Sun et al. *Advances in Difference Equations* (2018) 2018:10
https://doi.org/10.1186/s13662-017-1464-z

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Hopf bifurcation analysis in a fractional-order survival red blood cells model and PD^{α} control

Qingshan Sun[1], Min Xiao[1*], Binbin Tao[1], Guoping Jiang[1], Jinde Cao[2,3], Fuchen Zhang[4] and Chengdai Huang[5]

*Correspondence: candymanxm2003@aliyun.com
[1]College of Automation, Nanjing University of Posts and Telecommunications, Nanjing, 210003, China
Full list of author information is available at the end of the article

Abstract

In this paper, we put forward a fractional-order survival red blood cells model and study the dynamics through the Hopf bifurcation. When the delay transcends the threshold, a series of Hopf bifurcations occur at the positive equilibrium. Then, a fractional-order Proportional and Derivative (PD^{α}) controller is applied to the proposed model for the Hopf bifurcation control. It is discovered that by setting proper parameters, the PD^{α} controller can delay or advance the onset of Hopf bifurcations. Therefore the Hopf bifurcation of the fractional-order survival red blood cells model becomes controllable to achieve desirable behaviors. Finally, numerical examples are presented to demonstrate the theoretical analysis.

Keywords: Hopf bifurcation; bifurcation control; time delays; PD^{α} controller; survival red blood cells model

1 Introduction

Fractional calculus was born in 1695 as an important branch of mathematics, almost simultaneously with classical calculus. Compared with integer-order derivatives, it has been found that fractional derivatives have the superiority of accuracy and flexibility when used to describe some non-classical phenomena in natural science and engineering applications such as neurons [1], finance systems [2], biological systems [3], and so on. Especially in biological systems, fractional calculus has more advantages than traditional integer-order calculus in describing molecular dynamics with memory characteristics and historical dependence [4, 5]. Fractional calculus accumulates the global information of the function in a weighted form, which is also called memory. A large number of examples show that the fractional calculus has a more universal meaning than the integer calculus.

In recent years, there have been many papers about the stability and the Hopf bifurcation analysis of integer-order survival red blood cells models [6–9]. With the rapid development of biomedical and molecular biology, researchers have proposed some new research topics [10–12]. Many biological systems have shown the feature of fractal geometry, the characteristics of memory and diversity discharge activities which cannot be described accurately by using the classical calculus theory. It is known that the integer-order calculus is only determined by the local character of the function, while the fractional-order one can gather the global information of the function in the weighted form [13]. In the biological field, Magin argued that the fractional-order derivative can describe the activities of the organism more accurately [10, 11]. In [12], the dynamics of a red blood cells model is fully described by linear fractional-order differential equations, and the theory of fractional calculus provides a concise way to describe and quantify the biomechanical behaviors of membranes, cells and tissues. However, the qualitative theory of bifurcations in a fractional-order system is still a problem, which has not been researched thoroughly. It is more meaningful to investigate the fractional-order survival red blood cells model instead of the integer-order counterpart.

For getting better desirable stability domain, we can add some effective controllers, such as hybrid controller [14, 15], state feedback controller [16, 17] and delayed feedback controller [18, 19], for dynamics control in nonlinear systems. As we know, the PID controller consists of the proportional unit P, the integrating unit I and the differential unit D. We can adjust the stability of the systems by setting the three control parameters k_p, k_i and k_d in the PID controller. The PID controller is mainly applicable to systems with essentially linear and dynamic characteristics which are not changeable through time [20]. In consideration of the feature of fractional-order systems, we develop a fractional-order Proportional and Derivative (PD^{α}) scheme to control the bifurcation of the fractional-order survival red blood cells model in this paper. It is worth mentioning that such a control strategy has not been reported in the control of bifurcation for fractional-order systems. Motivated by the above discussions, we investigate the problem of bifurcation and control for the delayed fractional-order survival red blood cells model in the present paper.

2 Model description

In [21], Wazewska-Czyzewska and Lasota proposed the survival red blood cells model:

$$\frac{dX}{dt} = -aX(t) + be^{-cX(t-\tau)}, \quad a > 0, b > 0, c > 0, \tau > 0, t \geq 0, \tag{2.1}$$

where $X(t)$ represents the number of red blood cells in time t, a is the death rate of red blood cells, b and c describe the production of red blood cells per unit time and τ is the necessary time to produce a red blood cell.

There are many definitions of fractional derivatives. The Grünwald-Letnikov definition, the Riemann-Liouville definition and the Caputo definition are usually used to deal with fractional-order systems. Since the Caputo derivative only requires the initial conditions which are based on integer-order derivative and represents well-understood features of physical situation, it is more applicable to real world problems. Hence, the Caputo fractional-order derivative is employed in this paper.

The Caputo fractional-order derivative is defined as follows:

$${}_{e}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(d-\alpha)} \int_{e}^{t} (t-\tau)^{d-\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{2.2}$$

where $d - 1 < \alpha < d$, $d \in N$, and $\Gamma(\cdot)$ is the gamma function. The symbol α denotes the value of the fractional order that is usually chosen in the range $0 < \alpha \leq 1$.

