He et al. *Advances in Difference Equations* (2018) 2018:49
https://doi.org/10.1186/s13662-018-1465-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH
Open Access

Existence of positive solutions for a high order fractional differential equation integral boundary value problem with changing sign nonlinearity

Jiankun He, Mei Jia*, Xiping Liu and Hui Chen

*Correspondence: jiamei-usst@163.com
College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, China

Abstract

In this paper, we investigate the existence of positive solutions for a class of high order fractional differential equation integral boundary value problems with changing sign nonlinearity. By applying cone expansion and cone compression fixed point theorem, we have obtained and proved theorems related to the existence of positive solutions, which highlight the influences of the parameters in different ranges on the existence of positive solutions. Finally, we also give some examples to illustrate our main results.

Keywords: high order Riemann-Liouville fractional derivative; changing sign nonlinearity; Riemann-Stieltjes integral boundary value problems; fixed point theorem

1 Introduction

In this paper, we investigate the existence of positive solutions for a class of high order fractional differential equation integral boundary value problems with changing sign nonlinearity:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} u(1) = \int_0^1 D_{0^+}^{\beta} u(t)\, \mathrm{d}A(t), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, $n-1 < \alpha \le n$, $n \ge 3$, $0 < \beta \le 1$, $\lambda > 0$, and $\int_0^1 D_{0^+}^{\beta} u(t)\, \mathrm{d}A(t)$ denotes the Riemann-Stieltjes integrals with respect to A, in which $A(t)$ is a monotone increasing function and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ may change sign, $\mathbb{R}^+ = [0, +\infty)$.

In recent years, fractional differential equations arise in many engineering and scientific fields such as mathematical modeling of systems physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth. Researchers have reached a significance in ordinary and partial differential equations involving fractional derivatives, see [1–3] and the references therein. Since the boundary value problems play an important role in fractional differential equations theory, more attention has been paid and plenty of meaningful results have been obtained, see [4–26].

© The Author(s) 2018. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Generally speaking, in order to guarantee the existence of positive solutions of boundary value problems, the nonlinearity is usually nonnegative, see [6, 7, 9, 10, 16, 17, 20, 21, 24] and the references therein. And with the nonlinearity changing sign, it will bring much more difficulties to the study of the problem, and the papers studying this are relatively few, see [11, 12, 15, 17, 27] and the references therein. Compared with Riemann integral, Riemann-Stieltjes integral is more general. For example, the Riemann-Stieltjes integral $\int_0^1 u(t)\,\mathrm{d}A(t)$ will become the Riemann integral $\int_0^1 A'(t)u(t)\,\mathrm{d}t$ when A has a continuous derivative. Moreover, Riemann-Stieltjes integral will become more important when A is not differentiable or A is discontinuous. Any finite or infinite sum can be expressed as a Riemann-Stieltjes integral by a suitable choice of discontinuous A, see [28]. Since the nonlocal boundary value problems include the multi-point boundary value problem (A is a step function) and the Riemann integral boundary value problem (A has a continuous derivative), it has become a more general case where we study the boundary value problem with integral boundary conditions of Riemann-Stieltjes type. Many researchers have done a lot of work on this class of boundary value problems, see [12, 22, 25, 29, 30] and the references therein.

In [12], Zhang studied the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \\ D_{0^+}^{\beta} u(1) = \int_0^1 D_{0^+}^{\beta} u(t)\,\mathrm{d}A(t), \end{cases} \tag{1.2}$$

where $2 < \alpha \leq 3$, $0 < \beta \leq 1$, the nonlinearity $f(t,u)$ may change sign on some set. By means of the monotone iterative technique, the existence of nontrivial solutions or positive solutions is obtained. However, $f(t,u)$ is monotone increasing with respect to the fact that u in some interval is restricted, that is to say, $f(t,u)$ has local monotonicity.

In our work, it is not necessary to have monotonicity of the nonlinearity $f(t,u)$ since we derive the properties of the corresponding integral kernel function and get a more accurate inequality than the literature [12]. By a fixed point theorem, sufficient conditions for the existence of positive solutions of boundary value problem (1.1) are obtained. It is worth mentioning that the nonlinearity $f(t,u)$ does not need to be nonnegative and lower bounded. We also focus on studying the impact of the parameter λ on the existence of positive solutions, and we obtain sufficient conditions so that the problem has at least one positive solution when the parameter λ belongs to two intervals.

We say that f satisfies the L^1-Carathéodory conditions on $[0,1] \times \mathbb{R}^+$ if

(1) $f(\cdot,u)$ is measurable for all $u \in \mathbb{R}^+$;

(2) $f(t,\cdot)$ is continuous for a.e. $t \in [0,1]$;

(3) for each $r > 0$, there exists $\varphi_r \in L^1[0,1]$, $\varphi_r(t) \geq 0$, such that $f(t,u) \leq \varphi_r(t)$ for all $u \in [0,r]$ and a.e. $t \in [0,1]$.

Denote $\Delta = \int_0^1 t^{\alpha-\beta-1}\,\mathrm{d}A(t)$, in this paper, assume that $\int_0^1 t^{\alpha-\beta-1}\,\mathrm{d}A(t) < 1$ holds.

2 Preliminaries

The definitions of fractional integral and fractional derivative and the related lemmas can be found in [1–3].

Let the space $E = C[0,1]$, then E is a Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Definition 2.1 If $u \in E$, $D_{0+}^{\alpha} u \in L^1[0,1]$ and satisfies (1.1), u is called a solution of fractional boundary value problem (1.1). Furthermore, if $u(t) > 0, t \in (0,1)$, u is called a positive solution of fractional boundary value problem (1.1).

Lemma 2.1 (see[17]) *If $D_{0^+}^{\alpha} u \in L^1[0,1]$, then $I_{0^+}^{n-\alpha} u \in AC^n[0,1]$.*

Lemma 2.2 *For any $y \in L^1[0,1]$, the unique solution of the boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} u(1) = \int_0^1 D_{0^+}^{\beta} u(t)\, \mathrm{d}A(t) \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,s) y(s)\, \mathrm{d}s, \tag{2.2}$$

where

$$G(t,s) = H(t,s) + \frac{t^{\alpha-1}}{1-\Delta} \int_0^1 K(\tau,s)\, \mathrm{d}A(\tau), \tag{2.3}$$

$$H(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s < t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$K(\tau,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\tau(1-s))^{\alpha-\beta-1} - (\tau-s)^{\alpha-\beta-1}, & 0 \le s < \tau \le 1, \\ (\tau(1-s))^{\alpha-\beta-1}, & 0 \le \tau \le s \le 1. \end{cases} \tag{2.5}$$

Proof Suppose u is a solution to boundary value problem (2.1). Since $y \in L^1[0,1]$, then $D_{0^+}^{\alpha} u \in L^1[0,1]$. By Lemma 2.1, $I_{0^+}^{n-\alpha} u \in AC^n[0,1]$. Thus we have

$$I_{0^+}^{\alpha} \bigl(D_{0^+}^{\alpha} u\bigr)(t) = u(t) - \sum_{k=1}^{n} c_k t^{\alpha-k},$$

that is,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, \mathrm{d}s + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.6}$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$.

