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Some new integral inequalities with mixed nonlinearities for discontinuous functions

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Abstract

In this paper, we establish some new integral inequalities with mixed nonlinearities for discontinuous functions, which provide a handy tool in deriving the explicit bounds for the solutions of impulsive differential equations and differential-integral equations with impulsive conditions.

Keywords: integral inequalities; discontinuous functions; mixed nonlinearities; impulsive differential equations

1 Introduction

In recent years, the theory of impulsive differential systems has been attracting the attention of many mathematicians, and the interest in the subject is still growing. This is partly due to broad applications of it in many areas including threshold theory in biology, ecosystems management and orbital transfer of satellite, see [1]. One effective method for investigating the properties of solutions to impulsive differential systems is related to the integral inequalities for discontinuous functions (integro-sum inequalities). Up to now, a lot of integro-sum inequalities (for example, [2–18] and the references therein) have been discovered. For example, in 2003, Borysenko [3] considered the following integro-sum inequality:

$$x(t) \le a(t) + \int_{t_0}^{t} q(\tau)x^m(\tau)\,\mathrm{d}\tau + \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0), \quad m > 0, m \ne 1.$$

In 2009, Gallo and Piccirillo [8] further discussed the following nonlinear integro-sum inequality:

$$x(t) \le c(t) + h(t)\int_{t_0}^{t} f(s)w\bigl(x\bigl(b(s)\bigr)\bigr)\,\mathrm{d}s + \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0), \quad m > 0.$$

In 2012, Wang et al. [17] considered the nonlinear integro-sum inequality as follows:

$$\begin{aligned} x^m(t) \le{}& c(t) + 2\int_{\alpha(t_0)}^{\alpha(t)} \bigl[M_1 f_1(t,s)u^{\frac{m}{2}}(s) + N_1 g_1(t,s)u^m(s)\bigr]\,\mathrm{d}s \\ &+ 2\int_{t_0}^{t} \bigl[M_2 f_2(t,s)u^{\frac{m}{2}}(s) + N_2 g_2(t,s)u^m(s)\bigr]\,\mathrm{d}s + \sum_{t_0<t_i<t} \beta_i x(t_i - 0), \quad m > 0. \end{aligned}$$

Very recently, in 2016, Zheng et al. [18] considered the following nonlinear integro-sum inequality under the condition $p > q > 0$:

$$x^p(t) \le a_0(t) + \frac{p-q}{p} \sum_{i=1}^{N} \int_{t_0}^{t} g_i(s) x^q\big(\phi_i(s)\big)\,\mathrm{d}s + \sum_{j=1}^{L} \int_{t_0}^{t} b_j(s) \int_{t_0}^{s} c_j(\theta) x^q\big(w_j(s)\big)\,\mathrm{d}\theta\,\mathrm{d}s + \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0).$$

Motivated by [3, 8, 17, 18], in this paper, we investigate some new integro-sum inequality with mixed nonlinearities under the condition $p > 0$, $q > 0$ ($p \neq q$):

$$x^p(t) \le a(t) + \int_{t_0}^{t} f_1(s) x^q(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s) \int_{t_0}^{s} g_1(\tau) x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_0}^{t} f_3(s) \int_{t_0}^{s} g_2(\tau) x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + c(t) \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0)$$

and the more general form

$$x^p(t) \le a(t) + \int_{t_0}^{t} f(s) x^q(s)\,\mathrm{d}s + \sum_{j=1}^{L} \int_{t_0}^{t} b_j(s) \int_{t_0}^{s} g_j(\tau) x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \sum_{k=1}^{M} \int_{t_0}^{t} c_k(s) \int_{t_0}^{s} \theta_k(\tau) x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + d(t) \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0).$$

We also discuss some nonlinear integro-sum inequality with positive and negative coefficients under the condition $0 < q < p < r$:

$$x^p(t) \le a(t) + b(t) \int_{t_0}^{t} \big[f(s) x^p(s) + g(s) x^q(s) - h(s) x^r(s)\big]\,\mathrm{d}s + c(t) \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0),$$

and the more general form under the condition $0 < q_j < p < r_j$ ($j = 1, 2, \ldots, L$):

$$x^p(t) \le a(t) + \int_{t_0}^{t} f(s) x^p(s)\,\mathrm{d}s + \sum_{j=1}^{L} \int_{t_0}^{t} g_j(s) x^{q_j}(s)\,\mathrm{d}s - \sum_{j=1}^{L} \int_{t_0}^{t} h_j(s) x^{r_j}(s)\,\mathrm{d}s + c(t) \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0).$$

Based on these inequalities, we provide explicit bounds for unknown functions concerned and then apply the results to research the qualitative properties of solutions of certain impulsive differential equations.