The Laplace transformation of the Caputo fractional-order derivative is represented as

$$L\left\{{}_0^C D_t^{\alpha} f(t)\right\} = s^{\alpha} F(s) - \sum_{k=0}^{d-1} s^{\alpha-k-1} f^{(k)}(0). \tag{2.3}$$

If $f^{(k)}(0) = 0$, $k = 0, 1, \ldots, d-1$, then $L\{{}_0^C D_t^{\alpha} f(t)\} = s^{\alpha} F(s)$.

A class of n-dimensional linear fractional-order systems with multiple time delays can be represented in the following form [22]:

$$\begin{aligned} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} &= a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \cdots + a_{1n}x_n(t-\tau_{1n}), \\ \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} &= a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \cdots + a_{2n}x_n(t-\tau_{2n}), \\ &\cdots \\ \frac{d^{\alpha_n} x_n}{dt^{\alpha_n}} &= a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \cdots + a_{nn}x_n(t-\tau_{nn}), \end{aligned} \tag{2.4}$$

with the characteristic equation

$$\det \begin{pmatrix} s^{\alpha_1} - a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \cdots & -a_{1n}e^{-s\tau_{1n}} \\ -a_{21}e^{-s\tau_{21}} & s^{\alpha_2} - a_{22}e^{-s\tau_{22}} & \cdots & -a_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \cdots & s^{\alpha_n} - a_{nn}e^{-s\tau_{nn}} \end{pmatrix} = 0, \tag{2.5}$$

where $0 < \alpha_i \leq 1$ ($i = 1, 2, 3, \ldots, n$), and $d^{\alpha_i}/dt^{\alpha_i}$ is chosen as the Caputo fractional derivative (2.2).

Theorem 2.1 ([22]) *Given that all the roots of the characteristic equation* (2.5) *have negative real parts, the zero solution of system* (2.4) *is Lyapunov globally asymptotically stable.*

Remark 2.1 Theorem 2.1 indicates that the stability boundary for the delayed fractional-order system (2.4) is the imaginary axis.

Remark 2.2 If $\tau_{ij} = 0$, $i, j = 1, \ldots, n$, then Theorem 2.1 converts into Matignon criterion [23]: if all the roots λs of the equation $\det(\lambda I - A) = 0$ satisfy $|\arg(\lambda)| > \alpha\pi/2$, then the zero solution of system (2.4) is Lyapunov globally asymptotically stable, where $A = (a_{ij})_{n\times n}$ is the coefficient matrix and $\lambda = s^{\alpha}$. It can be seen that the stability boundary is described by $|\arg(\lambda)| = \alpha\pi/2$ (or $|\arg(s)| = \pi/2$) for the fractional-order system (2.4) without delays.

Remark 2.3 If all the eigenvalues λs of A satisfy $|\arg(\lambda)| > \alpha\pi/2$ and the characteristic equation (2.5) has no purely imaginary roots for any $\tau_{ij} > 0$, $i, j = 1, \ldots, n$, then the zero solution of system (2.4) is Lyapunov globally asymptotically stable [22].

For model (2.1), we let

$$m = \frac{a}{bc}, \qquad \upsilon = \tau bc, \qquad u(t) = cX\left(\frac{t}{bc}\right).$$

Then we get

$$\frac{du(t)}{dt} = -mu(t) + e^{-u(t-\upsilon)}, \quad t \geq 0, m > 0, \upsilon > 0. \tag{2.6}$$

In this paper, we focus on the dynamics of the following fractional-order survival red blood cells model with time delays:

$$\frac{d^\alpha u(t)}{d^\alpha t} = -mu(t) + e^{-u(t-\upsilon)}, \quad t \geq 0, m > 0, \upsilon > 0. \tag{2.7}$$

We can easily see that, for model (2.6), there is a unique u^* satisfying the following equation:

$$mu^* = e^{-u^*}. \tag{2.8}$$

It can be seen that $u^* > 1$ if and only if $0 < m < 1/e$.

3 Bifurcation analysis of the uncontrolled model

In this part, we investigate the stability of the fractional-order survival red blood cells model (2.7), and some existence conditions of Hopf bifurcations are addressed.

Let $u(t) - u^* = y(t)$, then the linearized model (2.7) is

$$\frac{d^\alpha y}{dt^\alpha} = -my(t) - mu^* y(t-\upsilon), \tag{3.1}$$

with the characteristic equation

$$s^q + m + mu^* e^{-s\upsilon} = 0. \tag{3.2}$$

In the following, we investigate the roots distribution of equation (3.2) by regarding the time delay υ as the bifurcation parameter.