From the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we can get $c_n = c_{n-1} = c_{n-2} = \cdots = c_2 = 0$. Thus,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, \mathrm{d}s + c_1 t^{\alpha-1}, \tag{2.7}$$

and from the boundary condition $D_{0^+}^{\beta} u(1) = \int_0^1 D_{0^+}^{\beta} u(t)\, \mathrm{d}A(t)$, we can get

$$c_1 = \frac{1}{\Gamma(\alpha)(1-\Delta)} \biggl(\int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, \mathrm{d}s - \int_0^1 \biggl(\int_0^{\tau} (\tau-s)^{\alpha-\beta-1} y(s)\, \mathrm{d}s \biggr) \mathrm{d}A(\tau) \biggr).$$

Therefore, the unique solution of boundary value problem (2.1) is

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)}\biggl(\int_0^1 (1-s)^{\alpha-\beta-1}y(s)\,\mathrm{d}s \\
&\quad - \int_0^1\biggl(\int_0^\tau (\tau-s)^{\alpha-\beta-1}y(s)\,\mathrm{d}s\biggr)\,\mathrm{d}A(\tau)\biggr) \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}y(s)\,\mathrm{d}s \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)}\biggl(\int_0^1\biggl(\int_0^1 \tau^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}y(s)\,\mathrm{d}s\biggr)\,\mathrm{d}A(\tau) \\
&\quad - \int_0^1\biggl(\int_0^\tau (\tau-s)^{\alpha-\beta-1}y(s)\,\mathrm{d}s\biggr)\,\mathrm{d}A(\tau)\biggr) \\
&= \int_0^1\biggl(H(t,s) + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)}\int_0^1 K(\tau,s)\,\mathrm{d}A(\tau)\biggr)y(s)\,\mathrm{d}s \\
&= \int_0^1 G(t,s)y(s)\,\mathrm{d}s,
\end{aligned}$$

where $G(t,s)$, $H(t,s)$, and $K(t,s)$ are defined by (2.3), (2.4), and (2.5), respectively.

On the other hand, if u satisfies (2.2), then u will also satisfy (2.6). Thus, we have

$$\bigl(D_{0^+}^\alpha u\bigr)(t) = \biggl(\frac{\mathrm{d}}{\mathrm{d}t}\biggr)^n I_{0^+}^{n-\alpha}u(t) = -\biggl(\frac{\mathrm{d}}{\mathrm{d}t}\biggr)^n \bigl(I_{0^+}^{n-\alpha}I_{0^+}^{\alpha}y\bigr)(t) = -\biggl(\frac{\mathrm{d}}{\mathrm{d}t}\biggr)^n I_{0^+}^{n}y(t) = -y(t),$$

which implies the equation of boundary value problem (2.1) is satisfied.

We can easily show that u satisfies the boundary condition of boundary value problem (2.1). □

Lemma 2.3 *The function $H(t,s)$, which is defined by* (2.4), *satisfies the following conditions*:

(1) $H(t,s) \geq 0$ *is continuous for $t,s \in [0,1]$ and $H(t,s) > 0$ for $t,s \in (0,1)$*;

(2) *For $t,s \in [0,1]$, we have*

$$H(t,s) \leq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1};$$

(3) *For $t,s \in [0,1]$, we have*

$$\frac{\beta}{\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-\beta-1} \leq H(t,s) \leq \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-\beta-1}.$$

Proof (1) By (2.4), it is clear that $H(t,s)$ is continuous on $[0,1]\times[0,1]$ and $H(t,s)\geq 0$ when $0\leq t\leq s\leq 1$.

For $0\leq s\leq t\leq 1$, we have

$$\begin{aligned}
t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} &\geq t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-s)^{\alpha-1} \\
&= t^{\alpha-1}(1-s)^{\alpha-\beta-1}\bigl(1-(1-s)^{\beta}\bigr) \geq 0.
\end{aligned}$$

So, we have $H(t,s) \geq 0$ for any $t,s \in [0,1]$. Similarly, for $t,s \in (0,1)$, we have $H(t,s) > 0$.

(2) It follows from (2.4) that $H(t,s) \leq \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-\beta-1}$.

(3) Since $n-1 < \alpha \leq n$, $n \geq 3$, it is easy to show that $H(t,s) \leq \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-\beta-1}$ for $0 \leq t \leq s \leq 1$.

For $0 \leq s \leq t \leq 1$, let

$$h(t,s) = \frac{1}{\Gamma(\alpha)}\bigl(t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\bigr) - \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-\beta-1}.$$

We have

$$h(1,s) \leq 0, \qquad h(s,s) \leq 0,$$

and

$$\frac{\partial h(t,s)}{\partial t} = \frac{\alpha-1}{\Gamma(\alpha)}\bigl(t^{\alpha-2}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-2}\bigr).$$

If there exists $t_0 \in (s,1)$ such that $\frac{\partial h(t_0,s)}{\partial t} = 0$, then

$${t_0}^{\alpha-2}(1-s)^{\alpha-\beta-1} = (t_0-s)^{\alpha-2},$$

which implies that

$$\begin{aligned} h(t_0,s) &= \frac{1}{\Gamma(\alpha)}\bigl({t_0}^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t_0-s)^{\alpha-1} - s(1-s)^{\alpha-\beta-1}\bigr) \\ &= \frac{1}{\Gamma(\alpha)}\bigl({t_0}^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t_0-s)^{\alpha-2}(t_0-s) - s(1-s)^{\alpha-\beta-1}\bigr) \\ &= \frac{1}{\Gamma(\alpha)}\bigl({t_0}^{\alpha-1}(1-s)^{\alpha-\beta-1} - {t_0}^{\alpha-2}(1-s)^{\alpha-\beta-1}(t_0-s) - s(1-s)^{\alpha-\beta-1}\bigr) \\ &= \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1}\bigl({t_0}^{\alpha-1} - {t_0}^{\alpha-2}(t_0-s) - s\bigr) \\ &= \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1} s\bigl({t_0}^{\alpha-2} - 1\bigr) \leq 0. \end{aligned}$$

Hence, we have

$$h(t,s) = \frac{1}{\Gamma(\alpha)}\bigl(t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\bigr) - \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-\beta-1} \leq 0,$$

which implies that $H(t,s) \leq \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-\beta-1}$ for $0 \leq s < t \leq 1$.

