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Some results on q-harmonic number sums

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Abstract

In this paper, we establish some relations involving q-Euler type sums, q-harmonic numbers and q-polylogarithms. Then, using the relations obtained with the help of q-analog of partial fraction decomposition formula, we develop new closed form representations of sums of q-harmonic numbers and reciprocal q-binomial coefficients. Moreover, we give explicit formulas for several classes of q-harmonic sums in terms of q-polylogarithms and q-harmonic numbers. The given representations are new.

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Keywords: q-harmonic number; q-binomial coefficient; q-polylogarithm function

1 Introduction and preliminaries

Let $k, r, m_1, m_2, \ldots, m_r$ be positive integers and $p \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ with $p + k > 1$. The Euler type sums $W_k(m_1, m_2, \ldots, m_r, p)$ involving harmonic numbers and binomial coefficients are defined by the convergent series [1]

$$W_k(\mathbf{m}; p) = W_k(m_1, m_2, \ldots, m_r; p) := \sum_{n=1}^{\infty} \frac{H_n^{(m_1)} H_n^{(m_2)} \cdots H_n^{(m_r)}}{n^p \binom{n+k}{k}}, \tag{1.1}$$

where $H_n^{(m)}$ stands for the nth generalized harmonic number defined by

$$H_n^{(m)} := \sum_{j=1}^{n} \frac{1}{j^m} \quad \bigl(n, k \in \mathbb{N} := \{1, 2, 3, \ldots\}\bigr), \tag{1.2}$$

the quantity $w := m_1 + m_2 + \cdots + m_r + p + k$ and the quantity r are called the weight and the degree of (1.1), respectively. The empty sum $H_0^{(m)}$ is conventionally understood to be zero. When $m = 1$, then $H_n := H_n^{(1)}$ is called a classical harmonic number. As usual, we let $\{a\}_k$ be the k repetitions such that

$$W\bigl(a, \{m\}_r, b; p\bigr) = W(a, \underbrace{m, \ldots, m}_{r}, b; p).$$

There are many results for sums of harmonic numbers with positive terms. For example, in [1], Xu et al. proved the result

$$W_k(\{1\}_2;1)=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left\{\begin{array}{l}3\zeta(3)+\frac{H_r^3+3H_rH_r^{(2)}+2H_r^{(3)}}{3}\\ -\frac{H_r^2+H_r^{(2)}}{r}-\sum_{i=1}^{r-1}\frac{H_i}{i^2}+\zeta(2)H_{r-1}\end{array}\right\}\quad (k\in\mathbb{N}). \tag{1.3}$$

Sofo also obtained many other identities involving harmonic numbers and central binomial coefficients. For instance, in [2], Sofo gave the following identity:

$$W_k(\{1\}_2;0)=\frac{k}{k-1}\left(\zeta(2)-H_{k-1}^{(2)}+\frac{2}{(k-1)^2}\right)\quad (2\le k\in\mathbb{N}), \tag{1.4}$$

where $\zeta(p)$ stands for the classical Riemann zeta function defined by [3]

$$\zeta(p):=\sum_{n=1}^{\infty}\frac{1}{n^p}\quad (\Re(p)>1).$$

There are many works investigating sums of both harmonic numbers and binomial coefficients (see, for example, [1, 2, 4–6] and the references therein).

If $k=0$ in (1.1), then

$$W_0(\mathbf{m};p)=W_0(m_1,m_2,\ldots,m_r;p):=\sum_{n=1}^{\infty}\frac{H_n^{(m_1)}H_n^{(m_2)}\cdots H_n^{(m_r)}}{n^p}, \tag{1.5}$$

which is just the classical Euler sums $S_{\mathbf{m},\mathbf{p}}$ defined in [7], where $\mathbf{m}:=(m_1,m_2,\ldots,m_r)$. The study of Euler sums $W_0(\mathbf{m};p)$ was started by Euler. Euler's original contribution was a method to reduce double sums $W_0(p;q)$ (or $S_{p,q}$) to certain rational linear combinations of products of zeta values. Examples for such evaluations, all due to Euler, are as follows:

$$W_0(1;3)=\frac{5}{4}\zeta(4),\qquad W_0(1;4)=3\zeta(5)-\zeta(2)\zeta(3),$$
$$W_0(2;4)=\zeta^2(3)-\frac{1}{3}\zeta(6),\qquad W_0(2;5)=5\zeta(2)\zeta(5)+2\zeta(3)\zeta(4)-10\zeta(7).$$

After that many different methods, including partial fraction expansions, Eulerian beta integrals, summation formulas for generalized hypergeometric functions and contour integrals, have been used to evaluate these sums. The relationship between the values of the Riemann zeta function and the classical Euler sums $W_0(\mathbf{m};p)$ (or $S_{\mathbf{m};p}$) has been studied by many authors (for example, see [7–16] and the references therein).

So far, surprisingly little work has been done on q-analogues of Euler sums. We begin with some basic notation. Let q be a real number with $0<q<1$. The q-analogue of a non-negative integer n is defined as

$$n_q=[n]_q:=\sum_{k=0}^{n-1}q^k=\frac{1-q^n}{1-q}.$$

For any real number a, put

$$(a)_0 := (a;q)_0 = 1 \quad \text{and} \quad (a)_n := (a;q)_n := \prod_{k=1}^{n} \left(1 - aq^{k-1}\right), \quad n \geq 1.$$

Let n, m denote integers. Then the Gaussian q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q)_n}{(q)_m (q)_{n-m}} = \frac{[n]_q!}{[m]_q![n-m]_q!}, \tag{1.6}$$

where $0 \leq m \leq n$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$ with $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right]_q = 1$.

For non-negative integers n, s and $m \in \mathbb{N}$, define q-analogues of harmonic numbers

$$\zeta_n\left[m, q^s\right] := \sum_{j=1}^{n} \frac{q^{sj}}{[j]_q^m},$$

with the convention that $\zeta_0[m, q^s] := 0$. When $s = 0$ and 1, we use the following notations (see [17]):

$$[H_n] := \zeta_n[1] = \zeta_n[1, 1]$$

and

$$\zeta_n[m] := \zeta_n[m, 1], \qquad \left[H_n^{(m)}\right] := \zeta_n[m, q].$$

Similar as in the definition of classical Euler sum $W_0(\mathbf{m}; p)$ (or $S_{\mathbf{m};p}$), the q-analogues of Euler sum of index $\mathbf{m} := (m_1, \ldots, m_r), \mathbf{s} := (s_1, \ldots, s_r)$ with $p > 1$ are defined by

$$W_{0,t}^{(\mathbf{s})}[\mathbf{m}; p] := \sum_{n=1}^{\infty} \frac{\zeta_n[m_1, q^{s_1}]\zeta_n[m_2, q^{s_2}] \cdots \zeta_n[m_r, q^{s_r}]}{([n]_q)^p} q^{tn}, \tag{1.7}$$