Let $s = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\omega > 0)$. Then equation (3.2) becomes

$$\omega^q \cos\frac{q\pi}{2} + \omega^q \sin\frac{q\pi}{2} \cdot i + m + m \cdot u^*(\cos\omega\upsilon - i\sin\omega\upsilon) = 0. \tag{3.3}$$

Separating the real and imaginary parts gives

$$\begin{cases} a_2\cos\omega\upsilon = a_1 - \omega^q\cos\frac{q\pi}{2}, \\ a_2\sin\omega\upsilon = \omega^q\sin\frac{q\pi}{2}, \end{cases} \tag{3.4}$$

where $a_1 = -m$, $a_2 = mu^*$. It can be obtained from (3.4) that

$$\omega^{2q} - 2a_1\omega^q\cos\frac{q\pi}{2} + a_1^2 - a_2^2 = 0. \tag{3.5}$$

Denote

$$h_1(\omega) = \omega^{2q} - 2a_1\omega^q\cos\frac{q\pi}{2} + a_1^2 - a_2^2.$$

Lemma 3.1 *For equation* (3.2), *we have the following*:

(i) *If* $m \geq 1/e$, *then all roots of the characteristic equation* (3.2) *have negative real parts*.

(ii) *If* $0 < m < 1/e$, *then equation* (3.2) *has a pair of purely imaginary roots* $\pm\omega_0$ *when* $\upsilon = \upsilon_j$, $j = 0, 1, \ldots$, *where*

$$\upsilon_j = \frac{1}{\omega_0} \arccos\left(\frac{a_1 - \omega_0 \cos\frac{q\pi}{2}}{a_2} + 2j\pi\right), \quad j = 0, 1, 2, \ldots, \tag{3.6}$$

where ω_0 *is the unique positive zero of the function* $h_1(\omega)$.

Proof (i) From $|a_1| > a_2$, then $h_1(0) > 0$, and the symmetry axis is $a_1 \cos\frac{q\pi}{2} < 0$. Combining $q > 0$, we can see that equation (3.5) has no real root, so equation (3.2) has no purely imaginary root. This finishes the proof of (i).

(ii) By means of $|a_1| < a_2$, it is easy to see that $h_1(0) < 0$. Combining $q > 0$, there exists a unique positive number ω_0 such that $h_1(\omega) = 0$. Then ω_0 is a root of equation (3.5). Hence, for υ_j as defined in (3.6), (ω_0, υ_j) is a root of equation (3.3). It can be seen that $\pm\omega_0$ is a pair of purely imaginary roots of equation (3.2), while $\upsilon = \upsilon_j$, $j = 0, 1, \ldots$. This completes the proof of (ii). □

Remark 3.1 The conclusion (ii) of Lemma 3.1 gives the onset of Hopf bifurcation of model (2.7).

Here we make the following assumption:

$$(H_1) \quad \frac{P_1 Q_1 + P_2 Q_2}{Q_1^2 + Q_2^2} > 0,$$

where

$$P_1 = mu^* \omega_0 \sin \omega_0 \upsilon_0,$$

$$P_2 = mu^* \omega_0 \cos \omega_0 \upsilon_0,$$

$$Q_1 = q\omega_0^{q-1} \cos \frac{(q-1)}{2}\pi - mu^* \upsilon \cos \omega_0 \upsilon_0,$$

$$Q_2 = q\omega_0^{q-1} \sin \frac{(q-1)}{2}\pi + mu^* \upsilon \sin \omega_0 \upsilon_0.$$

Lemma 3.2 *Let* $s(\upsilon) = \varsigma(\upsilon) + i\omega(\upsilon)$ *be the root of equation* (3.2). *It is easy to see* $\varsigma(\upsilon_j) = 0$, $\omega(\upsilon_j) = \omega_0$, *when* $\upsilon = \upsilon_j$. *If* (H_1) *holds, then we have*

$$\mathrm{Re}\left[\frac{ds}{d\tau}\right]_{\omega=\omega_0, \upsilon=\upsilon_0} > 0.$$

Proof Differentiating equation (3.2) implicitly with respect to υ, we obtain

$$\frac{ds}{d\upsilon} = \frac{smu^* e^{-s\upsilon}}{qs^{q-1} - \upsilon mu^* e^{-s\upsilon}}. \tag{3.7}$$

Hence, we deduce that

$$\operatorname{Re}\left[\frac{ds}{d\upsilon}\right]_{\omega=\omega_0,\upsilon=\upsilon_0} = \frac{P_1Q_1 + P_2Q_2}{Q_1^2 + Q_2^2}.$$

Obviously, hypothesis (H_1) means that the transversality condition is satisfied. □

Theorem 3.1 *For model* (2.7), *when* $0 < m < 1/e$, *the following results hold*:

(i) *The equilibrium* u^* *of model* (2.7) *is locally asymptotically stable for* $\upsilon \in [0, \upsilon_0)$, *and unstable when* $\upsilon > \upsilon_0$.

(ii) *Model* (2.7) *undergoes a Hopf bifurcation at the equilibrium* u^* *when* $\upsilon = \upsilon_0$.

Proof Note that the eigenvalue $\lambda = -(m + mu^*) < 0$ of the linearized system of (3.1) satisfies the inequality $|\arg(\lambda)| > q\pi/2$ when $\upsilon = 0$. Therefore, the condition for the Hopf bifurcation is satisfied.