Therefore, we can get that

$$H(t,s) \leq \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-\beta-1} \quad \text{for any } t,s \in [0,1].$$

On the other hand, for $0 \leq s < t \leq 1$, we can show

$$H(t,s) = \frac{1}{\Gamma(\alpha)}\bigl(t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\bigr)$$

$$\begin{aligned}
&\geq \frac{1}{\Gamma(\alpha)}\bigl(t^{\alpha-1}(1-s)^{\alpha-\beta-1}-t^{\alpha-1}(1-s)^{\alpha-1}\bigr)\\
&=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1}\bigl(1-(1-s)^{\beta}\bigr)\\
&\geq \frac{\beta}{\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-\beta-1}.
\end{aligned}$$

For $0\leq t\leq s\leq 1$, we have

$$H(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1}\geq \frac{\beta}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1}s.$$

Hence, (3) holds. □

Lemma 2.4 *The function $K(\tau,s)$, which is defined by* (2.5), *satisfies the following conditions*:

(1) *$K(\tau,s)\geq 0$ is continuous for $\tau,s\in[0,1]$ and $K(\tau,s)>0$ for $\tau,s\in(0,1)$;*

(2) *For $\tau\in[0,1], s\in[0,1]$, we have*

$$K(\tau,s)\leq \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-\beta-1};$$

(3) *For $\tau\in(0,1], s\in[0,1]$, we have*

$$\begin{aligned}
\frac{\min\{\alpha-\beta-1,1\}}{\Gamma(\alpha)}\tau^{\alpha-\beta-1}(1-\tau)s(1-s)^{\alpha-\beta-1}&\leq K(\tau,s)\\
&\leq \frac{\max\{\alpha-\beta-1,1\}}{\Gamma(\alpha)}\tau^{\alpha-\beta-2}s(1-s)^{\alpha-\beta-1}.
\end{aligned}$$

Proof By the expression of $K(\tau,s)$, it is easy to check that (1) and (2) hold.

(3) Similar to the proof of Lemma 2.8 in [12], we can prove

$$K(\tau,s)\geq \frac{\min\{\alpha-\beta-1,1\}}{\Gamma(\alpha)}\tau^{\alpha-\beta-1}(1-\tau)s(1-s)^{\alpha-\beta-1}.$$

In the following we will prove

$$K(\tau,s)\leq \frac{\max\{\alpha-\beta-1,1\}}{\Gamma(\alpha)}\tau^{\alpha-\beta-2}s(1-s)^{\alpha-\beta-1}.$$

We divide the proof into the following two cases for $\alpha-\beta-1\in(0,+\infty)$.

Case 1: $0<\alpha-\beta-1\leq 1$.

If $0\leq s<\tau\leq 1$, then

$$\begin{aligned}
K(\tau,s)&=\frac{1}{\Gamma(\alpha)}\bigl(\tau^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-(\tau-s)^{\alpha-\beta-1}\bigr)\\
&=\frac{\tau^{\alpha-\beta-2}(1-s)^{\alpha-\beta-2}}{\Gamma(\alpha)}\left(\tau(1-s)-(\tau-s)\frac{(1-\frac{s}{\tau})^{\alpha-\beta-2}}{(1-s)^{\alpha-\beta-2}}\right)\\
&\leq \frac{\tau^{\alpha-\beta-2}(1-s)^{\alpha-\beta-2}}{\Gamma(\alpha)}\bigl(\tau(1-s)-(\tau-s)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\tau^{\alpha-\beta-2}(1-s)^{\alpha-\beta-2}}{\Gamma(\alpha)} s(1-\tau) \\
&\leq \frac{1}{\Gamma(\alpha)} \tau^{\alpha-\beta-2} s(1-s)^{\alpha-\beta-1}.
\end{aligned}
$$

If $0 \leq \tau \leq s \leq 1$, then

$$
K(\tau,s) = \frac{1}{\Gamma(\alpha)} \tau^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} \leq \frac{1}{\Gamma(\alpha)} \tau^{\alpha-\beta-2} s(1-s)^{\alpha-\beta-1}.
$$

Case 2: $1 < \alpha - \beta - 1$.
If $0 \leq s < \tau \leq 1$, we have

$$
\begin{aligned}
K(\tau,s) &= \frac{1}{\Gamma(\alpha)} \left(\tau^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (\tau-s)^{\alpha-\beta-1}\right) \\
&\leq \frac{\alpha-\beta-1}{\Gamma(\alpha)} \big(\tau(1-s)\big)^{\alpha-\beta-2} \big(\tau(1-s) - (\tau-s)\big) \\
&\leq \frac{\alpha-\beta-1}{\Gamma(\alpha)} \big(\tau(1-s)\big)^{\alpha-\beta-2} s(1-\tau) \\
&\leq \frac{\alpha-\beta-1}{\Gamma(\alpha)} \tau^{\alpha-\beta-2} s(1-s)^{\alpha-\beta-1}.
\end{aligned}
$$

If $0 \leq \tau \leq s \leq 1$, we have

$$
K(\tau,s) = \frac{1}{\Gamma(\alpha)} \tau^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} \leq \frac{\alpha-\beta-1}{\Gamma(\alpha)} \tau^{\alpha-\beta-2} s(1-s)^{\alpha-\beta-1}.
$$

Therefore, we can get that (3) holds. □

Denote

$$
\begin{aligned}
M_1 &= \frac{1}{\Gamma(\alpha)} \left(\beta + \frac{\min\{\alpha-\beta-1, 1\} \int_0^1 \tau^{\alpha-\beta-1}(1-\tau)\,\mathrm{d}A(\tau)}{1-\Delta}\right), \\
M_2 &= \frac{1}{\Gamma(\alpha)} \left(1 + \frac{\max\{\alpha-\beta-1, 1\} \int_0^1 \tau^{\alpha-\beta-2}\,\mathrm{d}A(\tau)}{1-\Delta}\right), \\
M_3 &= \frac{1}{\Gamma(\alpha)} \left(1 + \frac{A(1)-A(0)}{1-\Delta}\right).
\end{aligned}
$$

Lemma 2.5 *The function $G(t,s)$, which is defined by* (2.3), *satisfies the following conditions*:

(1) $G(t,s) \geq 0$ *is continuous for* $t,s \in [0,1]$ *and* $G(t,s) > 0$ *for* $t,s \in (0,1)$;

(2) *For* $t,s \in [0,1]$, *we have*

$$
G(t,s) \leq M_3 t^{\alpha-1};
$$

(3) *For* $t,s \in [0,1]$, *we have*

$$
M_1 t^{\alpha-1} s(1-s)^{\alpha-\beta-1} \leq G(t,s) \leq M_2 s(1-s)^{\alpha-\beta-1} \leq M_2.
$$

Proof (1) From Lemmas 2.3 and 2.4, we obtain that $G(t,s) \geq 0$ is continuous for $t,s \in [0,1]$ and $G(t,s) > 0$ for $t,s \in (0,1)$.