where $s_i \in \mathbb{N}_0, m_i \in \mathbb{N}$ $(i = 1, 2, \ldots, r)$ and $r, t \in \mathbb{N}$, the quantities $w := m_1 + m_2 + \cdots + m_r + p + k$ and r are called the weight and the degree of $W_{0,t}^{(\mathbf{s})}[\mathbf{m}; p]$, respectively. There are fewer results for sums of the type (1.7). Some related results for q-Euler type sums and related sums (e.g. q-L-function and q-multiple zeta values) may be seen in the works of [17–37] and the references therein. For example, in [17], Xu et al. gave the following identity:

$$W_{0,1}^{(0)}[1; s] = \mathrm{Li}_{s+1}[q] + \frac{s}{2}\mathrm{Li}_{s+1}\left[q^2\right] - \frac{1}{2}\sum_{j=2}^{s-1} \mathrm{Li}_j[q]\mathrm{Li}_{s+1-j}[q], \quad 2 \leq s \in \mathbb{N}.$$

Examples for such evaluation are as follows:

$$\sum_{n=1}^{\infty} \frac{[H_n]}{n_q^2} q^n = \mathrm{Li}_3[q] + \mathrm{Li}_3\left[q^2\right], \qquad \sum_{n=1}^{\infty} \frac{[H_n]}{n_q^3} q^n = \frac{3}{2}\mathrm{Li}_4\left[q^2\right] + \mathrm{Li}_4[q] - \frac{1}{2}\mathrm{Li}_2^2[q].$$

Furthermore, they proved the following conclusion: for positive integer $s \geq 2$, the quadratic sum $W_{0,1}^{(0,0)}[\{1\}_2; s]$ and the cubic combination sum $W_{0,1}^{(0,0,0)}[\{1\}_3; s] - 3W_{0,1}^{(0,0)}[1, 2;$

$s]$ are reducible to linear q-Euler sums and to polynomials in q-polylogarithms. In particular, we have

$$W_{0,1}^{(0,0)}\big(\{1\}_2;2\big)=\frac{7}{2}\mathrm{Li}_4\big[q^2\big]+2\mathrm{Li}_4[q]-\frac{1}{2}\mathrm{Li}_2^2[q]-(1-q)\big(\mathrm{Li}_3\big[q^2\big]+\mathrm{Li}_3[q]\big). \tag{1.8}$$

The q-analogues of polylogarithm function $\mathrm{Li}_m[x]$ and linear q-Euler sum $W_{0,t}^{(s)}[m;p]$ are defined by

$$\mathrm{Li}_m[x]:=\sum_{n=1}^{\infty}\frac{x^n}{[n]_q^m},\quad |x|<1, \tag{1.9}$$

$$W_{0,t}^{(s)}[m;p]:=\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^s]}{[n]_q^p}q^{tn}, \tag{1.10}$$

where $t,m\in\mathbb{N}$, $s\in\mathbb{N}_0$ and $p>1$.

In this paper we will develop identities, closed form representations of q-harmonic numbers and reciprocal q-binomial coefficients of the form:

$$W_{k,t}^{(s_1,\ldots,s_r)}[m_1,m_2,\ldots,m_r;p]:=\sum_{n=1}^{\infty}\frac{\zeta_n[m_1,q^{s_1}]\zeta_n[m_2,q^{s_2}]\cdots\zeta_n[m_r,q^{s_r}]}{[n]_q^p\left[{n+k\atop k}\right]_q}q^{tn}, \tag{1.11}$$

for $p=0$ and 1 with $t=k$ and $k-1$. Here, $s_i\in\mathbb{N}_0, m_i\in\mathbb{N}$ $(i=1,2,\ldots,r)$ and $k,r\in\mathbb{N}$. We show that the linear sums $W_{k,k}^{(m-1)}[m;1]$ is a rational linear combination of products of q-harmonic numbers and q-polylogarithms, and we give an explicit formula. We also provide explicit evaluations of quadratic sum $W_{k,k}^{(0,0)}[\{1\}_2;1]$ in a closed form in terms of q-polylogarithms, q-harmonic numbers and q-rational series. Furthermore, we prove that the cubic sum $W_{k,k}^{(0,0,0)}[\{1\}_3;1]$ is expressible in terms of q-polylogarithms, q-harmonic numbers and q-rational series. Letting q approach 1, we can find that the q-Euler type sum $W_{k,t}^{(\mathbf{s})}[\mathbf{m};p]$ converges to the classical Euler type sums $W_k(\mathbf{m};p)$, namely

$$\lim_{q\to 1}W_{k,t}^{(s_1,\ldots,s_r)}[m_1,m_2,\ldots,m_r;p]=W_k(m_1,m_2,\ldots,m_r;p).$$

Next, we prove a lemma which will be useful in the development of the main theorems.

Lemma 1.1 *For positive integers m,r,k and $r<k$, then*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{q^{mn}}{[n+r]_q^m[n+k]_q}\\&\quad=\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{[k-r]_q^jq^{rj}}\mathrm{Li}_{m-j+1}\big[q^{m-j},r\big]+\frac{(-1)^{m-1}}{[k-r]_q^mq^{rm}}\big\{\big[H_k^{(1)}\big]-\big[H_r^{(1)}\big]\big\},\end{aligned} \tag{1.12}$$

where the q-special function $\mathrm{Li}_p[x,a]$ is defined by

$$\mathrm{Li}_p[x,a]:=\sum_{n=1}^{\infty}\frac{x^n}{[n+a]_q^p},\quad |x|<1,p\in\mathbb{N}. \tag{1.13}$$

Proof By a simple calculation, the sum on the left-hand side of (1.12) is equal to

$$\sum_{n=1}^{\infty}\frac{q^{mn}}{[n+r]_q^m[n+k]_q}=\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{[k-r]_q^j q^{rj}}\sum_{n=1}^{\infty}\frac{q^{(m-j)n}}{[n+r]_q^{m-j+1}} + \frac{(-1)^{m-1}}{[k-r]_q^{m-1}q^{r(m-1)}}\sum_{n=1}^{\infty}\frac{q^n}{[n+r]_q[n+k]_q}. \tag{1.14}$$

On the other hand, we note that, for $N > k > r \geq 1$ and $N, k, r \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n}{[n+r]_q[n+k]_q} &= \frac{1}{[k-r]_q q^r}\sum_{n=1}^{\infty}\left\{\frac{1}{[n+r]_q}-\frac{1}{[n+k]_q}\right\}\\ &= \frac{1}{[k-r]_q q^r}\lim_{N\to\infty}\sum_{n=1}^{N}\left\{\frac{1}{[n+r]_q}-\frac{1}{[n+k]_q}\right\}\\ &= \frac{1}{[k-r]_q q^r}\left\{\frac{1}{[r+1]_q}+\cdots+\frac{1}{[k]_q}-\lim_{N\to\infty}\sum_{j=r+1}^{k}\frac{1}{[N+j]_q}\right\}\end{aligned} \tag{1.15}$$

and

$$\lim_{N\to\infty}(N+j)_q=\lim_{N\to\infty}\frac{1-q^{N+j}}{1-q}=\frac{1}{1-q}. \tag{1.16}$$