(i) We can find that when $\upsilon = 0$ the roots of equation (3.2) have negative real parts. In Lemma 3.1, we can see that all the roots of equation (3.2) have negative real parts for $\upsilon \in [0, \upsilon_0)$ by the definition of υ_0. From Lemma 3.2, this implies that equation (3.2) has at least a positive root when $\upsilon > \upsilon_0$.

(ii) From the above discussion, it is obvious that the occurrence condition of the Hopf bifurcation is satisfied for (2.6). Therefore, near the equilibrium u^*, there occurs a Hopf bifurcation when $\upsilon = \upsilon_0$. □

4 Bifurcation analysis of the controlled model

In this part, by choosing the time delay υ as the bifurcation parameter, we are trying to control the Hopf bifurcation of (2.7) based on the fractional-order Proportional and Derivative (PD^α) control strategy.

For the delayed fractional-order model (2.7), we propose a single input and output PD^α controller as follows:

$$\rho(t) = k_p\bigl(u(t) - u^*\bigr) + k_d \frac{d^\alpha}{dt^\alpha}\bigl(u(t) - u^*\bigr), \tag{4.1}$$

where k_p is the proportional control parameter and k_d is the derivative control parameter.

Hence, the controlled fractional-order survival red blood cells model with time delays becomes

$$\begin{aligned}\frac{d^\alpha u}{dt^\alpha} &= -mu(t) + e^{-u(t-\upsilon)} + \rho(t)\\ &= -mu(t) + e^{-u(t-\upsilon)} + k_p\bigl(u(t) - u^*\bigr) + k_d \frac{d^\alpha}{dt^\alpha}\bigl(u(t) - u^*\bigr).\end{aligned} \tag{4.2}$$

Let $u(t) - u^* = y(t)$, The controlled model (4.2) becomes

$$\begin{aligned}\frac{d^\alpha y}{dt^\alpha} &= \frac{1}{1-k_d}\bigl[-my(t) - mu^*y(t-\upsilon) + k_py(t)\bigr]\\ &= \frac{1}{1-k_d}\bigl[(k_p - m)y(t) - mu^*y(t-\upsilon)\bigr],\end{aligned} \tag{4.3}$$

with the characteristic equation

$$s^q - \frac{1}{1-k_d}\left[(k_p - m) - mu^* e^{-s\upsilon}\right] = 0. \tag{4.4}$$

Here, we assume the value range of the control parameters: $k_p < m$, $k_d < 1$.

Remark 4.1 Comparing model (4.3) and model (3.1), it is obvious that the controlled fractional-order model and the uncontrolled one have the same equilibrium point.

Let $s = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ ($\omega > 0$), then equation (4.4) becomes

$$\omega^q\left(\cos\frac{q\pi}{2} + i\sin\frac{q\pi}{2}\right) - \frac{k_p - m}{1-k_d} + \frac{mu^*}{1-k_d}(\cos\omega\upsilon - i\sin\omega\upsilon) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{cases} a_4\cos\omega\upsilon = a_3 - \omega^q\cos\frac{q\pi}{2}, \\ a_4\sin\omega\tau = \omega^q\sin\frac{q\pi}{2}, \end{cases} \tag{4.5}$$

where $a_3 = \frac{k_p - m}{1-k_d}$, $a_4 = \frac{mu^*}{1-k_d}$. It can be obtained from (4.5) that

$$\omega^{2q} - 2a_1\omega^q\cos\frac{q\pi}{2} + a_3^2 - a_4^2 = 0. \tag{4.6}$$

Denote

$$h_2(\omega) = \omega^{2q} - 2a_1\omega^q\cos\frac{q\pi}{2} + a_3^2 - a_4^2.$$

Lemma 4.1 *If* $|a_3| < a_4$, *equation* (4.4) *has a pair of purely imaginary roots* $\pm\omega_0^c$ *when* $\upsilon = \upsilon_j^c$, $j = 0, 1, \ldots$, *where*

$$\upsilon_j^c = \frac{1}{\omega_0^c}\arccos\left(\frac{a_3 - \omega_0^c\cos\frac{q\pi}{2}}{a_4} + 2j\pi\right), \quad j = 0, 1, 2, \ldots, \tag{4.7}$$

where ω_0^c *is the unique positive zero of the function* $h_2(\omega)$.

Proof By means of $|a_3| < a_4$, it is easy to see that $h_2(0) < 0$. Combining with $q > 0$, there exists a unique positive number ω_0^c such that $h_2(\omega) = 0$. Then ω_0^c is a root of (4.6). Hence, for υ_j^c as defined in (4.7), $(\omega_0^c, \upsilon_j^c)$ is a root of equation (4.5). It can be seen that $\pm\omega_0^c$ is a pair of purely imaginary roots of equation (4.4), while $\upsilon = \upsilon_j^c$, $j = 0, 1, \ldots$. □

Remark 4.2 Lemma 4.1 obtains the onset of the delayed fractional-order model's Hopf bifurcations.

