^{[\mu(A)-\lambda(1-\alpha)](t-t_0)} E\left\|x(t_0)\right\|^2. \end{aligned}$$

Since $x(t) = \phi(t, t_0)x(t_0)$, we obtain

$$\begin{aligned} &\left\|\phi(t, t_0)\right\| = \sup_{\|x(t_0)\|\neq 0}\frac{e^{[\mu(A)-\lambda(1-\alpha)](t-t_0)}E\|x(t_0)\|^2}{E\|x(t_0)\|^2}, \\ &\left\|\phi(t, t_0)\right\| \leq e^{[\mu(A)-\lambda(1-\alpha)](t-t_0)}, \quad t \geq t_0. \end{aligned}$$

This completes the proof. □

Now consider the linear uncertain random impulsive control system (3).

Theorem 1 *Assume that hypothesis* (H_1) *holds, then the zero solution of system* (3) *is robustly exponentially stable provided* $2a^2 + 2(b + \|B\|)^2 + 2k < 0$, *where* $k = [\mu(A) - \lambda(1-\alpha)]$.

Proof Since $2a^2 + 2(b + \|B\|)^2 + 2k < 0$, we choose small enough $\gamma \in (0, \eta)$ such that $2a^2 + 2(b + \|B\|)^2 e^{\gamma(\tau - t_0)} + 2k + \gamma < 0$, and $e^{-\gamma t_0} \leq 1$.

Furthermore, for any $\epsilon \in (0, \gamma)$, we have

$$0 \leq 2a^2 + 2\bigl(b + \|B\|\bigr)^2 e^{(\gamma-\epsilon)(\tau - t_0)} \leq -(2k + \gamma - \epsilon). \tag{6}$$

By the formula for variation of parameters, the solution of (3) can be presented as follows:

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t,s)\bigl[\Delta A x(s) + (B + \Delta B)x\bigl(s - r(s)\bigr)\bigr]\,ds,$$

where $\phi(t, t_0)$ is the Cauchy matrix of the impulsive linear system (5). Then we have

$$\begin{aligned}
\bigl\|x(t)\bigr\|^2 &\leq 2\bigl\|\phi(t,t_0)\bigr\|^2 \bigl\|x(t_0)\bigr\|^2 \\
&\quad + 2\biggl[\int_{t_0}^{t} \bigl\|\phi(t,s)\bigr\| \bigl(a\bigl\|x(s)\bigr\| + \bigl(b + \|B\|\bigr)\bigl\|x\bigl(s - r(s)\bigr)\bigr\|\bigr)\,ds\biggr]^2, \\
E\bigl\|x(t)\bigr\|^2 &\leq 2\bigl\|\phi(t,t_0)\bigr\|^2 E\bigl\|x(t_0)\bigr\|^2 \\
&\quad + 2\biggl[\int_{t_0}^{t} \bigl\|\phi(t,s)\bigr\| \bigl(aE\bigl\|x(s)\bigr\| + \bigl(b + \|B\|\bigr)E\bigl\|x\bigl(s - r(s)\bigr)\bigr\|\bigr)\,ds\biggr]^2, \\
E\bigl\|x(t)\bigr\|^2 &\leq 2e^{2k(t-t_0)} E\bigl\|x(t_0)\bigr\|^2 \\
&\quad + 2\int_{t_0}^{t} e^{2k(t-s)}\bigl(2a^2 E\bigl\|x(s)\bigr\|^2 + 2\bigl(b + \|B\|\bigr)^2 E\bigl\|x\bigl(s - r(s)\bigr)\bigr\|^2\bigr)\,ds, \quad t \geq t_0.
\end{aligned} \tag{7}$$

Without loss of generality, we assume that $E\|\varphi\|^2 > 0$. From $\gamma > \epsilon$, we get

$$E\bigl\|x(t)\bigr\|^2 \leq E\|\varphi\|^2 < E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t-t_0)} \quad \text{for } t \leq t_0. \tag{8}$$

In the following, we shall prove that

$$E\bigl\|x(t)\bigr\|^2 < E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t-t_0)} \quad \text{for } t \geq t_0. \tag{9}$$