By using the definition of q-harmonic numbers, we have the relations

$$\left[H_n^{(m)}\right]=(q-1)\zeta_n[m-1]+\zeta_n[m], \quad m\in\mathbb{N}, \tag{1.17}$$

$$\left[H_n^{(1)}\right]=[H_n]+n(q-1). \tag{1.18}$$

Combining (1.15), (1.16) and (1.18) yields

$$\sum_{n=1}^{\infty}\frac{q^n}{[n+r]_q[n+k]_q}=\frac{1}{[k-r]_q q^r}\left\{\left[H_k^{(1)}\right]-\left[H_r^{(1)}\right]\right\}. \tag{1.19}$$

Substituting (1.19) into (1.14) yields the desired result. The proof of Lemma 1.1 is finished. □

2 Main conclusions and proofs

In this section, we will give the main results of the present paper. Firstly, we establish a q-analog of partial fraction decomposition formula by the method of mathematical induction. Secondly, using the Jackson's q-integral, we prove some relations between q-Euler type sums, q-harmonic numbers and q-polylogarithms. Then we use the formulas obtained to evaluate several infinite series involving q-harmonic numbers.

It is clear that the conclusions which we present here can be seen as an extension of classical Euler type sums given by Sofo and Xu. Letting $q \to 1$, we obtain many well-known results which are given by Sofo and Xu.

2.1 *q*-analog of partial fraction decomposition formula

Theorem 2.1 *For positive integer m and real x with $x \neq -1, -2, \dots, -m$, the following identity holds*:

$$\prod_{i=1}^{m}\left(1-q^{x+i}\right)^{-1} = \sum_{i=1}^{m}\frac{q^{(1-m)x}}{1-q^{x+i}}\prod_{j=1,j\neq i}^{m}\frac{1}{q^i-q^j}. \tag{2.1}$$

Notice that the term in the sum for $m = 1$ is the empty product which is 1, *namely, when* $m = 1$, *we set* $\prod_{j=1,j\neq 1}^{1}(q^i-q^j)^{-1} := 1$.

Proof The proof is by induction on m. For $m = 1$, we have $\prod_{i=1}^{1}(1-q^{x+i})^{-1} = (1-q^{x+1})^{-1}$, and the formula is true. For $m > 1$, we proceed as follows. First assume that formula (2.1) holds for $m \leq k-1$, we note that

$$\begin{aligned}\frac{1}{\prod_{i=1}^{k}(1-q^{x+i})} &= \frac{1}{1-q^{x+k}}\cdot\frac{1}{\prod_{i=1}^{k-1}(1-q^{x+i})}\\ &= \sum_{i=1}^{k-1}\frac{q^{(2-k)x}}{(1-q^{x+k})(1-q^{x+i})}\prod_{j=1,j\neq i}^{k-1}\frac{1}{q^i-q^j}\\ &= \sum_{i=1}^{k-1}\frac{q^{(2-k)x}}{q^{x+i}-q^{x+k}}\left\{\frac{1}{1-q^{x+i}}-\frac{1}{1-q^{x+k}}\right\}\prod_{j=1,j\neq i}^{k-1}\frac{1}{q^i-q^j}\\ &= \sum_{i=1}^{k-1}\frac{q^{(1-k)x}}{1-q^{x+i}}\prod_{j=1,j\neq i}^{k}\frac{1}{q^i-q^j}-\frac{q^{(1-k)x}}{1-q^{x+k}}\sum_{i=1}^{k-1}\prod_{j=1,j\neq i}^{k}\frac{1}{q^i-q^j}.\end{aligned} \tag{2.2}$$

Then, by the induction hypothesis, we have that

$$\begin{aligned}&\frac{1}{\prod_{i=1}^{k-1}(1-q^{x+i})}\\ &\quad= \sum_{i=1}^{k-1}\frac{q^{(2-k)x}}{1-q^{x+i}}\prod_{j=1,j\neq i}^{k-1}\frac{1}{q^i-q^j},\quad x\neq -1,-2,\dots,-(k-1).\end{aligned} \tag{2.3}$$

Setting $x = -k$ in the above equation, we deduce that

$$\begin{aligned}\frac{1}{\prod_{i=1}^{k-1}(1-q^{i-k})} = \frac{q^{k(k-1)}}{\prod_{i=1}^{k-1}(q^k-q^i)} &= \sum_{i=1}^{k-1}\frac{q^{(k-2)k}}{1-q^{i-k}}\prod_{j=1,j\neq i}^{k-1}\frac{1}{q^i-q^j}\\ &= -\sum_{i=1}^{k-1}\frac{q^{k(k-1)}}{q^i-q^k}\prod_{j=1,j\neq i}^{k-1}\frac{1}{q^i-q^j}.\end{aligned}$$

Hence, we obtain

$$\prod_{i=1}^{k-1}\left(q^k-q^i\right)^{-1} = -\sum_{i=1}^{k-1}\prod_{j=1,j\neq i}^{k}\frac{1}{q^i-q^j}. \tag{2.4}$$

Substituting (2.4) into (2.2), we arrive at the conclusion that

$$\prod_{i=1}^{k}\left(1-q^{x+i}\right)^{-1}=\sum_{i=1}^{k}\frac{q^{(1-k)x}}{1-q^{x+i}}\prod_{j=1,j\neq i}^{k}\frac{1}{q^i-q^j}.$$

The proof of Theorem 2.1 is completed. □

Next, we give a q-analog of partial fraction decomposition formula.

Corollary 2.2 *For integer $m>0$ and real x with $x\neq\{-1,-2,\dots,-m\}$, we have*

$$\prod_{i=1}^{m}[x+i]_q^{-1}=\sum_{i=1}^{m}(-1)^{i-1}[i]_q\begin{bmatrix}m\\ i\end{bmatrix}_q\frac{q^{(1-m)x-\frac{i}{2}(2m-1-i)}}{[m]_q!}\cdot\frac{1}{[x+i]_q},\tag{2.5}$$

where $[x+i]_q:=\frac{1-q^{x+i}}{1-q}$.