2 Preliminaries

Throughout the present paper, R denotes the set of real numbers; $\mathrm{R}_+ = [0, +\infty)$ is the subset of R; $C(D, E)$ denotes the class of all continuous functions defined on the set D with range in the set E.

Lemma 2.1 ([19]) *Assume that the following conditions for $t \geq t_0$ hold*:

(i) x_0 *is a nonnegative constant,*

(ii)
$$x(t) \leq x_0 + \int_{t_0}^{t} \bigl[e(s)x(s) + l(s)x^{\alpha}(s)\bigr]\,\mathrm{d}s,$$

where x, e and l are nonnegative continuous functions and $\alpha \neq 1$ is a positive constant.

If

$$1 + (1-\alpha)x_0^{(\alpha-1)} \int_{t_0}^{t} l(s)\exp\biggl((\alpha-1)\int_{t_0}^{s} e(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s > 0$$

holds, then

$$x(t) \leq x_0 \exp\biggl(\int_{t_0}^{t} e(s)\,\mathrm{d}s\biggr) \times \biggl\{1 + (1-\alpha)x_0^{(\alpha-1)} \int_{t_0}^{t} l(s)\exp\biggl((\alpha-1)\int_{t_0}^{s} e(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\biggr\}^{\frac{1}{1-\alpha}}, \quad t \geq t_0.$$

Lemma 2.2 ([20]) *Let x be a nonnegative function, $0 < q < p < r$, $c_1 \geq 0$, $k_2 \geq 0$, $c_2 > 0$ and $k_1 > 0$. Then*

$$c_1 x^q - c_2 x^r \leq (k_1 - k_2)x^p + \theta_1(p,q,c_1,k_1) + \theta_2(p,r,c_2,k_2),$$

where

$$\theta_1(p,q,c_1,k_1) := \frac{p-q}{q}\biggl(\frac{q}{p}\biggr)^{\frac{p}{p-q}} c_1^{\frac{p}{p-q}} k_1^{\frac{-q}{p-q}}, \qquad \theta_2(p,r,c_2,k_2) := \frac{r-p}{r}\biggl(\frac{p}{r}\biggr)^{\frac{p}{r-p}} c_2^{\frac{-p}{r-p}} k_2^{\frac{r}{r-p}}.$$

3 Main results

Theorem 3.1 *Suppose that x is a nonnegative piecewise continuous function defined on $[t_0,\infty)$ with discontinuities of the first kind in the points t_i ($i = 1,2,\ldots$) and satisfies the integro-sum inequality*

$$x^p(t) \leq a(t) + \int_{t_0}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_0}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + c(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i-0), \quad t \geq t_0, \tag{3.1}$$

where $0 \leq t_0 < t_1 < t_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, functions $a(t) \geq 0$ and $c(t) \geq 0$ are defined on $[t_0,\infty)$, $f_1,f_2,f_3,g_1,g_2 \in C(\mathrm{R}_+,\mathrm{R}_+)$, $\beta_i \geq 0$ ($i = 1,2,\ldots$), $p > 0$, $q > 0$, $p \neq q$ and $m > 0$ are constants. If

$$1 + \frac{p-q}{p} r_i^{\frac{q-p}{p}}(t) \int_{t_{i-1}}^{t} l(s)\exp\biggl(\frac{q-p}{p}\int_{t_{i-1}}^{s} e(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s > 0, \quad i = 1,2,\ldots,$$

then, for $t \geq t_0$, the following estimates hold:

$$x(t) \leq v_1(t), \quad t \in [t_0, t_1], \tag{3.2}$$

$$x(t) \leq v_i(t), \quad t \in (t_{i-1}, t_i], i = 2, 3, \ldots, \tag{3.3}$$

where

$$v_i(t) = r_i^{\frac{1}{p}}(t) \exp\left(\frac{1}{p}\int_{t_{i-1}}^{t} e(s)\,\mathrm{d}s\right)$$
$$\times \left\{1 + \frac{p-q}{p} r_i^{\frac{q-p}{p}}(t) \int_{t_{i-1}}^{t} l(s) \exp\left(\frac{q-p}{p}\int_{t_{i-1}}^{s} e(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}^{\frac{1}{p-q}}, \quad i = 1, 2, \ldots,$$