We make the following assumption:

$$(H_2)\quad \frac{M_1N_1 + M_2N_2}{N_1^2 + N_2^2} > 0,$$

where

$$M_1 = \frac{\omega_0^c m u^*}{1-k_d}\sin\omega_0^c \upsilon_0^c,$$
$$M_2 = \frac{\omega_0^c m u^*}{1-k_d}\cos\omega_0^c \upsilon_0^c,$$
$$N_1 = q(\omega_0^c)^{q-1}\cos\frac{(q-1)}{2}\pi - \frac{\upsilon m u^*}{1-k_d}\cos\omega_0^c \upsilon_0^c,$$
$$N_2 = q(\omega_0^c)^{q-1}\sin\frac{(q-1)}{2}\pi + \frac{\upsilon m u^*}{1-k_d}\sin\omega_0^c \upsilon_0^c.$$

Lemma 4.2 *Let $s(\upsilon) = \varsigma(\upsilon) + i\omega(\upsilon)$ be the root of equation* (4.4). *It is easy to see $\varsigma(\upsilon_j^c) = 0$, $\omega(\upsilon_j^c) = \omega_0^c$ when $\upsilon = \upsilon_j^c$. If (H_2) holds, then we have*

$$\operatorname{Re}\left[\frac{ds}{d\upsilon}\right]_{\omega=\omega_0^c,\upsilon=\upsilon_0^c} > 0.$$

Proof Differentiating equation (4.4) relative to υ, we get

$$\frac{ds}{d\upsilon} = \frac{\frac{smu^*}{1-k_d}e^{-s\upsilon}}{qs^{q-1} - \frac{\upsilon mu^*}{1-k_d}e^{-s\upsilon}}. \tag{4.8}$$

Hence, we obtain

$$\operatorname{Re}\left[\frac{ds}{d\upsilon}\right]_{\omega=\omega_0^c,\upsilon=\upsilon_0^c} = \frac{M_1N_1 + M_2N_2}{N_1^2 + N_2^2}.$$ □

Theorem 4.1 *For model* (4.2), *when $|a_3| < a_4$, we get the following results.*

(i) *The equilibrium u^* of model* (4.2) *is locally asymptotically stable for $\upsilon \in [0, \upsilon_0^c)$, and unstable when $\upsilon > \upsilon_0^c$.*

(ii) *Model* (4.2) *undergoes a Hopf bifurcation at the equilibrium u^* when $\upsilon = \upsilon_0^c$.*

Proof Note that the eigenvalue $\lambda = \frac{1}{1-k_d}[(k_p - m) - mu^*] = a_1 - a_2 < 0$ of the linearized system of (4.3) satisfies the inequality $|\arg(\lambda)| > q\pi/2$ when $\upsilon = 0$. Therefore, the condition for the Hopf bifurcation is satisfied.

(i) It is easy to see that all the roots of equation (4.4) with $\upsilon = 0$ have negative real parts. The definition of υ_0^c implies that all the roots of equation (4.4) have negative real parts for $\upsilon \in [0, \upsilon_0^c)$. From Lemma 4.2, it indicates that equation (4.4) has at least a root with positive real parts when $\upsilon > \upsilon_0^c$.

(ii) From the above discussion, it is clear that the occurrence condition of the Hopf bifurcation is satisfied for model (4.2). Therefore, near the equilibrium u^*, there occurs a Hopf bifurcation when $\upsilon = \upsilon_0^c$. □

5 Numerical simulations

In this section, we provide numerical simulations to confirm our theoretical analysis and display the Hopf bifurcation phenomenon of the delayed fractional-order model.

For the uncontrolled model (2.7), we take $\alpha = 0.92$, $m = 0.2$ used in [7]. Then (2.7) has a positive equilibrium $u^* = 1.3267$. From (3.6), we can obtain $\upsilon_0 = 19.6749$. The positive

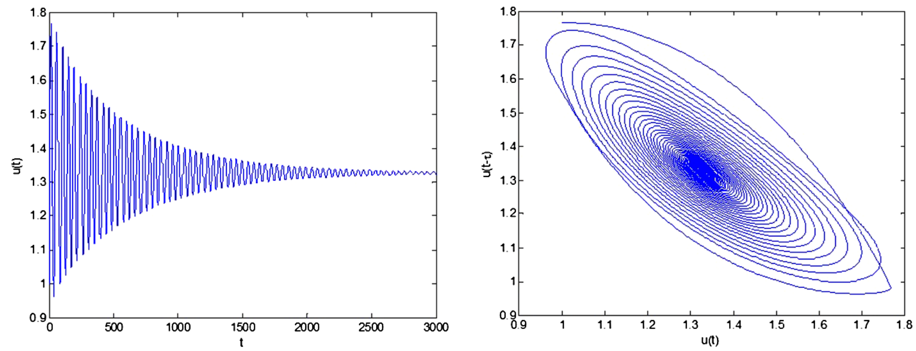

Figure 1 Waveform plot and phase portrait of the fractional-order model (2.7) with $\alpha = 0.92$, $m = 0.2$, and the initial value $u_0 = 1$. The equilibrium u^* is asymptotically stable when $\upsilon = 18 < \upsilon_0 = 19.6749$.