Proof Multiplying (2.1) by $(1-q)^m$ and using the definition of $[x+i]_q$, we obtain

$$\begin{aligned}\frac{1}{\prod_{i=1}^{m}[x+i]_q}&=\frac{1}{\prod_{i=1}^{m}\frac{1-q^{x+i}}{1-q}}=\sum_{i=1}^{m}\frac{q^{(1-m)x}}{\frac{1-q^{x+i}}{1-q}}\prod_{j=1,j\neq i}^{m}\frac{1}{\frac{q^i-q^j}{1-q}}\\&=\sum_{i=1}^{m}\frac{q^{(1-m)x}}{[x+i]_q}\prod_{j=1,j\neq i}^{m}\frac{1}{\frac{q^i-q^j}{1-q}}.\end{aligned}\tag{2.6}$$

We may rewrite the product on the right-hand side of (2.6) as follows:

$$\begin{aligned}\prod_{j=1,j\neq i}^{m}\frac{1}{\frac{q^i-q^j}{1-q}}&=\prod_{j=1}^{i-1}\frac{1}{-q^j\frac{1-q^{i-j}}{1-q}}\cdot\prod_{j=i+1}^{m}\frac{1}{q^i\frac{1-q^{j-i}}{1-q}}\\&=\frac{(-1)^{i-1}}{q^{\frac{i}{2}(2m-1-i)}}\cdot\frac{1}{[i-1]_q![m-i]_q!}.\end{aligned}\tag{2.7}$$

Combining (2.6) with (2.7), we may deduce the desired result. This completes the proof of Corollary 2.2. □

By using the definition of q-binomial coefficient in (1.6) and combining (2.5) with $m=k,x=n$ $(k,n\in\mathbb{N})$, we have the following expansion:

$$\frac{1}{\begin{bmatrix}n+k\\ k\end{bmatrix}_q}=\frac{[k]_q!}{\prod_{i=1}^{k}[n+i]_q}=\sum_{i=1}^{k}(-1)^{i-1}[i]_q\begin{bmatrix}k\\ i\end{bmatrix}_q\frac{q^{(1-k)n-\frac{i}{2}(2k-1-i)}}{[n+i]_q}.\tag{2.8}$$

Similarly, using a similar argument, we can get

$$\frac{1}{\begin{bmatrix}n+k+r\\ k\end{bmatrix}_q}=[k]_q\sum_{i=1}^{k-1}(-1)^{i-1}[i]_q\begin{bmatrix}k-1\\ i\end{bmatrix}_q\frac{q^{(2-k)(n+r+1)-\frac{i}{2}(2k-3-i)}}{[n+r+1]_q[n+r+1+i]_q},\tag{2.9}$$

where $r\in\mathbb{N}_0$ and $2\leq k\in\mathbb{N}$.

2.2 Identities for q-Euler type sums

Theorem 2.3 *For positive integers m and k, the following identity holds*:

$$\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^{m-1}]}{[n]_q[n+k]_q}q^n=\frac{1}{[k]_q}\left\{\begin{array}{c}\mathrm{Li}_{m+1}[q^m]+(-1)^{m-1}\sum_{i=1}^{k-1}\frac{[H_i^{(1)}]}{[i]_q^m}q^i\\ +\sum_{j=1}^{m-1}(-1)^{j-1}\mathrm{Li}_{m+1-j}[q^{m-j}][H_{k-1}^{(j)}]\end{array}\right\}. \tag{2.10}$$

Proof By using the Cauchy product of power series and the definition of q-harmonic numbers, we can find that

$$\sum_{n=1}^{\infty}\zeta_n\bigl[m,q^p\bigr]x^n=\frac{\mathrm{Li}_m[q^px]}{1-x},\quad |x|<1, m,p\in\mathbb{N}. \tag{2.11}$$

Multiplying (2.11) by $x^{-1}-x^{k-1}$ and q-integrating over $(0,q)$ yield

$$[k]_q\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^p]}{[n]_q[n+k]_q}q^n=\mathrm{Li}_{m+1}\bigl[q^{p+1}\bigr]+\sum_{i=1}^{k-1}\sum_{n=1}^{\infty}\frac{q^{(p+1)n+i}}{[n]_q^m[n+i]_q}, \tag{2.12}$$

where the generalized q-integral is defined by ($a\le x\ne\infty$) (see [3, 17, 38, 39])

$$\int_a^x f(t)d_qt=\int_0^x f(t)d_qt-\int_0^a f(t)d_qt=(1-q)\sum_{i=0}^{\infty}q^i\bigl[xf\bigl(q^ix\bigr)-af\bigl(q^ia\bigr)\bigr] \tag{2.13}$$

and

$$\int_0^x f(t)d_qt=(1-q)x\sum_{i=0}^{\infty}q^if\bigl(q^ix\bigr),\quad x\ne\infty. \tag{2.14}$$

Taking $p=m-1$ in (2.12) and $r=0$ in (1.12), we obtain

$$[k]_q\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^{m-1}]}{[n]_q[n+k]_q}q^n=\mathrm{Li}_{m+1}\bigl[q^m\bigr]+\sum_{i=1}^{k-1}q^i\sum_{n=1}^{\infty}\frac{q^{mn}}{[n]_q^m[n+i]_q}, \tag{2.15}$$

$$\sum_{n=1}^{\infty}\frac{q^{mn}}{[n]_q^m[n+k]_q}=\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{[k]_q^j}\mathrm{Li}_{m-j+1}\bigl[q^{m-j}\bigr]+(-1)^{m-1}\frac{[H_k^{(1)}]}{[k]_q^m}. \tag{2.16}$$

Substituting (2.16) into (2.15) yields the desired result. We finish the proof of Theorem 2.3. □

In fact, by a similar argument as in the proof of Theorem 2.3, we obtain the more general identity

$$\begin{aligned}[k-r]_q\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^{m-1}]}{[n+r]_q[n+k]_q}q^n&=\sum_{j=1}^{m-1}(-1)^{j-1}\mathrm{Li}_{m-j+1}\bigl[q^{m-j}\bigr]\frac{[H_{k-1}^{(j)}]-[H_{r-1}^{(j)}]}{q^r}\\&\quad+(-1)^{m-1}\sum_{i=1}^{k-r}\frac{[H_{r+i-1}^{(1)}]}{[r+i-1]_q^m}q^{i-1},\end{aligned} \tag{2.17}$$

where $m, k, r \in \mathbb{N}$ and $r < k$. Putting $m = 1$ and 2 in (2.10), we give the following two examples:

$$\sum_{n=1}^{\infty} \frac{[H_n]}{[n]_q[n+k]_q} q^n = \frac{1}{[k]_q} \left\{ \mathrm{Li}_2[q] + \frac{[H_k^{(1)}]^2 + \zeta_k[2, q^2]}{2} - \frac{[H_k^{(1)}]}{[k]_q} q^k \right\}, \tag{2.18}$$

$$\sum_{n=1}^{\infty} \frac{[H_n^{(2)}]}{[n]_q[n+k]_q} q^n = \frac{1}{[k]_q} \left\{ \mathrm{Li}_3\left[q^2\right] + \mathrm{Li}_2[q]\left[H_{k-1}^{(1)}\right] - \sum_{i=1}^{k-1} \frac{[H_i^{(1)}]}{[i]_q^2} q^i \right\}. \tag{2.19}$$