(2) For any $t,s \in [0,1]$, from (2) of Lemma 2.3 and (2) of Lemma 2.4, we have

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{t^{\alpha-1}}{1-\Delta}\int_0^1 K(\tau,s)\,\mathrm{d}A(\tau) \\ &\leq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1} + \frac{t^{\alpha-1}(A(1)-A(0))}{\Gamma(\alpha)(1-\Delta)}(1-s)^{\alpha-\beta-1} \\ &\leq \frac{1}{\Gamma(\alpha)}\left(1+\frac{A(1)-A(0)}{1-\Delta}\right)t^{\alpha-1}(1-s)^{\alpha-\beta-1} \\ &\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(1+\frac{A(1)-A(0)}{1-\Delta}\right) \\ &= M_3 t^{\alpha-1}. \end{aligned}$$

(3) For any $t,s \in [0,1]$, from (3) of Lemma 2.3 and (3) of Lemma 2.4, we have

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{t^{\alpha-1}}{1-\Delta}\int_0^1 K(\tau,s)\,\mathrm{d}A(\tau) \\ &\leq \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-\beta-1} + \frac{t^{\alpha-1}\max\{\alpha-\beta-1,1\}\int_0^1 \tau^{\alpha-\beta-2}\,\mathrm{d}A(\tau)}{\Gamma(\alpha)(1-\Delta)}s(1-s)^{\alpha-\beta-1} \\ &\leq \frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\left(1+\frac{\max\{\alpha-\beta-1,1\}\int_0^1 \tau^{\alpha-\beta-2}\,\mathrm{d}A(\tau)}{1-\Delta}\right) \\ &= M_2 s(1-s)^{\alpha-\beta-1} \\ &\leq M_2. \end{aligned}$$

On the other hand, for any $t,s \in [0,1]$, we have

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{t^{\alpha-1}}{1-\Delta}\int_0^1 K(\tau,s)\,\mathrm{d}A(\tau) \\ &\geq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}s\beta(1-s)^{\alpha-\beta-1} \\ &\quad + \frac{t^{\alpha-1}\min\{\alpha-\beta-1,1\}\int_0^1 \tau^{\alpha-\beta-1}(1-\tau)\,\mathrm{d}A(\tau)}{\Gamma(\alpha)(1-\Delta)}s(1-s)^{\alpha-\beta-1} \\ &= \frac{t^{\alpha-1}s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\left(\beta+\frac{\min\{\alpha-\beta-1,1\}\int_0^1 \tau^{\alpha-\beta-1}(1-\tau)\,\mathrm{d}A(\tau)}{1-\Delta}\right) \\ &= M_1 t^{\alpha-1}s(1-s)^{\alpha-\beta-1}. \end{aligned}$$

□

3 Existence of positive solutions of the boundary value problem

We make the following assumption throughout this paper.

(H1) There exists a nonnegative function $p \in L^1[0,1]$ and $\int_0^1 p(s)\,\mathrm{d}s > 0$ such that

$$f(t,u) \geq -p(t);$$

(H2) There exists $[\theta_1, \theta_2] \subset (0,1)$ such that

$$\liminf_{u\to+\infty} \inf_{t\in[\theta_1,\theta_2]} \frac{f(t,u)}{u} = +\infty;$$

(H3) There exists $[\theta_3, \theta_4] \subset (0,1)$ such that

$$\liminf_{u\to+\infty} \inf_{t\in[\theta_3,\theta_4]} f(t,u) > \frac{2M_2M_3\int_0^1 p(s)\,\mathrm{d}s}{M_1^2\theta_3^{\alpha-1}\int_{\theta_3}^{\theta_4} s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s},$$

and

$$\limsup_{u\to+\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u} = 0.$$

Define $P = \{u \in E : u(t) \geq \frac{M_1 t^{\alpha-1}}{M_2}\|u\|, t \in [0,1]\}$. Obviously, $P \subset E$ is a cone of E.

We denote $B_r = \{u \in E : \|u\| < r\}$, $P_r = P \cap B_r$, and $\partial P_r = P \cap \partial B_r$.

Let $\omega(t)$ be the solution of the boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda p(t) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} u(1) = \int_0^1 D_{0^+}^{\beta} u(t)\,\mathrm{d}A(t). \end{cases} \tag{3.1}$$

By Lemma 2.2, we have

$$\omega(t) = \lambda \int_0^1 G(t,s)p(s)\,\mathrm{d}s$$

is the unique solution of boundary value problem (3.1).

Denote $[u(t) - \omega(t)]^+ = \max\{u(t) - \omega(t), 0\}$, let

$$f^*\bigl(t, u(t)\bigr) = f\bigl(t, \bigl[u(t) - \omega(t)\bigr]^+\bigr) + p(t), \quad t \in (0,1).$$

Next we will consider the following boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda f^*(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} u(1) = \int_0^1 D_{0^+}^{\beta} u(t)\,\mathrm{d}A(t). \end{cases} \tag{3.2}$$

We define an operator $T : P \to E$ by

$$Tu(t) = \lambda \int_0^1 G(t,s) f^*\bigl(s, u(s)\bigr)\,\mathrm{d}s.$$

Lemma 3.1 *Assume* (H1) *holds. Then u^* is a positive solution of boundary value problem* (1.1) *if and only if $u = u^* + \omega$ is a positive solution of boundary value problem* (3.2) *and $u(t) \geq \omega(t)$ for $t \in [0,1]$.*

Proof If u^* is a positive solution of boundary value problem (1.1), then

$$\begin{aligned} D_{0+}^{\alpha}\big(u^*(t)+\omega(t)\big) &= D_{0+}^{\alpha}u^*(t)+D_{0+}^{\alpha}\omega(t) \\ &= -\lambda f\big(t,u^*(t)\big)-\lambda p(t) \\ &= -\lambda\big(f\big(t,u^*(t)\big)+p(t)\big) \\ &= -\lambda f^*\big(t,\big(u^*(t)+\omega(t)\big)\big), \end{aligned}$$

which implies that

$$D_{0+}^{\alpha}u(t)=-\lambda f^*\big(t,u(t)\big).$$

Since u^* is a positive solution, then $u(t)\geq\omega(t)\geq 0$ for $t\in[0,1]$. It is easy to show that $u(t)$ satisfies the boundary conditions. Therefore, $u(t)$ is a positive solution of boundary value problem (3.2).