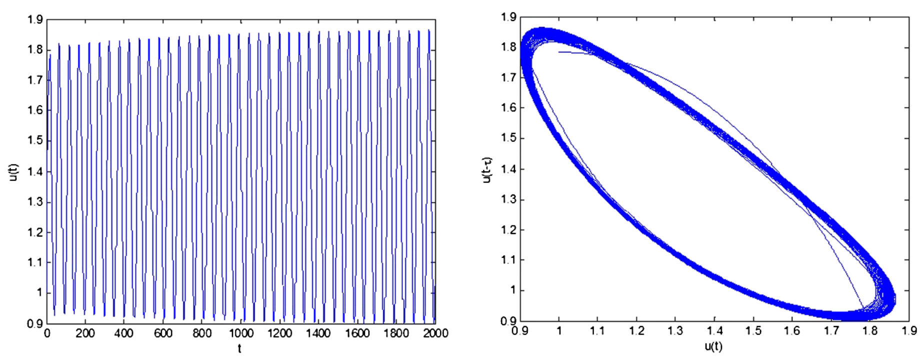

Figure 2 Waveform plot and phase portrait of the fractional-order model (2.7) with $\alpha = 0.92$, $m = 0.2$, and the initial value $u_0 = 1$. A Hopf bifurcation occurs and the periodic oscillation bifurcates from u^*, where $\upsilon = 21 > \upsilon_0 = 19.6749$.

Table 1 Bifurcation point υ_0 versus fractional order α for model (2.7)

Fractional order α	Bifurcation point υ_0
0.9	21.5435
0.92	19.6749
0.94	17.9960
0.96	16.4929
0.98	15.1312
1.0	13.9045

equilibrium point u^* is asymptotically stable when $\upsilon = 18 < \upsilon_0 = 19.6749$ as illustrated in Figure 1, and when $\upsilon = 21 > \upsilon_0 = 19.6749$, there occurs a Hopf bifurcation at the positive equilibrium point u^* as demonstrated in Figure 2.

The effect of the order α from 0.9 to 1 on the values of υ_0 for model (2.7) is shown in Table 1.

In order to make a comparison with the uncontrolled fractional-order model (2.7), we discuss the controlled model (4.2) with the parameters $\alpha = 0.92$, $m = 0.2$. When we choose the control parameters $k_p = -0.05$, $k_d = 0.5$, from (4.7) we can obtain $\upsilon_0^c = 27.7129$. It can be seen that the stable region has been enlarged and the critical value υ_0^c has been increased to a larger value than that of the uncontrolled model. This indicates that the

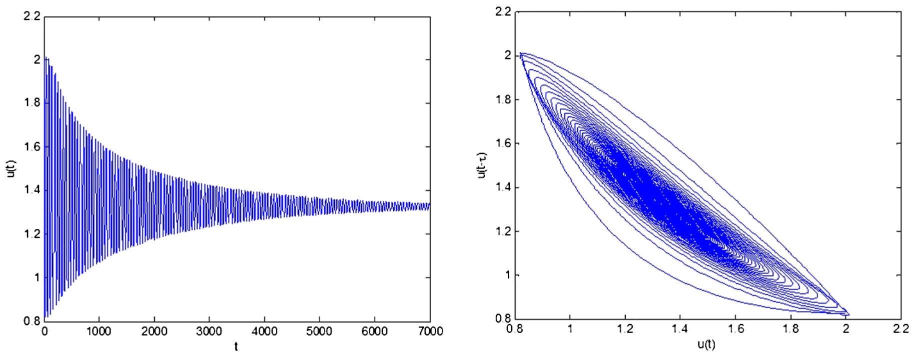

Figure 3 Waveform plot and phase portrait of the controlled fractional-order model (4.2) with $\alpha = 0.92$, $m = 0.2$, $k_p = -0.05$, $k_d = 0.5$ and the initial value $u_0 = 2$. The equilibrium u^* is asymptotically stable when $\upsilon = 24 < \upsilon_0^c = 27.7129$.

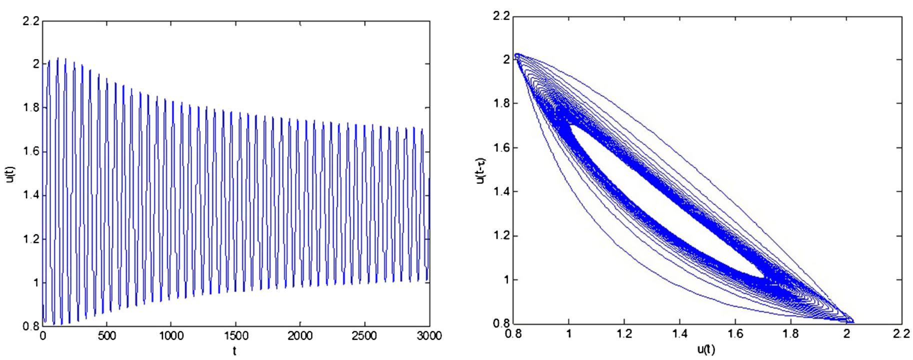

Figure 4 Waveform plot and phase portrait of the controlled fractional-order model (4.2) with $\alpha = 0.92$, $m = 0.2$, $k_p = -0.05$, $k_d = 0.5$ and the initial value $u_0 = 2$. A Hopf bifurcation occurs and the periodic oscillation bifurcates from u^*, where $\upsilon = 30 > \upsilon_0^c = 27.7129$.