Corollary 2.4 *For $m \in \mathbb{N}$, we have*

$$\begin{aligned} &\frac{1}{2} W_{0,m}^{(1,1)}[1,1;m+1] - (-1)^{m-1} W_{0,1}^{(0,1)}[1,1;m+1] \\ &\quad = W_{0,m+1}^{(1)}[1;m+2] + \sum_{j=1}^{m-1} (-1)^{j-1} \mathrm{Li}_{m-j+1}\left[q^{m-j}\right] W_{0,1}^{(0)}[1;j+1] \\ &\qquad - \frac{1}{2} W_{0,m}^{(2)}[2;m+1] - \mathrm{Li}_2[q]\mathrm{Li}_{m+1}\left[q^m\right] \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\frac{1}{2} \left\{ W_{0,m}^{(0,1,1)}\left[\{1\}_3; m+1\right] + W_{0,m}^{(0,2)}[1,2;m+1] \right\} \\ &\qquad - \frac{(-1)^{m-1}}{2} \left\{ W_{0,1}^{(0,1,1)}\left[\{1\}_3; m+1\right] + W_{0,1}^{(0,2)}[1,2;m+1] \right\} \\ &\quad = \left\{ W_{0,m+1}^{(0,1)}[1,1;m+2] - (-1)^{m-1} W_{0,2}^{(0,1)}[1,1;m+2] \right\} \\ &\qquad - \mathrm{Li}_2[q] \left\{ W_{0,m}^{(0)}[1;m+1] - (-1)^{m-1} W_{0,1}^{(0)}[1;m+1] \right\} \\ &\qquad + \sum_{j=1}^{m-1} (-1)^{j-1} W_{0,1}^{(0)}[1;j+1] W_{0,m-j}^{(0)}[1;m-j+1]. \end{aligned} \tag{2.21}$$

Proof Formula (2.20) shows that multiplying (2.18) by $\frac{q^{mk}}{[k]_q^m}$ and summing with respect to k, then using (2.16) yield

$$\begin{aligned} &\sum_{n,k=1}^{\infty} \frac{[H_n]}{[k]_q^m [n]_q [n+k]_q} q^n q^{mk} \\ &\quad = \sum_{k=1}^{\infty} \frac{q^{mk}}{[k]_q^{m+1}} \left\{ \mathrm{Li}_2[q] + \frac{[H_k^{(1)}]^2 + \zeta_k[2,q^2]}{2} - \frac{[H_k^{(1)}]}{[k]_q} q^k \right\} \\ &\quad = \mathrm{Li}_2[q]\mathrm{Li}_{m+1}\left[q^m\right] + \frac{1}{2} \left\{ W_{0,m}^{(1,1)}[1,1;m+1] + W_{0,m}^{(2)}[2;m+1] \right\} - W_{0,m+1}^{(1)}[1;m+2] \\ &\quad = \sum_{n=1}^{\infty} \frac{[H_n]}{[n]_q} q^n \sum_{k=1}^{\infty} \frac{q^{mk}}{[k]_q^m [n+k]_q} \\ &\quad = \sum_{n=1}^{\infty} \frac{[H_n]}{[n]_q} q^n \left\{ \sum_{j=1}^{m-1} \frac{(-1)^{j-1}}{[n]_q^j} \mathrm{Li}_{m-j+1}\left[q^{m-j}\right] + (-1)^{m-1} \frac{[H_n^{(1)}]}{[n]_q^m} \right\} \end{aligned}$$

$$= (-1)^{m-1} W_{0,1}^{(0,1)}[1,1;m+1] + \sum_{j=1}^{m-1} (-1)^{j-1} \mathrm{Li}_{m-j+1}\big[q^{m-j}\big] W_{0,1}^{(0)}[1;j+1]. \tag{2.22}$$

By a direct calculation, we obtain the result (2.20).

Similarly, to prove (2.21), multiplying (2.18) by $\frac{[H_k]}{[k]_q^m} q^{mk}$ and summing with respect to k, then applying the same arguments as in the proof of (2.20), we may easily deduce the desired result. □

Theorem 2.5 *For $x, y \in [-1,1]$ and positive integers m_1, m_2, k, r with $r < k$, the following identity holds*:

$$\begin{aligned}
&[k-r]_q \sum_{n=1}^{\infty} \frac{\zeta_n[m_1,x]\zeta_n[m_2,y] - \zeta_n[m_1+m_2,xy]}{[n+r]_q[n+k]_q} q^{n+r} \\
&= \sum_{i=1}^{k-r} \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n[m_1,x]y^n}{[n]_q^{m_2}[n+r+i-1]_q} + \frac{\zeta_n[m_2,y]x^n}{[n]_q^{m_1}[n+r+i-1]_q} \right. \\
&\quad \left. - \frac{2(xy)^n}{[n]_q^{m_1+m_2}[n+r+i-1]_q} \right\} q^{n+r+i-1},
\end{aligned} \tag{2.23}$$

where the partial sum $\zeta_n[m,x]$ is defined by

$$\zeta_n[m,x] := \sum_{j=1}^{n} \frac{x^j}{[j]_q^m}.$$

Proof To prove identity (2.23), we consider the generating function

$$F[x,y,z] := \sum_{n=1}^{\infty} \big\{ \zeta_n[m_1,x]\zeta_n[m_2,y] - \zeta_n[m_1+m_2,xy] \big\} z^{n-1}, \quad z \in (-1,1). \tag{2.24}$$

By the definition of $\zeta_n[m,x]$, we can rewrite (2.24) as follows:

$$\begin{aligned}
F[x,y,z] &= \sum_{n=1}^{\infty} \left\{ \begin{array}{c} (\zeta_n[m_1,x] + \frac{x^{n+1}}{[n+1]_q^{m_1}})(\zeta_n[m_2,y] + \frac{x^{n+1}}{[n+1]_q^{m_2}}) \\ - (\zeta_n[m_1+m_2,xy] + \frac{x^{n+1}y^{n+1}}{[n+1]_q^{m_1+m_2}}) \end{array} \right\} z^{n-1} \\
&= zF[x,y,z] + \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n[m_1,x]}{[n+1]_q^{m_2}} y^{n+1} + \frac{\zeta_n[m_2,y]}{[n+1]_q^{m_1}} x^{n+1} \right\} z^n \\
&= zF[x,y,z] + \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n[m_1,x]}{[n]_q^{m_2}} y^n + \frac{\zeta_n[m_2,y]}{[n]_q^{m_1}} x^n - 2\frac{x^n y^n}{[n]_q^{m_1+m_2}} \right\} z^{n-1}.
\end{aligned}$$