If this is not true, by (8) and the piecewise continuity of $x(t)$, there must exist $t^* > t_0$ such that

$$E\bigl\|x\bigl(t^*\bigr)\bigr\|^2 \geq E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t^*-t_0)}, \tag{10}$$

$$E\bigl\|x(t)\bigr\|^2 \leq E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t-t_0)}, \quad t < t^*. \tag{11}$$

From (6), (7) and (11), we get

$$\begin{aligned}
E\|x(t^*)\|^2 &\le 2e^{2k(t^*-t_0)}E\|\varphi\|^2 \\
&\quad + 2\int_{t_0}^{t^*} e^{2k(t^*-s)}\bigl(2a^2e^{-(\gamma-\epsilon)s}E\|\varphi\|^2 + 2(b+\|B\|)^2e^{-(\gamma-\epsilon)(s-r(s))}E\|\varphi\|^2\bigr)\,ds \\
&\le 2e^{2k(t^*-t_0)}E\|\varphi\|^2 \\
&\quad + 2\int_{t_0}^{t^*} e^{2k(t^*-s)}e^{-(\gamma-\epsilon)s}E\|\varphi\|^2\bigl(2a^2 + 2(b+\|B\|)^2e^{(\gamma-\epsilon)(r(s))}\bigr)\,ds \\
&\le 2e^{2k(t^*-t_0)}E\|\varphi\|^2 \\
&\quad + 2e^{2kt^*}E\|\varphi\|^2\bigl(2a^2 + 2(b+\|B\|)^2e^{(\gamma-\epsilon)(r(s))}\bigr)\int_{t_0}^{t^*} e^{-(2k+(\gamma-\epsilon))s}\,ds \\
&\le 2e^{2k(t^*-t_0)}E\|\varphi\|^2 \\
&\quad \times\left[1 + \bigl(2a^2 + 2(b+\|B\|)^2e^{(\gamma-\epsilon)(\tau-t_0)}\bigr)e^{2kt_0}\int_{t_0}^{t^*} e^{-(2k+\gamma-\epsilon)s}\,ds\right] \\
&\le 2e^{2k(t^*-t_0)}E\|\varphi\|^2\bigl(1 + \bigl(2a^2 + 2(b+\|B\|)^2e^{(\gamma-\epsilon)(\tau-t_0)}\bigr) \times \bigl(-(2k+\gamma-\epsilon)\bigr)^{-1} \\
&\quad \times \bigl\{e^{2kt_0}\bigl[e^{-(2k+\gamma-\epsilon)t^*} - e^{-(2k+\gamma-\epsilon)t_0}\bigr]\bigr\}\bigr) \\
&\le 2e^{2k(t^*-t_0)}E\|\varphi\|^2\bigl[1 + e^{-2k(t^*-t_0)-(\gamma-\epsilon)t^*} - 1\bigr] \\
&\le 2e^{2k(t^*-t_0)}E\|\varphi\|^2\bigl(e^{-2k(t^*-t_0)-(\gamma-\epsilon)t^*}\bigr), \\
E\|x(t^*)\|^2 &\le 2E\|\varphi\|^2e^{-(\gamma-\epsilon)t^*}.
\end{aligned}$$

This contradicts (10), and so estimate (9) holds. Letting $\epsilon \to 0$, we have

$$E\|x(t)\|^2 \le 2E\|\varphi\|^2e^{-\gamma t}, \quad t \ge t_0.$$

This completes the proof. □

In the following theorem, we prove that the nonlinear uncertain random impulsive control system (1) is robust exponentially mean square stable.

Theorem 2 *Assume that hypothesis* (H_1) *holds, then the zero solution of system* (1) *is robustly exponentially stable provided* $4a^2 + 4(b+\|B\|)^2E\|\varphi\|^{2(\mu_1-1)} + 2(c+\|C\|)^2 \times E\|\varphi\|^{2(\mu_2-1)}M + 2k < 0$, *where* $k = [\mu(A) - \lambda(1-\alpha)]$ *and* $M = \int_0^\infty \|h(s)\|e^{\mu_2\eta s}\,ds$.