PD^α controller can delay the onset of Hopf bifurcations effectively. The equilibrium u^* is asymptotically stable when $\upsilon = 24 < \upsilon_0^c = 27.7129$ as illustrated in Figure 3, and when $\upsilon = 30 > \upsilon_0^c = 27.7129$, there occurs a Hopf bifurcation at the positive equilibrium point u^* as demonstrated in Figure 4.

Next, we select other control parameters to validate the effectiveness of our proposed PD^α scheme in the bifurcation control. We also take the same parameters $\alpha = 0.92$, $m = 0.2$ for the original model (2.7). When $k_p = 0.05$, $k_d = 0.5$, from (4.7) one can obtain $\upsilon_0^c = 6.1794$. The critical value υ_0^c is smaller than that of the uncontrolled model (2.7), which shows that the PD^α controller can advance the onset of the Hopf bifurcation effectively. The equilibrium u^* is asymptotically stable when $\upsilon = 6 < \upsilon_0^c = 6.1794$ as illustrated in Figure 5, and when $\upsilon = 7 > \upsilon_0^c = 6.1794$, there occurs a Hopf bifurcation at the positive equilibrium point u^* as demonstrated in Figure 6.

6 Conclusions

In this paper, we have studied the Hopf bifurcation of a fractional-order red blood cells model with time delay and have proposed the configuration for the stable region. In order to control the Hopf bifurcation of the delayed fractional-order red blood cells model, we have designed a fractional-order Proportional and Derivative (PD^α) controller, which

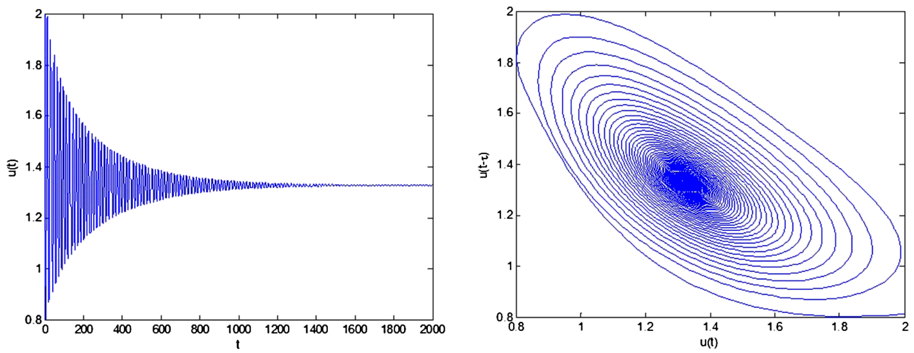

Figure 5 Waveform plot and phase portrait of the controlled fractional-order model (4.2) with $\alpha = 0.92$, $m = 0.2$, $k_p = 0.05$, $k_d = 0.5$ and the initial value $u_0 = 2$. The equilibrium u^* is asymptotically stable when $\upsilon = 6 < \upsilon_0^c = 6.1794$.

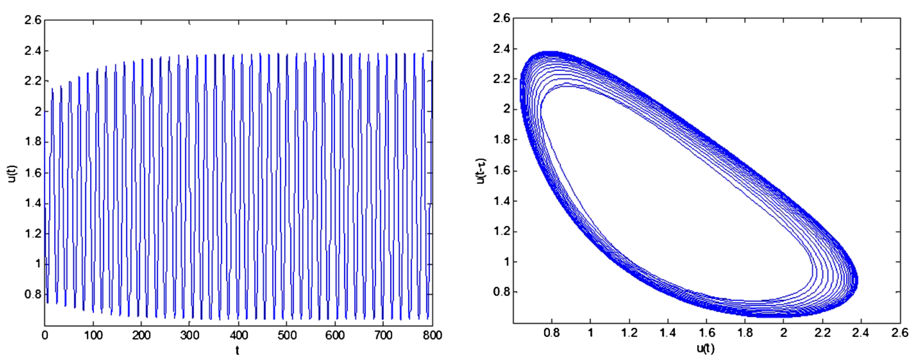

Figure 6 Waveform plot and phase portrait of the controlled fractional-order model (4.2) with $\alpha = 0.92$, $m = 0.2$, $k_p = 0.05$, $k_d = 0.5$ and the initial value $u_0 = 2$. A Hopf bifurcation occurs and the periodic oscillation bifurcates from u^*, where $\upsilon = 7 > \upsilon_0^c = 6.1794$.

can successfully delay or advance the onset of Hopf bifurcation. Therefore, we can choose appropriate values of the Proportional and Derivative parameters to change the characteristics of Hopf bifurcation embedding in fractional-order systems with time delays.