Hence, we obtain the formula

$$\begin{aligned}
&\sum_{n=1}^{\infty} \big\{ \zeta_n[m_1,x]\zeta_n[m_2,y] - \zeta_n[m_1+m_2,xy] \big\} z^{n-1} \\
&= \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n[m_1,x]}{[n]_q^{m_2}} y^n + \frac{\zeta_n[m_2,y]}{[n]_q^{m_1}} x^n - 2\frac{x^n y^n}{[n]_q^{m_1+m_2}} \right\} \frac{z^{n-1}}{1-z}.
\end{aligned} \tag{2.25}$$

Multiplying (2.25) by $z^r - z^k$ and q-integrating over $(0,q)$, then using the identity

$$\frac{q^{n+r}}{[n+r]_q} - \frac{q^{n+k}}{[n+k]_q} = [k-r]_q \frac{q^{n+r}}{[n+r]_q[n+k]_q},$$

we can deduce (2.23). The proof is completed. □

In fact, using the Cauchy product of power series, (2.25) can be rewritten as

$$\begin{aligned}&\sum_{n=1}^{\infty}\bigl\{\zeta_n[m_1,x]\zeta_n[m_2,y]-\zeta_n[m_1+m_2,xy]\bigr\}z^{n-1}\\&\quad=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\biggl\{\frac{\zeta_k[m_1,x]}{[k]_q^{m_2}}y^k+\frac{\zeta_n[m_2,y]}{[k]_q^{m_1}}x^k-2\frac{x^ky^k}{[k]_q^{m_1+m_2}}\biggr\}z^{n-1}.\end{aligned} \tag{2.26}$$

Thus, comparing the coefficients of z^{n-1} in (2.26), we obtain

$$\sum_{k=1}^{n}\biggl\{\frac{\zeta_k[m_1,x]}{[k]_q^{m_2}}y^k+\frac{\zeta_k[m_2,y]}{[k]_q^{m_1}}x^k\biggr\}=\zeta_n[m_1,x]\zeta_n[m_2,y]+\zeta_n[m_1+m_2,xy]. \tag{2.27}$$

Similarly, considering the following function

$$F[x,y]:=\sum_{n=1}^{\infty}\bigl\{\zeta_n^3[m,x]-3\zeta_n[m,x]\zeta_n\bigl[2m,x^2\bigr]+2\zeta_n\bigl[3m,x^3\bigr]\bigr\}y^{n-1},\quad y\in(-1,1),$$

by a similar argument as in the proof of (2.23), we deduce that

$$\begin{aligned}&\sum_{n=1}^{\infty}\bigl\{\zeta_n^3[m,x]-3\zeta_n[m,x]\zeta_n\bigl[2m,x^2\bigr]+2\zeta_n\bigl[3m,x^3\bigr]\bigr\}y^{n-1}\\&\quad=3\sum_{n=1}^{\infty}\biggl\{\frac{\zeta_n^2[m,x]}{[n]_q^m}x^n-\frac{\zeta_n[2m,x^2]}{[n]_q^m}x^n-2\frac{\zeta_n[m,x]}{[n]_q^{2m}}x^{2n}+2\frac{x^{3n}}{[n]_q^{3m}}\biggr\}\frac{y^{n-1}}{1-y}.\end{aligned} \tag{2.28}$$

By using the Cauchy product of power series again and then comparing the coefficients of y^{n-1}, we obtain

$$\begin{aligned}&3\sum_{k=1}^{n}\biggl\{\frac{\zeta_k^2[m,x]-\zeta_k[2m,x^2]}{[k]_q^m}x^k-2\frac{\zeta_k[m,x]}{[k]_q^{2m}}x^{2k}+2\frac{x^{3k}}{[k]_q^{3m}}\biggr\}\\&\quad=\zeta_n^3[m,x]-3\zeta_n[m,x]\zeta_n\bigl[2m,x^2\bigr]+2\zeta_n\bigl[3m,x^3\bigr].\end{aligned} \tag{2.29}$$

Combining (2.27) and (2.29), we have the result

$$\sum_{k=1}^{n}\frac{\zeta_k^2[m,x]+\zeta_k[2m,x^2]}{[k]_q^m}x^k=\frac{1}{3}\bigl\{\zeta_n^3[m,x]+3\zeta_n[m,x]\zeta_n\bigl[2m,x^2\bigr]+2\zeta_n\bigl[3m,x^3\bigr]\bigr\}. \tag{2.30}$$

Setting $x=q, m=1$ in the above equation, we obtain

$$\sum_{k=1}^{n}\frac{[H_k^{(1)}]^2+\zeta_k[2,q^2]}{[k]_q}q^k=\frac{1}{3}\bigl\{\bigl[H_n^{(1)}\bigr]^3+3\bigl[H_n^{(1)}\bigr]\zeta_n\bigl[2,q^2\bigr]+2\zeta_n\bigl[3,q^3\bigr]\bigr\}.$$

Theorem 2.6 *For positive integers m,r,k with $r<k$ and $x\in[-1,1]$, the following identity holds*:

$$\begin{aligned}&[k-r]_q\sum_{n=1}^{\infty}\frac{\zeta_n^3[m,x]-\zeta_n[3m,x^3]}{[n+r]_q[n+k]_q}q^{n+r}\\&=3\sum_{i=1}^{k-r}\sum_{n=1}^{\infty}\left\{\frac{\zeta_n^2[m,x]}{[n]_q^m[n+r+i-1]_q}x^n-\frac{\zeta_n[m,x]}{[n]_q^{2m}[n+r+i-1]_q}x^{2n}\right\}q^{n+r+i-1}.\end{aligned}\tag{2.31}$$

Proof Similar as in the proof of Theorem 2.5, we consider the power series

$$F[x,y]:=\sum_{n=1}^{\infty}\left\{\zeta_n^3[m,x]-\zeta_n\left[3m,x^3\right]\right\}y^{n-1},\quad y\in(-1,1)$$

and apply the same arguments as in the proof of Theorem 2.5. We may deduce formula (2.31). □

Taking $r=0$ and $m_1=m_2=x=y=1$ in (2.23), we get

$$\begin{aligned}&[k]_q\sum_{n=1}^{\infty}\frac{[H_n]^2-\zeta_n[2]}{[n]_q[n+k]_q}q^n\\&=2\sum_{i=1}^{k-1}q^i\sum_{n=1}^{\infty}\left\{\frac{[H_n]}{[n]_q[n+i]_q}q^n-\frac{q^n}{[n]_q^2[n+i]_q}\right\}+2\sum_{n=1}^{\infty}\left\{\frac{[H_n]}{[n]_q^2}q^n-\frac{q^n}{[n]_q^3}\right\}\\&=2\mathrm{Li}_3\left[q^2\right]+2\mathrm{Li}_2[q]\left[H_{k-1}^{(1)}\right]+\sum_{i=1}^{k-1}\frac{\left[H_i^{(1)}\right]^2+\zeta_i[2,q^2]}{[i]_q}q^i\\&\quad-2\sum_{i=1}^{k-1}\frac{[H_i^{(1)}]}{[i]_q^2}q^{2i}-2\sum_{i=1}^{k-1}q^i\sum_{n=1}^{\infty}\frac{q^n}{[n]_q^2[n+i]_q}.\end{aligned}\tag{2.32}$$