Proof Since $4a^2 + 4(b+\|B\|)^2E\|\varphi\|^{2(\mu_1-1)} + 2(c+\|C\|)^2E\|\varphi\|^{2(\mu_2-1)}M + 2k < 0$, we choose small enough $\gamma \in (0,\eta)$ such that

$$4a^2 + 4(b+\|B\|)^2E\|\varphi\|^{2(\mu_1-1)}e^{\mu_1\gamma(\tau-t_0)} + 2(c+\|C\|)^2E\|\varphi\|^{2(\mu_2-1)}M + (2k+\gamma) < 0,$$

and

$$e^{-\gamma t_0} \le 1.$$

Furthermore, for any $\epsilon \in (0,\gamma)$, we have

$$\begin{aligned} 0 &\le 4a^2 + 4\big(b + \|B\|\big)^2 E\|\varphi\|^{2(\mu_1-1)} e^{\mu_1(\gamma-\epsilon)(\tau-t_0)} + 2\big(c + \|C\|\big)^2 E\|\varphi\|^{2(\mu_2-1)} M \\ &\le -(2k + \gamma - \epsilon). \end{aligned} \tag{12}$$

By the formula for variation of parameters, the solution $x(t)$ can be represented as

$$\begin{aligned} x(t) = \phi(t,t_0)x(t_0) + \int_{t_0}^{t} \phi(t,s)\bigg(&\Delta A x(s) + (B+\Delta B)x^{\mu_1}\big(s - r(s)\big) \\ &+ (C+\Delta C)\int_0^\infty h(\eta)x^{\mu_2}(s-\eta)\,d\eta\bigg)\,ds, \end{aligned}$$

where $\phi(t,t_0)$ is the Cauchy matrix of impulsive linear system (5). Then we have

$$\begin{aligned} \big\|x(t)\big\|^2 \le{}& 2\big\|\phi(t,t_0)\big\|^2\big\|x(t_0)\big\|^2 \\ &+ 2\bigg(\int_{t_0}^{t}\big\|\phi(t,s)\big\|\bigg(a\big\|x(s)\big\| + \big(b+\|B\|\big)\big\|x^{\mu_1}\big(s-r(s)\big)\big\| \\ &+ \big(c+\|C\|\big)\int_0^\infty \big\|h(\eta)\big\|\big\|x^{\mu_2}(s-\eta)\big\|\,d\eta\bigg)\,ds\bigg)^2, \\ E\big\|x(t)\big\|^2 \le{}& 2e^{2k(t-t_0)}E\big\|x(t_0)\big\|^2 \\ &+ 2\bigg(\int_{t_0}^{t} e^{k(t-s)}\bigg(aE\big\|x(s)\big\| + \big(b+\|B\|\big)E\big\|x^{\mu_1}\big(s-r(s)\big)\big\| \\ &+ \big(c+\|C\|\big)\int_0^\infty \big\|h(\eta)\big\|E\big\|x^{\mu_2}(s-\eta)\big\|\,d\eta\bigg)\,ds\bigg)^2 \\ \le{}& 2e^{2k(t-t_0)}E\big\|x(t_0)\big\|^2 \\ &+ 2\int_{t_0}^{t} e^{2k(t-s)}\,ds\bigg(aE\big\|x(s)\big\| + \big(b+\|B\|\big)E\big\|x^{\mu_1}\big(s-r(s)\big)\big\| \\ &+ \big(c+\|C\|\big)\int_0^\infty \big\|h(\eta)\big\|E\big\|x^{\mu_2}(s-\eta)\big\|\,d\eta\bigg)^2, \\ E\big\|x(t)\big\|^2 \le{}& 2e^{2k(t-t_0)}E\big\|x(t_0)\big\|^2 \\ &+ 2\bigg(4a^2E\big\|x(s)\big\|^2 + 4\big(b+\|B\|\big)^2E\big\|x^{\mu_1}\big(s-r(s)\big)\big\|^2 \\ &+ 2\big(c+\|C\|\big)^2\int_0^\infty \big\|h(\eta)\big\|E\big\|x^{\mu_2}(s-\eta)\big\|^2\,d\eta\bigg)\int_{t_0}^{t} e^{2k(t-s)}\,ds. \end{aligned} \tag{13}$$