On the other hand, if $u=u^*+\omega$ is a positive solution of boundary value problem (3.2) and $u(t)\geq\omega(t)$ for $t\in[0,1]$, we can easily prove that u^* is a positive solution of boundary value problem (1.1). □

Lemma 3.2 *Assume that* (H1) *holds and f satisfies the L^1-Carathéodory conditions, then $T:P\to P$ is completely continuous.*

Proof (1) We can show that $T:P\to P$.

According to Lemmas 2.2 and 2.5, we have $Tu(t)\geq 0$ on $[0,1]$ for $u\in P$ and

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G(t,s)f^*\big(s,u(s)\big)\,\mathrm{d}s \\ &\geq \lambda t^{\alpha-1}M_1\int_0^1 s(1-s)^{\alpha-\beta-1}f^*\big(s,u(s)\big)\,\mathrm{d}s. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G(t,s)f^*\big(s,u(s)\big)\,\mathrm{d}s \\ &\leq \lambda M_2\int_0^1 s(1-s)^{\alpha-\beta-1}f^*\big(s,u(s)\big)\,\mathrm{d}s. \end{aligned}$$

Then $Tu(t)\geq\frac{M_1t^{\alpha-1}}{M_2}\|Tu\|$, which implies $T:P\to P$.

(2) We show that T is a continuous operator.

Let $\{u_n\}\subset P, u_0\in P$, and $\|u_n-u_0\|\to 0$ as $n\to\infty$, there exists a constant $r_0>0$ such that $\|u_n\|\leq r_0$ and $\|u_0\|\leq r_0$. Therefore, for a.e. $s\in[0,1]$, we have

$$\big|f^*\big(s,u_n(s)\big)-f^*\big(s,u_0(s)\big)\big|\to 0,\quad n\to\infty,$$

and

$$\big|f^*\big(s,u_n(s)\big)-f^*\big(s,u_0(s)\big)\big|\leq 2\varphi_{r_0}(s).$$

By the Lebesgue dominated convergence theorem, we can get

$$\lim_{n\to\infty} \|Tu_n - Tu_0\| = 0, \quad n \to \infty.$$

Hence, $T : P \to P$ is continuous.

(3) $T : P \to P$ is relatively compact.

Let $\Omega \subset P$ be any bounded set, then there exists a constant $l > 0$ such that $\|u\| \leq l$ for each $u \in \Omega$, we have

$$\bigl|Tu(t)\bigr| = \left|\lambda \int_0^1 G(t,s)f^*\bigl(s,u(s)\bigr)\,\mathrm{d}s\right| \leq \lambda M_2 \int_0^1 \bigl(\varphi_l(s) + p(s)\bigr)\,\mathrm{d}s < +\infty,$$

which implies that $\|Tu\| \leq \lambda M_2 \int_0^1 (\varphi_l(s) + p(s))\,\mathrm{d}s$. Hence, $T(\Omega)$ is uniformly bounded.

In addition, for any given $u \in \Omega$, because $G(t,s)$ is continuous for $(t,s) \in [0,1] \times [0,1]$, then it must be uniformly continuous. So, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that, for any $t_1, t_2, s \in [0,1]$, as $|t_1 - t_2| < \delta$, we can get

$$\bigl|G(t_1,s) - G(t_2,s)\bigr| < \frac{\varepsilon}{\lambda \int_0^1 (\varphi_l(s) + p(s))\,\mathrm{d}s + 1}.$$

Then

$$\bigl|Tu(t_1) - Tu(t_2)\bigr| \leq \lambda \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr|\bigl|f^*\bigl(s,u(s)\bigr)\bigr|\,\mathrm{d}s < \varepsilon.$$

Thus, we prove $T(\Omega)$ is equicontinuous.

According to the Arzela-Ascoli theorem, we conclude that $T(\Omega)$ is relatively compact.

Therefore, $T : P \to P$ is completely continuous. □

Theorem 3.3 *Assume that* (H1) *and* (H2) *hold. Then there exists a constant* $\lambda^* > 0$ *such that boundary value problem* (1.1) *has at least one positive solution for any* $\lambda \in (0, \lambda^*)$.

Proof By assumption (H2), there exists a constant $N > 0$ such that, for any $t \in [\theta_1, \theta_2]$ and $u > N$, we have $f(t,u) > \tilde{L}u$, where $\tilde{L} = \frac{2M_2}{\lambda M_1^2 \theta_1^{2(\alpha-1)} \int_{\theta_1}^{\theta_2} s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s}$.

For any $\lambda > 0$, let $r_1 > \max\{\frac{2\lambda M_2 M_3}{M_1} \int_0^1 p(s)\,\mathrm{d}s, \frac{2M_2 N}{M_1 \theta_1^{\alpha-1}}\}$.

Then, for any $u \in \partial P_{r_1}$, $t \in [0,1]$, we have

$$\begin{aligned}
u(t) - \omega(t) &\geq \frac{M_1 t^{\alpha-1}}{M_2} r_1 - \lambda \int_0^1 G(t,s)p(s)\,\mathrm{d}s \\
&\geq \frac{M_1 t^{\alpha-1}}{M_2} r_1 - \lambda M_3 t^{\alpha-1} \int_0^1 p(s)\,\mathrm{d}s \\
&\geq t^{\alpha-1}\left(\frac{M_1}{M_2} r_1 - \lambda M_3 \int_0^1 p(s)\,\mathrm{d}s\right) \\
&\geq t^{\alpha-1}\left(\frac{M_1}{M_2} r_1 - \frac{M_1}{2M_2} r_1\right) \\
&= \frac{M_1}{2M_2} r_1 t^{\alpha-1} \geq 0.
\end{aligned}$$