From (1.17) and (2.19), we have

$$\begin{aligned}[k]_q\sum_{n=1}^{\infty}\frac{\zeta_n[2]}{[n]_q[n+k]_q}q^n&=[k]_q\sum_{n=1}^{\infty}\frac{[H_n^{(2)}]}{[n]_q[n+k]_q}q^n+(1-q)[k]_q\sum_{n=1}^{\infty}\frac{[H_n]}{[n]_q[n+k]_q}q^n\\&=\mathrm{Li}_3\left[q^2\right]+\mathrm{Li}_2[q]\left[H_{k-1}^{(1)}\right]-\sum_{i=1}^{k-1}\frac{[H_i^{(1)}]}{[i]_q^2}q^i\\&\quad+(1-q)\left\{\mathrm{Li}_2[q]+\frac{\left[H_k^{(1)}\right]^2+\zeta_k[2,q^2]}{2}-\frac{[H_k^{(1)}]}{[k]_q}q^k\right\}.\end{aligned}\tag{2.33}$$

Substituting (2.33) into (2.32) results in

$$\begin{aligned}&[k]_q\sum_{n=1}^{\infty}\frac{[H_n]^2}{[n]_q[n+k]_q}q^n\\&=3\mathrm{Li}_3\left[q^2\right]+3\mathrm{Li}_2[q]\left[H_{k-1}^{(1)}\right]+\sum_{i=1}^{k-1}\frac{\left[H_i^{(1)}\right]^2+\zeta_i[2,q^2]}{[i]_q}q^i\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{i=1}^{k-1}\frac{[H_i^{(1)}]}{[i]_q^2}q^{2i}-\sum_{i=1}^{k-1}\frac{[H_i^{(1)}]}{[i]_q^2}q^i-2\sum_{i=1}^{k-1}q^i\sum_{n=1}^{\infty}\frac{q^n}{[n]_q^2[n+i]_q}\\
&+(1-q)\left\{\operatorname{Li}_2[q]+\frac{[H_k^{(1)}]^2+\zeta_k[2,q^2]}{2}-\frac{[H_k^{(1)}]}{[k]_q}q^k\right\}. \qquad (2.34)
\end{aligned}
$$

Similarly, putting $r=0, m=1, x=1$ in (2.31), we get

$$
\begin{aligned}
[k]_q\sum_{n=1}^{\infty}\frac{[H_n]^3-\zeta_n[3]}{[n]_q[n+k]_q}q^n &= 3\sum_{n=1}^{\infty}\left\{\frac{[H_n]^2}{[n]_q^2}q^n-\frac{[H_n]}{[n]_q^3}q^n\right\}\\
&\quad+3\sum_{i=1}^{k-1}q^i\sum_{n=1}^{\infty}\left\{\frac{[H_n]^2}{[n]_q[n+i]_q}-\frac{[H_n]}{[n]_q^2[n+i]_q}\right\}q^n. \qquad (2.35)
\end{aligned}
$$

From (2.10), (2.34) and (2.35), we know that the cubic q-sums

$$\sum_{n=1}^{\infty}\frac{[H_n]^3}{[n]_q[n+k]_q}q^n$$

are reducible to q-polylogarithms, q-harmonic numbers and q-rational series. Letting q tend to 1 in (2.34), we get the expression for quadratic sums

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+k)}$$

in terms of harmonic numbers and zeta values:

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+k)}=\frac{1}{k}\left\{\begin{array}{l}3\zeta(3)+\frac{H_k^3+3H_kH_k^{(2)}+2H_k^{(3)}}{3}\\ \quad-\frac{H_k^2+H_k^{(2)}}{k}-\sum_{i=1}^{k-1}\frac{H_i}{i^2}+\zeta(2)H_{k-1}\end{array}\right\}. \qquad (2.36)$$

Note that result (2.36) is given in Sofo's paper [40] and Xu's paper [1] with Zhang and Zhu. It should be emphasized that the papers [1, 2, 4, 5, 40] also contain many other types of results. For example, from [1], we have the result

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n(n+k)}=\frac{1}{k}\left\{\begin{array}{l}10\zeta(4)+\frac{H_k^4+8H_kH_k^{(3)}+6H_k^2H_k^{(2)}+3(H_k^{(2)})^2+6H_k^{(4)}}{4}\\ \quad-\frac{H_k^3+3H_kH_k^{(2)}+2H_k^{(3)}}{k}+4\zeta(3)H_{k-1}+\frac{1}{2}\zeta(2)H_{k-1}^{(2)}\\ \quad+\frac{3}{2}\zeta(2)H_{k-1}^2+\sum_{i=1}^{k-1}\frac{H_i}{i^3}-\frac{3}{2}\sum_{i=1}^{k-1}\frac{H_i^2+H_i^{(2)}}{i^2}-3\sum_{i=1}^{k-1}\frac{1}{i}\sum_{j=1}^{i}\frac{H_j}{j^2}\end{array}\right\}.$$

The result above can also be obtained by using (2.35) with letting $q\to 1$.

3 Some expressions of series involving q-harmonic numbers and q-binomial coefficients

In this section, we give some closed form sums of $W_{k,t}^{(\mathbf{s})}[\mathbf{m};p]$ through q-polylogarithms, q-harmonic numbers and other q-series.

From (2.8) and (2.9), we have the expansions

$$\begin{aligned} & W_{k,k+p-1}^{(s_1,\ldots,s_r)}[m_1,m_2,\ldots,m_r;p] \\ &= \sum_{i=1}^{k}(-1)^{i-1}[i]_q \begin{bmatrix} k \\ i \end{bmatrix}_q q^{-\frac{i}{2}(2k-1-i)} \\ &\quad \times \sum_{n=1}^{\infty}\frac{\zeta_n[m_1,q^{s_1}]\zeta_n[m_2,q^{s_2}]\cdots\zeta_n[m_r,q^{s_r}]}{[n]_q^p[n+i]_q}q^{pn} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & W_{k,k-1}^{(s_1,\ldots,s_r)}[m_1,m_2,\ldots,m_r;0] \\ &= [k]_q\sum_{i=1}^{k-1}(-1)^{i-1}[i]_q \begin{bmatrix} k-1 \\ i \end{bmatrix}_q q^{(2-k)-\frac{i}{2}(2k-3-i)} \\ &\quad \times \sum_{n=1}^{\infty}\frac{\zeta_n[m_1,q^{s_1}]\zeta_n[m_2,q^{s_2}]\cdots\zeta_n[m_r,q^{s_r}]}{[n+1]_q[n+1+i]_q}q^{n}. \end{aligned} \tag{3.2}$$

Hence, combining formulas (2.10), (2.17), (2.34), (3.1) and (3.2), by direct calculations, we can get the following three results.