Without loss of generality, we assume that $E\|\varphi\|^2 > 0$. From $\gamma > \epsilon$, it is easily observed that

$$E\big\|x(t)\big\|^2 \le E\|\varphi\|^2 < E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t-t_0)} \quad \text{for } t \le t_0. \tag{14}$$

We shall prove that

$$E\big\|x(t)\big\|^2 \le E\|\varphi\|^2 < E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t-t_0)} \quad \text{for } t \ge t_0. \tag{15}$$

If this is not true, by (14) and the piecewise continuity of $x(t)$, there must exist $t^* > t_0$ such that

$$E\|x(t^*)\|^2 \geq E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t^*-t_0)}, \tag{16}$$

$$E\|x(t)\|^2 \leq E\|\varphi\|^2 e^{-(\gamma-\epsilon)(t-t_0)}, \quad t < t^*. \tag{17}$$

From (12), (13) and (17), we get

$$\begin{aligned}
E\|x(t^*)\|^2 \leq{}& 2e^{2k(t^*-t_0)}E\|\varphi\|^2 + 2\int_{t_0}^{t^*} e^{2k(t^*-s)} \\
&\times \Big(4a^2 e^{-(\gamma-\epsilon)s}E\|\varphi\|^2 + 4(b+\|B\|)^2 e^{-\mu_1(\gamma-\epsilon)(s-r(s))}E\|\varphi\|^{2\mu_1} \\
&+ 2(c+\|C\|)^2\int_0^\infty \|h(\eta)\| e^{-\mu_2(\gamma-\epsilon)(s-\eta)}E\|\varphi\|^{2\mu_1}\,d\eta\Big)\,ds \\
\leq{}& 2e^{2k(t^*-t_0)}E\|\varphi\|^2 \\
&+ 2e^{2kt^*}E\|\varphi\|^2\int_{t_0}^{t^*} e^{-(2k+\gamma-\epsilon)s} \\
&\times \Big(4a^2 + 4(b+\|B\|)^2 E\|\varphi\|^{2(\mu_1-1)}e^{(\gamma-\epsilon)(s-\mu_1 s+\mu_1 r(s))} \\
&+ 2(c+\|C\|)^2 E\|\varphi\|^{2(\mu_2-1)}\int_0^\infty \|h(\eta)\| e^{(\gamma-\epsilon)(s-\mu_2 s+\mu_2\eta)}\,d\eta\Big)\,ds \\
\leq{}& 2e^{2k(t^*-t_0)}E\|\varphi\|^2 \\
&+ 2e^{2kt^*}E\|\varphi\|^2\big(4a^2 + 4(b+\|B\|)^2 E\|\varphi\|^{2(\mu_1-1)}e^{\mu_1(\gamma-\epsilon)(\tau-t_0)} \\
&+ 2(c+\|C\|)^2 E\|\varphi\|^{2(\mu_2-1)}M\big)\int_{t_0}^{t^*} e^{-(2k+\gamma-\epsilon)s}\,ds \\
\leq{}& 2e^{2k(t^*-t_0)}E\|\varphi\|^2 \\
&+ 2e^{2kt^*}E\|\varphi\|^2\big(4a^2 + 4(b+\|B\|)^2 E\|\varphi\|^{2(\mu_1-1)}e^{\mu_1(\gamma-\epsilon)(\tau-t_0)} \\
&+ 2(c+\|C\|)^2 E\|\varphi\|^{2(\mu_2-1)}M \\
&\times \big(-(2k+\gamma-\epsilon)\big)^{-1}\big)\big[e^{-(2k+\gamma-\epsilon)t^*} - e^{-(2k+\gamma-\epsilon)t_0}\big] \\
\leq{}& 2e^{2k(t^*-t_0)}E\|\varphi\|^2\big[1 + e^{2kt_0}\big(e^{-(2k+\gamma-\epsilon)t^*} - e^{-(2k+\gamma-\epsilon)t_0}\big)\big] \\
\leq{}& 2e^{2k(t^*-t_0)}E\|\varphi\|^2\big[1 + e^{-2k(t^*-t_0)-(\gamma-\epsilon)t^*} - 1\big], \\
E\|x(t^*)\|^2 \leq{}& 2E\|\varphi\|^2 e^{-(\gamma-\epsilon)t^*}.
\end{aligned}$$

This contradicts (16), and so estimate (15) holds. Letting $\epsilon \to 0$, we have

$$E\|x(t)\|^2 \leq 2E\|\varphi\|^2 e^{-\gamma t}, \quad t \geq t_0.$$

This completes the proof. □

Especially, for the linear case, we have the following result.