Thus, for any $t \in [\theta_1, \theta_2]$, we have

$$u(t) - \omega(t) \geq \frac{M_1 \theta_1^{\alpha-1}}{2M_2} r_1 \geq N.$$

Hence, we get

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s) f^*\bigl(s, u(s)\bigr)\,\mathrm{d}s \\
&= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)\bigl(f\bigl(s, \bigl[u(s) - \omega(s)\bigr]^+\bigr) + p(s)\bigr)\,\mathrm{d}s \\
&= \max_{t\in[0,1]} \lambda \int_0^1 G(t,s)\bigl(f\bigl(s, u(s) - \omega(s)\bigr) + p(s)\bigr)\,\mathrm{d}s \\
&\geq \max_{t\in[\theta_1,\theta_2]} \lambda \int_{\theta_1}^{\theta_2} G(t,s) f\bigl(s, u(s) - \omega(s)\bigr)\,\mathrm{d}s \\
&\geq \max_{t\in[\theta_1,\theta_2]} \lambda \tilde{L} \int_{\theta_1}^{\theta_2} G(t,s)\bigl(u(s) - \omega(s)\bigr)\,\mathrm{d}s \\
&\geq \lambda \tilde{L} \frac{M_1 \theta_1^{\alpha-1}}{2M_2} r_1 \int_{\theta_1}^{\theta_2} M_1 \theta_1^{\alpha-1} s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s \\
&= \frac{\lambda M_1^2 \theta_1^{2(\alpha-1)} \int_{\theta_1}^{\theta_2} s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s}{2M_2} \tilde{L} r_1 = r_1.
\end{aligned}$$

Thus

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \partial P_{r_1}.$$

Take $0 < r_2 < r_1$. Choose $\lambda^* = \min\{\frac{M_1 r_2}{M_2 M_3 \int_0^1 p(s)\,\mathrm{d}s}, \frac{r_2}{M_2 \int_0^1 (\varphi_{r_2}(s) + p(s))\,\mathrm{d}s}\}$. For any $\lambda \in (0, \lambda^*)$, $u \in \partial P_{r_2}$, $u(t) \geq \frac{M_1 t^{\alpha-1}}{M_2} r_2$ and

$$\omega(t) = \lambda \int_0^1 G(t,s) p(s)\,\mathrm{d}s \leq \lambda M_3 t^{\alpha-1} \int_0^1 p(s)\,\mathrm{d}s.$$

We have

$$0 \leq \frac{M_1 t^{\alpha-1}}{M_2} r_2 - \lambda M_3 t^{\alpha-1} \int_0^1 p(s)\,\mathrm{d}s \leq u(t) - \omega(t) \leq r_2.$$

Therefore,

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s) f^*\bigl(s, u(s)\bigr)\,\mathrm{d}s \\
&= \lambda \int_0^1 G(t,s)\bigl(f\bigl(s, u(s) - \omega(s)\bigr) + p(s)\bigr)\,\mathrm{d}s \\
&\leq \lambda M_2 \int_0^1 \bigl(\varphi_{r_2}(s) + p(s)\bigr)\,\mathrm{d}s \\
&< \lambda^* M_2 \int_0^1 \bigl(\varphi_{r_2}(s) + p(s)\bigr)\,\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}&\le \frac{r_2}{M_2\int_0^1(\varphi_{r_2}(s)+p(s))\,\mathrm{d}s}M_2\int_0^1\bigl(\varphi_{r_2}(s)+p(s)\bigr)\,\mathrm{d}s\\&=r_2.\end{aligned}$$

Thus,

$$\|Tu\|\le\|u\|,\quad \text{for } u\in\partial P_{r_2}.$$

By Lemma 3.2, we know that T is a completely continuous operator.

According to cone expansion and cone compression fixed point theorem (see [31]), we can obtain that T has a fixed point u such that $r_1\le\|u\|\le r_2$ in $P\cap(\bar{B}_{r_2}\setminus B_{r_1})$.

Let $u^*=u-\omega$. For $t\in(0,1)$, we have

$$u^*(t)=\frac{M_1t^{\alpha-1}}{M_2}\|u\|-\lambda M_3t^{\alpha-1}\int_0^1p(s)\,\mathrm{d}s>t^{\alpha-1}\biggl(\frac{M_1}{M_2}r_2-\lambda^*M_3\int_0^1p(s)\,\mathrm{d}s\biggr)>0.$$

Then u^* is the positive solution of (1.1).

Hence, there exists a constant $\lambda^*>0$ such that boundary value problem (1.1) has at least one positive solution for any $\lambda\in(0,\lambda^*)$. □

Theorem 3.4 *Assume that* (H1) *and* (H3) *hold. Then there exists a constant* $\bar{\lambda}^*>0$ *such that boundary value problem* (1.1) *has at least one positive solution for any* $\lambda\in(\bar{\lambda}^*,+\infty)$.

Proof By the first limit of (H3), there exists a constant $N>0$ such that, for any $t\in[\theta_3,\theta_4]$, $u>N$, we have

$$f(t,u)\ge\frac{2M_2M_3\int_0^1p(s)\,\mathrm{d}s}{M_1^2\theta_3^{\alpha-1}\int_{\theta_3}^{\theta_4}s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s}.$$

Choose $\bar{\lambda}^*=\frac{N\theta_3^{1-\alpha}}{M_3\int_0^1p(s)\,\mathrm{d}s}$, $R_1=\frac{2\lambda M_2M_3\int_0^1p(s)\,\mathrm{d}s}{M_1}$.

Then, for any $\lambda\in(\bar{\lambda}^*,+\infty)$, $u\in\partial P_{R_1}$, we have

$$\begin{aligned}u(t)-\omega(t)&\ge\frac{M_1t^{\alpha-1}}{M_2}R_1-\lambda M_3t^{\alpha-1}\int_0^1p(s)\,\mathrm{d}s\\&=\lambda M_3t^{\alpha-1}\int_0^1p(s)\,\mathrm{d}s.\end{aligned}$$

Hence, for any $t\in[\theta_3,\theta_4]$, $u(t)-\omega(t)\ge Nt^{\alpha-1}\theta_3^{1-\alpha}\ge N$, we get

$$\begin{aligned}Tu(t)&=\lambda\int_0^1G(t,s)f^*\bigl(s,u(s)\bigr)\,\mathrm{d}s\\&\ge\lambda\int_{\theta_3}^{\theta_4}G(t,s)f\bigl(s,u(s)-\omega(s)\bigr)\,\mathrm{d}s\\&\ge\frac{2\lambda M_2M_3\int_0^1p(s)\,\mathrm{d}s}{M_1^2\theta_3^{\alpha-1}\int_{\theta_3}^{\theta_4}s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s}\int_{\theta_3}^{\theta_4}G(t,s)\,\mathrm{d}s\end{aligned}$$

$$\geq \frac{2\lambda M_2 M_3 \int_0^1 p(s)\,\mathrm{d}s}{M_1^2 \theta_3^{\alpha-1} \int_{\theta_3}^{\theta_4} s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s} \int_{\theta_3}^{\theta_4} t^{\alpha-1} M_1 s(1-s)^{\alpha-\beta-1}\,\mathrm{d}s$$
$$\geq \frac{2\lambda M_2 M_3 \int_0^1 p(s)\,\mathrm{d}s}{M_1} = R_1.$$