$$e(t) = f_2(t)\int_{t_0}^{t} g_1(\tau)\,\mathrm{d}\tau, \qquad l(t) = f_1(t) + f_3(t)\int_{t_0}^{t} g_2(\tau)\,\mathrm{d}\tau, \tag{3.4}$$

$$r_1(t) = \max_{t_0 \leq \tau \leq t} \bigl|a(\tau)\bigr|, \qquad h(t) = \max_{t_0 \leq \tau \leq t} \bigl|c(\tau)\bigr|, \tag{3.5}$$

$$r_{i+1}(t) = r_i(t) + \int_{t_{i-1}}^{t_i} f_1(s) v_i^q(s)\,\mathrm{d}s + \int_{t_{i-1}}^{t} \left(f_2(s)\int_{t_{i-1}}^{t_i} g_1(\tau) v_i^p(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$+ \int_{t_{i-1}}^{t} \left(f_3(s)\int_{t_{i-1}}^{t_i} g_2(\tau) v_i^q(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + h(t)\beta_i v_i^m(t_i - 0), \quad i = 1, 2, \ldots.$$

Proof From (3.1) and (3.5), we have, for $t \in I_0 = [t_0, t_1]$,

$$x^p(t) \leq r_1(t) + \int_{t_0}^{t} f_1(s) x^q(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau) x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$
$$+ \int_{t_0}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau) x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \tag{3.6}$$

and $r_1(t)$ is non-decreasing on $[t_0, \infty)$. Take any fixed $T \in [t_0, t_1]$, and for arbitrary $t \in [t_0, T]$, we have

$$x^p(t) \leq r_1(T) + \int_{t_0}^{t} f_1(s) x^q(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau) x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$
$$+ \int_{t_0}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau) x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s. \tag{3.7}$$

Let $u(t) = x^p(t)$. Inequality (3.7) is equivalent to

$$u(t) \leq r_1(T) + \int_{t_0}^{t} f_1(s) u^{\frac{q}{p}}(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau) u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$
$$+ \int_{t_0}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau) u^{\frac{q}{p}}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s. \tag{3.8}$$

Let

$$V(t) = r_1(T) + \int_{t_0}^{t} f_1(s) u^{\frac{q}{p}}(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau) u(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$
$$+ \int_{t_0}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau) u^{\frac{q}{p}}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$u(t) \le V(t), \qquad V(t_0) = r_1(T), \tag{3.10}$$

$V(t)$ is non-decreasing and

$$V'(t) = f_1(t)u^{\frac{q}{p}}(t) + f_2(t)\int_{t_0}^{t} g_1(\tau)u(\tau)\,\mathrm{d}\tau + f_3(t)\int_{t_0}^{t} g_2(\tau)u^{\frac{q}{p}}(\tau)\,\mathrm{d}\tau. \tag{3.11}$$

Since $V(t)$ is non-decreasing, from (3.11) we have

$$\begin{aligned} V'(t) &\le f_1(t)V^{\frac{q}{p}}(t) + f_2(t)\int_{t_0}^{t} g_1(\tau)V(\tau)\,\mathrm{d}\tau + f_3(t)\int_{t_0}^{t} g_2(\tau)V^{\frac{q}{p}}(\tau)\,\mathrm{d}\tau \\ &\le f_1(t)V^{\frac{q}{p}}(t) + f_2(t)\int_{t_0}^{t} g_1(\tau)\,\mathrm{d}\tau\, V(t) + f_3(t)\int_{t_0}^{t} g_2(\tau)\,\mathrm{d}\tau\, V^{\frac{q}{p}}(t) \\ &\le e(t)V(t) + l(t)V^{\frac{q}{p}}(t), \end{aligned} \tag{3.12}$$

where $e(t)$ and $l(t)$ are defined as in (3.4). Integrating (3.12) from t_0 to t yields

$$V(t) \le r_1(T) + \int_{t_0}^{t} \big[e(s)V(s) + l(s)V^{\frac{q}{p}}(s)\big]\,\mathrm{d}s.$$