Corollary 1 *Assume that hypothesis* (H_1) *holds, then the zero solution of system* (2) *is robustly exponentially stable provided* $4a^2 + 4(b + \|B\|)^2 + 2(c + \|C\|)^2 M + 2k < 0$, *where* $k = [\mu(A) - \lambda(1 - \alpha)]$ *and* $M = \int_0^\infty \|h(\eta)\| e^{(\gamma-\epsilon)\eta}\, d\eta$.

Proof The proof is similar to that of Theorem 2, when $\mu_1 = 1$, $\mu_2 = 1$. □

4 Example

In this section, we will give four numerical examples to illustrate that our results can be applied to stabilize the unstable continuous systems by using random impulsive control.

Example 1 Consider the following linear uncertain random impulsive control system with infinite delays:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)x\bigl(t - \tau(t)\bigr) \\ &\quad + (C + \Delta C)\int_0^\infty h(\eta)x(t - \eta)\, d\eta, \quad t \neq \xi_k, t \geq t_0, \\ x\bigl(\xi_k^+\bigr) &= b_i(\tau_i)x\bigl(\xi_k^-\bigr), \quad k = 1, 2, \ldots, \end{aligned} \tag{18}$$

where $\tau(t)$ is a time-varying delay function with $\tau(t) \in [0, \tau]$, $h(\eta) = 0.1e^{-1.2\eta}$, $\eta > 0$ and with the following parameter matrices:

$$A = \begin{bmatrix} 1.2 & -1.1 \\ 0.7 & 0.8 \end{bmatrix}, \qquad B = \begin{bmatrix} -1.3 & 0.7 \\ -0.9 & 0.5 \end{bmatrix}, \qquad C = \begin{bmatrix} -0.1 & -0.22 \\ 0.43 & 0.65 \end{bmatrix}.$$

The zero solution of system (18) is robust exponentially mean square stable provided $\lambda(1 - \alpha) > 9.4263$.

Proof By Corollary 1, and let us take $\eta = 0.2$, then $M = 0.1\int_0^\infty e^{-\eta}\, d\eta = 0.1$. Now, the eigenvalues of A are $1 + 0.8544i$, $1 - 0.8544i$. Further, we use the defined matrix norm and the matrix measure to get $\mu(A) = 1.2828$, $\|B\| = 1.7999$, $\|C\| = 0.8151$ and $\|\Delta A\| \leq a = 0.1$, $\|\Delta B\| \leq b = 0.2$, $\|\Delta C\| \leq c = 0.3$.

$$\begin{aligned} &4a^2 + 4\bigl(b + \|B\|\bigr)^2 + 2\bigl(c + \|C\|\bigr)^2 M + 2k < 0, \\ &4a^2 + 4\bigl(b + \|B\|\bigr)^2 + 2\bigl(c + \|C\|\bigr)^2 M + 2\bigl(\mu(A) - \lambda(1 - \alpha)\bigr) < 0, \\ &4(0.1)^2 + 4(0.2 + 1.7999)^2 + 2(0.3 + 0.8151)^2(0.1) + 2\bigl(1.2828 - \lambda(1 - \alpha)\bigr) < 0, \\ &18.8526 - 2\lambda(1 - \alpha) < 0, \\ &\lambda(1 - \alpha) > 9.4263. \end{aligned}$$