Thus

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \partial P_{R_1}.$$

On the other hand, let $\varepsilon = \frac{1}{3\lambda M_2}$, by (H3), $\limsup_{u\to+\infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u} = 0$, there exists a constant $M > R_1$ such that

$$f(t,u) \leq \varepsilon u \quad \text{for } t \in [0,1] \text{ and } u > M.$$

Since f satisfies the L^1-Carathéodory conditions on $[0,1] \times [0,+\infty)$, we have

$$f(t,u) \leq \varphi_M(t) \quad \text{for } t \in [0,1] \text{ and } 0 \leq u \leq M.$$

Let

$$R_2 > \max\left\{M, \frac{\lambda M_2 M_3 \int_0^1 p(s)\,\mathrm{d}s}{M_1}, 3\lambda M_2 \int_0^1 p(s)\,\mathrm{d}s, 3\lambda M_2 \int_0^1 \varphi_M(s)\,\mathrm{d}s\right\}.$$

Then, for any $u \in \partial P_{R_2}$ and $t \in [0,1]$, we have

$$u(t) - \omega(t) \geq \frac{M_1 t^{\alpha-1}}{M_2} R_2 - \lambda M_3 t^{\alpha-1} \int_0^1 p(s)\,\mathrm{d}s \geq 0.$$

Hence, for any $t \in [0,1]$, we get

$$\begin{aligned}
Tu(t) &= \lambda \int_0^1 G(t,s) f^*\bigl(s,u(s)\bigr)\,\mathrm{d}s \\
&= \lambda \int_0^1 G(t,s)\bigl(f\bigl(s,u(s)-\omega(s)\bigr) + p(s)\bigr)\,\mathrm{d}s \\
&\leq \lambda M_2 \int_0^1 \bigl(f\bigl(s,u(s)-\omega(s)\bigr) + p(s)\bigr)\,\mathrm{d}s \\
&= \lambda M_2 \int_0^1 f\bigl(s,u(s)-\omega(s)\bigr)\,\mathrm{d}s + \lambda M_2 \int_0^1 p(s)\,\mathrm{d}s \\
&= \lambda M_2 \biggl(\int_{u(s)-\omega(s)>M} f\bigl(s,u(s)-\omega(s)\bigr)\,\mathrm{d}s + \int_{0\leq u(s)-\omega(s)\leq M} f\bigl(s,u(s)-\omega(s)\bigr)\,\mathrm{d}s\biggr) \\
&\quad + \lambda M_2 \int_0^1 p(s)\,\mathrm{d}s \\
&\leq \lambda M_2 \varepsilon R_2 + \lambda M_2 \int_0^1 \varphi_M(s)\,\mathrm{d}s + \lambda M_2 \int_0^1 p(s)\,\mathrm{d}s \\
&\leq \frac{R_2}{3} + \frac{R_2}{3} + \frac{R_2}{3} = R_2.
\end{aligned}$$

Thus,

$$\|Tu\| \le \|u\| \quad \text{for } u \in \partial P_{R_2}.$$

By Lemma 3.2, we know that T is a completely continuous operator.

According to cone expansion and cone compression fixed point theorem (see [31]), we can obtain that T has a fixed point u such that $R_1 \le \|u\| \le R_2$ in $P \cap (\bar{B}_{R_2} \setminus B_{R_1})$.

Let $u^*(t) = u(t) - \omega(t), t \in [0,1]$. We have

$$u^*(t) = \frac{M_1 t^{\alpha-1}}{M_2}\|u\| - \lambda M_3 t^{\alpha-1}\int_0^1 p(s)\,\mathrm{d}s \ge \left(\lambda M_3 \int_0^1 p(s)\,\mathrm{d}s\right)t^{\alpha-1}.$$

Then u^* is the positive solution of (1.1).

Therefore, there exists a constant $\bar{\lambda}^* > 0$ such that boundary value problem (1.1) has at least one positive solution for any $\lambda \in (\bar{\lambda}^*, +\infty)$. □

4 Illustration

To illustrate our main results, we present the following examples.

Example 4.1 We consider the boundary value problem

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t) + \lambda(u^{\frac{3}{2}}(t) + \ln t) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, \\ D_{0^+}^{\frac{1}{2}} u(1) = \int_0^1 D_{0^+}^{\frac{1}{2}} u(t)\,\mathrm{d}t, \end{cases} \tag{4.1}$$

where $\alpha = \frac{7}{2}$, $n = 4$, $\beta = \frac{1}{2}$, $A(t) = t$. Choose $f(t,u) = u^{\frac{3}{2}} + \ln t$. Then

$$-|\ln t| = \ln t \le f(t,u) = u^{\frac{3}{2}} + \ln t, \quad t \in (0,1),$$

where

$$p(t) = |\ln t|, \qquad q(t) = 1, \qquad g(u) = u^{\frac{1}{2}}.$$

By direct calculation, we have

$$M_1 = 0.2256758, \qquad M_2 = M_3 = 0.7522528, \qquad \int_0^1 p(s)\,\mathrm{d}s = 1,$$

and

$$\liminf_{u\to+\infty} \inf_{t\in[\theta_1,\theta_2]} \frac{f(t,u)}{u} = +\infty.$$

So all the conditions of Theorem 3.3 are satisfied. By Theorem 3.3, there exists $\lambda^* > 0$ such that boundary value problem (4.1) has at least one positive solution provided $\lambda \in (0, \lambda^*)$.