Acknowledgements
This work is partly supported by the National Natural Science Foundation of China (Nos. 61573194, 61573096), the Six Talent Peaks High Level Project of Jiangsu Province (No. 2014-ZNDW-004) and the Science Foundation of Nanjing University of Posts and Telecommunications (No. NY213095).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]College of Automation, Nanjing University of Posts and Telecommunications, Nanjing, 210003, China. [2]School of Mathematics, Southeast University, Nanjing, 210096, China. [3]School of Mathematics and Statistics, Shandong Normal University, Ji'nan, 250014, China. [4]College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing, 400067, China. [5]School of Mathematics and Computer Science, Hubei University of Arts and Science, Xiangyang, 441053, China.

Received: 19 October 2017 Accepted: 28 December 2017 Published online: 10 January 2018

References

1. Xiao, M, Zheng, WX, Jiang, GP, Cao, JD: Undamped oscillations generated by Hopf bifurcations in fractional-order recurrent neural networks with Caputo derivative. IEEE Trans. Neural Netw. Learn. Syst. **26**, 3201-3214 (2015)
2. Laskin, N: Fractional market dynamics. Physica A **287**, 482-492 (2000)
3. Djordjević, VD, Jarić, J, Fabry, B, Fredberg, JJ, Stamenović, D: Fractional derivatives embody essential features of cell rheological behavior. Ann. Biomed. Eng. **31**, 692-699 (2003)
4. Arafa, AAM, Rida, SZ, Khalil, M: Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection. Nonlinear Biomed. Phys. **6**, 1 (2012)
5. Rihan, FA: Numerical modeling of fractional-order biological systems. Abstr. Appl. Anal. **2013**, 8 (2013)
6. Zhang, CR, Zu, YG, Zheng, BD: Stability and bifurcation of a discrete red blood cell survival model. Chaos Solitons Fractals **28**, 386-394 (2006)
7. Song, YL, Wei, JJ, Yuan, Y: Bifurcation analysis on a survival red blood cells model. J. Math. Anal. Appl. **316**, 459-471 (2006)
8. Fan, DJ, Wei, JJ: Bifurcation analysis of discrete survival red blood cells model. Commun. Nonlinear Sci. Numer. Simul. **14**, 3358-3368 (2009)
9. Yin, XW, Thomas, T, Zhang, JF: Multiple red blood cell flows through microvascular bifurcations: cell free layer, cell trajectory, and hematocrit separation. Microvasc. Res. **89**, 47-56 (2013)
10. Magin, RL: Fractional calculus in bioengineering. Crit. Rev. Biomed. Eng. **32**, 1-104 (2004)
11. Magin, RL: Fractional calculus models of complex dynamics in biological tissues. Comput. Math. Appl. **59**, 1586-1593 (2010)
12. Craiem, D, Magin, RL: Fractional order models of viscoelasticity as an alternative in the analysis of red blood cell (RBC) membrane mechanics. Phys. Biol. **7**(1), 13001 (2010)
13. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
14. Cheng, ZS, Cao, JD: Hybrid control of Hopf bifurcation in complex networks with delays. Neurocomputing **133**, 164-170 (2014)
15. Huang, CD, Cao, JD, Xiao, M: Hybrid control on bifurcation for a delayed fractional gene regulatory network. Chaos Solitons Fractals **87**, 19-29 (2016)
16. Yang, XS, Cao, JD, Ho, DWC: Exponential synchronization of discontinuous neural networks with time-varying mixed delays via state feedback and impulsive control. Cogn. Neurodyn. **9**, 113-128 (2015)
17. Wang, H, Zhu, QX: Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form. Automatica **54**, 284-291 (2015)
18. Zhao, HY, Xie, W: Hopf bifurcation for a small-world network model with parameters delay feedback control. Nonlinear Dyn. **63**, 345-357 (2011)
19. Fiedler, B, Oliva, SM: Delayed feedback control of a delay equation at Hopf bifurcation. J. Dyn. Differ. Equ. **28**, 1357-1391 (2016)
20. Zhusubaliyev, ZT, Medvedev, A, Silva, MM: Bifurcation analysis of PID-controlled neuromuscular blockade in closed-loop anesthesia. J. Process Control **25**, 152-163 (2015)
21. Wazewska-Czyzewska, M, Lasota, A: Mathematical problems of the dynamics of the red blood cells system. Mat. Stosow. **6**, 23-40 (1976)
22. Deng, WH, Li, CP, Lü, JH: Stability analysis of linear fractional differential system with multiple time delays. Nonlinear Dyn. **48**, 409-416 (2007)
23. Matignon, D: Stability results for fractional differential equations with applications to control processing. In: IMACS-SMC Multi-Conf. Comput. Eng. Syst. Appl., pp. 963-968 (1996)