From the above and Lemma 2.1, we get

$$V(t) \le r_1(T)\exp\biggl(\int_{t_0}^{t} e(s)\,\mathrm{d}s\biggr)\biggl\{1 + \frac{p-q}{p} r_1^{\frac{q-p}{p}}(T)\int_{t_0}^{t} l(s)\exp\biggl(\frac{q-p}{p}\int_{t_0}^{s} e(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\biggr\}^{\frac{p}{p-q}},$$

and then from (3.10) and the assumption $u(t) = x^p(t)$, we have

$$\begin{aligned} x(t) &\le r_1^{\frac{1}{p}}(T)\exp\biggl(\frac{1}{p}\int_{t_0}^{t} e(s)\,\mathrm{d}s\biggr) \\ &\quad\times \biggl\{1 + \frac{p-q}{p} r_1^{\frac{q-p}{p}}(T)\int_{t_0}^{t} l(s)\exp\biggl(\frac{q-p}{p}\int_{t_0}^{s} e(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\biggr\}^{\frac{1}{p-q}}. \end{aligned}$$

Since the above inequality is true for any $t \in [t_0, T]$, we obtain

$$\begin{aligned} x(T) &\le r_1^{\frac{1}{p}}(T)\exp\biggl(\frac{1}{p}\int_{t_0}^{T} e(s)\,\mathrm{d}s\biggr) \\ &\quad\times \biggl\{1 + \frac{p-q}{p} r_1^{\frac{q-p}{p}}(T)\int_{t_0}^{T} l(s)\exp\biggl(\frac{q-p}{p}\int_{t_0}^{s} e(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\biggr\}^{\frac{1}{p-q}}. \end{aligned}$$

Replacing T by t yields

$$\begin{aligned} x(t) &\le r_1^{\frac{1}{p}}(t)\exp\biggl(\frac{1}{p}\int_{t_0}^{t} e(s)\,\mathrm{d}s\biggr) \\ &\quad\times \biggl\{1 + \frac{p-q}{p} r_1^{\frac{q-p}{p}}(t)\int_{t_0}^{t} l(s)\exp\biggl(\frac{q-p}{p}\int_{t_0}^{s} e(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\biggr\}^{\frac{1}{p-q}} \\ &= v_1(t), \quad t \in I_0 = [t_0, t_1]. \end{aligned} \tag{3.13}$$

This means that (3.1) is true.

For $t \in I_1 = (t_1, t_2]$, from (3.1), (3.2), (3.5) and (3.13), we get

$$\begin{aligned}
x^p(t) &\le r_1(t) + \int_{t_0}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + h(t)\beta_1 x^m(t_1-0) \\
&= r_1(t) + \int_{t_0}^{t_1} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_1}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_0}^{t_1} f_2(s)\int_{t_0}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_0}^{t_1} f_3(s)\int_{t_0}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + h(t)\beta_1 x^m(t_1-0) \\
&\le r_1(t) + \int_{t_0}^{t_1} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_1}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_0}^{t_1} f_2(s)\int_{t_0}^{t_1} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_2(s)\int_{t_0}^{t_1} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_1}^{t} f_2(s)\int_{t_1}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t_1} f_3(s)\int_{t_0}^{t_1} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_1}^{t} f_3(s)\int_{t_0}^{t_1} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_3(s)\int_{t_1}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + h(t)\beta_1 x^m(t_1-0) \\
&\le r_1(t) + \int_{t_0}^{t_1} f_1(s)v_1^q(s)\,\mathrm{d}s + \int_{t_1}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_0}^{t_1} f_2(s)\int_{t_0}^{t_1} g_1(\tau)v_1^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_2(s)\int_{t_0}^{t_1} g_1(\tau)v_1^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_1}^{t} f_2(s)\int_{t_1}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t_1} f_3(s)\int_{t_0}^{t_1} g_2(\tau)v_1^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_1}^{t} f_3(s)\int_{t_0}^{t_1} g_2(\tau)v_1^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_3(s)\int_{t_1}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + h(t)\beta_1 v_1^m(t_1-0) \\
&= r_1(t) + \int_{t_0}^{t_1} f_1(s)v_1^q(s)\,\mathrm{d}s + \int_{t_1}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{t_1} g_1(\tau)v_1^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_2(s)\int_{t_1}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_0}^{t} f_3(s)\int_{t_0}^{t_1} g_2(\tau)v_1^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_3(s)\int_{t_1}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + h(t)\beta_1 v_1^m(t_1-0) \\
&= r_2(t) + \int_{t_1}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_1}^{t} f_2(s)\int_{t_1}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} f_3(s)\int_{t_1}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned} \tag{3.14}$$