Example 4.2 We consider the boundary value problem

$$\begin{cases} D_{0^+}^{\frac{8}{3}}u(t) + \lambda(-\frac{\sin(2\pi u(t))}{\sqrt{t(1-t)}} + u^{\frac{1}{2}}(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, \\ D_{0^+}^{\frac{1}{8}}u(1) = D_{0^+}^{\frac{1}{8}}u(\frac{1}{4}) + D_{0^+}^{\frac{1}{8}}u(\frac{3}{4}), \end{cases} \tag{4.2}$$

where $\alpha = \frac{8}{3}$, $n = 3$, $\beta = \frac{1}{8}$, $f(t,u) = -\frac{\sin(2\pi u)}{\sqrt{t(1-t)}} + u^{\frac{1}{2}}$,

$$A(t) = \begin{cases} 0, & 0 \le t < \frac{1}{4}, \\ 1, & \frac{1}{4} \le t < \frac{3}{4}, \\ 2, & \frac{3}{4} \le t < 1. \end{cases}$$

Let $p(t) = \frac{2}{\sqrt{t(1-t)}}$. Clearly, for $t \in (0,1)$, we have

$$f(t,u) > -p(t).$$

By direct calculation, we have

$$M_1 = 2.461073, \qquad M_2 = 6.3273, \qquad M_3 = 6.197864, \qquad \int_0^1 p(s)\,\mathrm{d}s = 6.283185,$$

and

$$\limsup_{u \to +\infty} \sup_{t \in [0,1]} \frac{f(t,u)}{u} = 0.$$

So, all the conditions of Theorem 3.4 are satisfied. By Theorem 3.4, there exists $\bar{\lambda}^* > 0$ such that boundary value problem (4.2) has at least one positive solution provided $\lambda \in (\bar{\lambda}^*, +\infty)$.

Acknowledgements
This work is supported by the National Natural Science Foundation of China (No. 11171220) and the Hujiang Foundation of China (B14005).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 November 2017 Accepted: 2 January 2018 Published online: 05 February 2018

References
1. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivations. Gordon & Breach, Yverdon (1993)
2. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
3. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
4. Jia, M, Liu, X: Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions. Appl. Math. Comput. **232**, 313-323 (2014)
5. Dong, X, Bai, Z, Zhang, S: Positive solutions to boundary value problems of p-Laplacian with fractional derivative. Bound. Value Probl. **2017**, Article ID 5 (2017)

6. Feng, M, Zhang, X, Ge, W: New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions. Bound. Value Probl. **2011**, Article ID 720702 (2011)
7. Zhang, X: Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions. Appl. Math. Lett. **39**, 22-27 (2015)
8. Liu, X, Jia, M, Ge, W: The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator. Appl. Math. Lett. **65**, 56-62 (2017)
9. Li, C, Luo, X, Zhou, Y: Existence of positive solutions of the boundary value problem for nonlinear fractional differential equation. Comput. Math. Appl. **59**, 1363-1375 (2010)
10. Wang, X, Liu, X, Deng, X: Existence and nonexistence of positive solutions for fractional integral boundary value problem with two disturbance parameters. Bound. Value Probl. **2015**, Article ID 186 (2015)
11. Wang, Y, Liu, L, Wu, Y: Positive solutions for a class of fractional boundary value problem with changing sign nonlinearity. Nonlinear Anal. **74**, 6434-6441 (2011)
12. Zhang, H: Iterative solutions fractional nonlocal boundary value problems involving integral condition. Bound. Value Probl. **2016**, 3 (2016)
13. Zhang, L, Bashir, A, Wang, G, Agarwal, RP,: Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. J. Comput. Appl. Math. **2013**(249), 51-56 (2013)
14. Liu, X, Jia, M, Ge, W: Multiple solutions of a p-Laplacian model involving a fractional derivative. Adv. Differ. Equ. **2013**, Article ID 126 (2013)
15. Cheng, J: Two positive solutions for $(n-1,1)$-type semipositone integral boundary value problems for coupled systems of nonlinear fractional differential equations. Bound. Value Probl. **2012**(17), 930 (2012)
16. Liu, X, Jia, M: The positive solutions for integral boundary value problem of fractional p-Laplacian equation with mixed derivatives. Mediterr. J. Math. **14**, Article ID 94 (2017)
17. Su, X, Jia, M, Li, M: The existence and nonexistence of positive solutions for fractional differential equations with nonhomogeneous boundary conditions. Adv. Differ. Equ. **2016**, Article ID 30 (2016)
18. Zhang, X, Liu, L, Wu, Y: Existence results for multiple positive solution of nonlinear higher order perturbed fractional differential equations with derivative. Appl. Math. Comput. **219**, 1420-1433 (2012)
19. Wu, J, Zhang, X: Eigenvalue problem of nonlinear semipositone higher order fractional differential equations. Abstr. Appl. Anal. **2012**, Article ID 740760 (2012)
20. Bai, Z, Lü, H: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**, 495-505 (2005)
21. Bai, Z: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. **72**, 916-924 (2010)
22. Wu, J, Zhang, X, Liu, L, Wu, Y: Positive solutions of higher-order nonlinear fractional differential equations with changing-sign measure. Adv. Differ. Equ. **2012**, Article ID 71 (2012)
23. Ahmad, B, Ntouyas, SK, Tariboon, J: Nonlocal fractional-order boundary value problems with generalized Riemann-Liouville integral boundary conditions. J. Comput. Anal. Appl. **23**, 1281-1296 (2017)
24. Yang, L, Xie, D, Bai, C, Yang, D: Multiple positive solutions for a coupled system of fractional multi-point BVP with p-Laplacian operator. Adv. Differ. Equ. **2017**, Article ID 168 (2017)
25. Ren, T, Li, S, Zhang, X, Liu, L: Maximum and minimum solutions for a nonlocal p-Laplacian fractional differential system from eco-economical processes. Bound. Value Probl. **2017**, Article ID 118 (2017)
26. Liu, X, Jia, M, Xiang, X: On the solvability of a fractional differential equation model involving the p-Laplacian operator. Comput. Math. Appl. **64**, 3267-3275 (2012)
27. Feng, H, Liu, J: Multiple positive solutions for m-point boundary value problems with one-dimensional p-Laplacian systems and sign changing nonlinearity. J. Comput. Anal. Appl. **20**, 1066-1077 (2016)
28. Apostol, TM: Mathematical Analysis, 2nd edn. Addison-Wesley, Massachusetts (1974)
29. Webb, JRL, Infante, G: Positive solution of nonlocal boundary value problems involving integral conditions. Nonlinear Differ. Equ. Appl. **2008**(15), 45-67 (2008)
30. Webb, JRL, Infante, G: Positive solution of nonlinear boundary value problems: a unified approach. J. Lond. Math. Soc. **2016**(74), 673-693 (2016)
31. Guo, D, Lakshmikantham, V: Nonlinear Problem in Abstract Cones. Academic Press, Orlando (1988)

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com