Theorem 3.1 *For positive integers m, k and p,*

$$\begin{aligned} & W_{k,k+p-1}^{(m-1)}[m;p] \\ &= \sum_{i=1}^{k}(-1)^{i-1}[i]_q \begin{bmatrix} k \\ i \end{bmatrix}_q q^{-\frac{i}{2}(2k-1-i)}\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^{m-1}]}{[n]_q^p[n+i]_q}q^{pn} \\ &= \sum_{i=1}^{k}(-1)^{i-1}[i]_q \begin{bmatrix} k \\ i \end{bmatrix}_q q^{-\frac{i}{2}(2k-1-i)}\left\{\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{[i]_q^j}\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^{m-1}]}{[n]_q^{p+1-j}}q^{(p-j)n}\right\} \\ &\quad + \sum_{i=1}^{k}(-1)^{i-1}\begin{bmatrix} k \\ i \end{bmatrix}_q q^{-\frac{i}{2}(2k-1-i)} \\ &\quad \times \frac{(-1)^{p-1}}{[i]_q^{p-1}}\left\{\begin{array}{l} \mathrm{Li}_{m+1}[q^m] + (-1)^{m-1}\sum_{j=1}^{i-1}\frac{[H_j^{(1)}]}{[j]_q^m}q^j \\ \quad + \sum_{j=1}^{m-1}(-1)^{j-1}\mathrm{Li}_{m+1-j}[q^{m-j}][H_{i-1}^{(j)}] \end{array}\right\}. \end{aligned} \tag{3.3}$$

Theorem 3.2 *For positive integers m and k,*

$$\begin{aligned} & W_{k,k-1}^{(m-1)}[m;0] \\ &= [k]_q\sum_{i=1}^{k-1}(-1)^{i-1}[i]_q \begin{bmatrix} k-1 \\ i \end{bmatrix}_q q^{(2-k)-\frac{i}{2}(2k-3-i)}\sum_{n=1}^{\infty}\frac{\zeta_n[m,q^{m-1}]}{[n+1]_q[n+1+i]_q}q^{n} \\ &= [k]_q\sum_{i=1}^{k-1}(-1)^{i-1}\begin{bmatrix} k-1 \\ i \end{bmatrix}_q q^{(2-k)-\frac{i}{2}(2k-3-i)}\left\{\begin{array}{l} \sum_{j=1}^{m-1}(-1)^{j-1}\mathrm{Li}_{m-j+1}[q^{m-j}]\frac{[H_i^{(j)}]}{q} \\ \quad + (-1)^{m-1}\sum_{j=1}^{i}\frac{[H_j^{(1)}]}{[j]_q^m}q^{j-1} \end{array}\right\}. \end{aligned} \tag{3.4}$$

Theorem 3.3 *For positive integer k,*

$$\begin{aligned}
&W_{k,k}^{(0,0)}[1,1;1] \\
&= \sum_{i=1}^{k}(-1)^{i-1}[i]_q \begin{bmatrix} k \\ i \end{bmatrix}_q q^{-\frac{i}{2}(2k-1-i)} \sum_{n=1}^{\infty} \frac{[H_n]^2}{[n]_q[n+i]_q} q^n \\
&= \sum_{i=1}^{k}(-1)^{i-1} \begin{bmatrix} k \\ i \end{bmatrix}_q q^{-\frac{i}{2}(2k-1-i)} \left\{ \begin{array}{l} 3\mathrm{Li}_3[q^2] + 3\mathrm{Li}_2[q][H_{i-1}^{(1)}] \\ \quad + \sum_{j=1}^{i-1} \frac{[H_j^{(1)}]^2 + \zeta_j[2,q^2]}{[j]_q} q^j \\ \quad - 2\sum_{j=1}^{i-1} \frac{[H_j^{(1)}]}{[j]_q^2} q^{2j} - \sum_{j=1}^{i-1} \frac{[H_j^{(1)}]}{[j]_q^2} q^j \\ \quad - 2\sum_{j=1}^{i-1} q^j \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^2[n+j]_q} + (1-q)\mathrm{Li}_2[q] \\ \quad + (1-q)(\frac{[H_i^{(1)}]^2 + \zeta_i[2,q^2]}{2} - \frac{[H_i^{(1)}]}{[i]_q} q^i) \end{array} \right\}. \quad (3.5)
\end{aligned}$$

Taking $p = 1$ in (3.3), we have

$$\begin{aligned}
&W_{k,k}^{(m-1)}[m;1] \\
&= \sum_{i=1}^{k}(-1)^{i-1} \begin{bmatrix} k \\ i \end{bmatrix}_q q^{-\frac{i}{2}(2k-1-i)} \left\{ \begin{array}{l} \mathrm{Li}_{m+1}[q^m] + (-1)^{m-1} \sum_{j=1}^{i-1} \frac{[H_j^{(1)}]}{[j]_q^m} q^j \\ \quad + \sum_{j=1}^{m-1} (-1)^{j-1} \mathrm{Li}_{m+1-j}[q^{m-j}][H_{i-1}^{(j)}] \end{array} \right\}. \quad (3.6)
\end{aligned}$$

Moreover, from (2.10), (2.34), (2.35) and (3.1), we know that the cubic q-Euler type sum $W_{k,k}^{(0,0,0)}[\{1\}_3;1]$ is reducible to q-polylogarithms, q-harmonic numbers and q-rational series. Letting $q \to 1$ in (3.5) and (3.6), we obtain the well-known identities [1]

$$W_k(m;0) = k\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k-1}{i}\left\{\sum_{j=1}^{m-1}(-1)^{j-1}\zeta(m-j+1)H_i^{(j)} + (-1)^{m-1}\sum_{j=1}^{i}\frac{H_j}{j^m}\right\},$$

$$W_k(m;1) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left\{ \begin{array}{l} \zeta(m+1) + (-1)^{m-1}\sum_{i=1}^{r-1}\frac{H_i}{i^m} \\ \quad + \sum_{j=1}^{m-1}(-1)^{j-1}\zeta(m+1-j)H_{r-1}^{(j)} \end{array} \right\},$$

$$W_k(\{1\}_2;1) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left\{ \begin{array}{l} 3\zeta(3) + \frac{H_r^3 + 3H_rH_r^{(2)} + 2H_r^{(3)}}{3} \\ \quad - \frac{H_r^2 + H_r^{(2)}}{r} - \sum_{i=1}^{r-1}\frac{H_i}{i^2} + \zeta(2)H_{r-1} \end{array} \right\}.$$

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The author declares that they have no competing interests.

Authors' contributions
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