Hence (18) is robust exponentially mean square stable. □

Example 2 Consider the following nonlinear uncertain random impulsive infinite delay differential system:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)x^{\mu_1}\big(t - \tau(t)\big) \\ &\quad + (C + \Delta C)\int_0^\infty h(\eta)x^{\mu_2}(t - \eta)\,d\eta, \quad t \neq \xi_k, t \geq t_0, \\ x\big(\xi_k^+\big) &= b_i(\tau_i)x\big(\xi_k^-\big), \quad k = 1, 2, \dots, \end{aligned} \tag{19}$$

where $\mu_1, \mu_2 \geq 1$ with A, B and C defined as in Example 1. Then there exists a constant $\eta > 0$ such that $M = \int_0^\infty |h(s)|e^{\mu_2 \eta s}\,ds < \infty$. Then the zero solution of system (19) is robust exponentially mean square stable provided $2.6056 + 15.9984E\|\varphi\|^{2(\mu_1-1)} + 2.4868ME\|\varphi\|^{2(\mu_2-1)} - 2\lambda(1-\alpha) < 0$.

Proof By Theorem 2, we get

$$\begin{aligned} &4a^2 + 4\big(b + \|B\|\big)^2 E\|\varphi\|^{2(\mu_1-1)} + 2\big(c + \|C\|\big)^2 E\|\varphi\|^{2(\mu_2-1)}M + 2k < 0, \\ &2.6056 + 15.9984E\|\varphi\|^{2(\mu_1-1)} + 2.4868ME\|\varphi\|^{2(\mu_2-1)} - 2\lambda(1-\alpha) < 0. \end{aligned}$$

Hence (19) is robust exponentially mean square stable. □

Example 3 Consider the following uncertain random impulsive control system with infinite delays:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)x\big(t - \tau(t)\big), \quad t \neq \xi_k, t \geq t_0, \\ x\big(\xi_k^+\big) &= b_i(\tau_i)x\big(\xi_k^-\big), \quad k = 1, 2, \dots. \end{aligned} \tag{20}$$

We take the following parameter matrices:

$$A = \begin{bmatrix} -1 & 0 & 0.5 \\ 0.5 & 2.5 & -1.5 \\ 0 & 3 & -1.5 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.5 & 0.1 & 0.3 \\ 0.2 & -0.5 & 0.1 \\ -0.3 & 0 & 0.2 \end{bmatrix}.$$

Then the zero solution of system (20) is robust exponentially mean square stable provided $\lambda(1-\alpha) > 6.4562$.

Proof Checking the eigenvalues of A, we find that they are –0.5, 0, 0.5. By Theorem 1, the matrix norm and the matrix measure are defined as $\mu(A) = 2.6591$, $\|B\| = 0.7353$, and $\|\Delta A\| \leq a = 1.2$, $\|\Delta B\| \leq b = 0.8$. Then

$$\begin{aligned} &2a^2 + 2\big(b + \|B\|\big)^2 + 2k < 0, \\ &2a^2 + 2\big(b + \|B\|\big)^2 + 2\big(\mu(A) - \lambda(1-\alpha)\big) < 0, \end{aligned}$$

$$2(1.2)^2 + 2(2.3571) + 2\big(2.6591 - \lambda(1-\alpha)\big) < 0,$$
$$12.9124 - 2\lambda(1-\alpha) < 0,$$
$$\lambda(1-\alpha) > 6.4562.$$

Hence (20) is robust exponentially mean square stable. □

Example 4 Consider the following linear random impulsive delay differential system of the form (4):

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -0.012 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.12 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t-0.05) \\ x_2(t-0.05) \end{bmatrix}.$$

The zero solution of the system is exponentially mean square stable provided $\lambda(1-\alpha) > 1.2173$.

Proof Checking the eigenvalues of A, we find that they are 0.0121 and 0.9879. By Theorem 1, we take the values are $\mu(A) = 1.2029$, $\|B\| = 0.12$ and $\|\Delta A\| \leq a = 0$, $\|\Delta B\| \leq b = 0$, using the matrix norm and the matrix measure. Then

$$2a^2 + 2\big(b + \|B\|\big)^2 + 2k < 0,$$
$$2a^2 + 2\big(b + \|B\|\big)^2 + 2\big(\mu(A) - \lambda(1-\alpha)\big) < 0,$$
$$2(0.12)^2 + 2(1.2029) - 2\lambda(1-\alpha) < 0,$$
$$\lambda(1-\alpha) > 1.2173.$$

This system without impulses is unstable, but by Theorem 1 this system can be exponentially mean square stable. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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