Inequality (3.14) is the same as (3.6) if we replace $r_1(t)$ and t_0 with $r_2(t)$ and t_1 in (3.6), respectively. Thus, by (3.14), we have, for $t \in I_1 = (t_1, t_2]$,

$$\begin{aligned}
x(t) &\le r_2^{\frac{1}{p}}(t)\exp\left(\frac{1}{p}\int_{t_1}^{t} e(s)\,\mathrm{d}s\right) \\
&\quad \times \left\{1 + \frac{p-q}{p} r_2^{\frac{q-p}{p}}(t)\int_{t_1}^{t} l(s)\exp\left(\frac{q-p}{p}\int_{t_1}^{s} e(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}^{\frac{1}{p-q}} = v_2(t).
\end{aligned}$$

Suppose that

$$x(t) \le r_i^{\frac{1}{p}}(t)\exp\left(\frac{1}{p}\int_{t_{i-1}}^{t} e(s)\,\mathrm{d}s\right) \times \left\{1+\frac{p-q}{p} r_i^{\frac{q-p}{p}}(t)\int_{t_{i-1}}^{t} l(s)\exp\left(\frac{q-p}{p}\int_{t_{i-1}}^{s} e(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}^{\frac{1}{p-q}} = v_i(t) \tag{3.15}$$

holds for $t \in I_{i-1} = (t_{i-1}, t_i]$, $i = 2, 3, \ldots$. Then, for $t \in I_i = (t_i, t_{i+1}]$, from (3.1), (3.2), (3.5) and (3.15) we obtain

$$\begin{aligned}
x^p(t) &\le r_1(t) + \int_{t_0}^{t} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_0}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + h(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i-0) \\
&= r_1(t) + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_i}^{t} f_1(s)x^q(s)\,\mathrm{d}s \\
&\quad + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} f_2(s)\int_{t_0}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_i}^{t} f_2(s)\int_{t_0}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} f_3(s)\int_{t_0}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{t_i}^{t} f_3(s)\int_{t_0}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + h(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i-0) \\
&\le r_1(t) + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} f_1(s)x^q(s)\,\mathrm{d}s + \int_{t_i}^{t} f_1(s)x^q(s)\,\mathrm{d}s \\
&\quad + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}}\left(f_2(s)\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \int_{t_i}^{t}\left(f_2(s)\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_{t_i}^{t} f_2(s)\int_{t_i}^{s} g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}}\left(f_3(s)\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} g_2(\tau)x^q(\tau)\right)\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + \int_{t_i}^{t}\left(f_3(s)\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_{t_i}^{t} f_3(s)\int_{t_i}^{s} g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad + h(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i-0) \\
&\le r_1(t) + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} f_1(s)v_{k+1}^q(s)\,\mathrm{d}s + \int_{t_i}^{t} f_1(s)x^q(s)\,\mathrm{d}s \\
&\quad + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}}\left(f_2(s)\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} g_1(\tau)v_{j+1}^p(\tau)\right)\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}$$

$$+\int_{t_i}^{t}\left(f_2(s)\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}g_1(\tau)v_{j+1}^{p}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s+\int_{t_i}^{t}f_2(s)\int_{t_i}^{s}g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

$$+\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}}\left(f_3(s)\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}g_2(\tau)v_{j+1}^{q}(\tau)\right)\mathrm{d}\tau\,\mathrm{d}s$$

$$+\int_{t_i}^{t}\left(f_3(s)\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}g_2(\tau)v_{j+1}^{q}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s+\int_{t_i}^{t}f_3(s)\int_{t_i}^{s}g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

$$+h(t)\sum_{t_0<t_i<t}\beta_i v_i^m(t_i-0)$$

$$=r_{i+1}(t)+\int_{t_i}^{t}f_1(s)x^q(s)\,\mathrm{d}s+\int_{t_i}^{t}f_2(s)\int_{t_i}^{s}g_1(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

$$+\int_{t_i}^{t}f_3(s)\int_{t_i}^{s}g_2(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s. \tag{3.16}$$

Inequality (3.16) is the same as (3.6) if we replace $r_1(t)$ and t_0 with $r_{i+1}(t)$ and t_i in (3.6), respectively. Thus, by (3.16), we have, for $t \in I_i = (t_i, t_{i+1}]$,

$$x(t)\le r_{i+1}^{\frac{1}{p}}(t)\exp\left(\frac{1}{p}\int_{t_i}^{t}e(s)\,\mathrm{d}s\right)$$

$$\times\left\{1+\frac{p-q}{p}r_{i+1}^{\frac{q-p}{p}}(t)\int_{t_i}^{t}l(s)\exp\left(\frac{q-p}{p}\int_{t_i}^{s}e(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}^{\frac{1}{p-q}}.$$

By induction, we know that (3.3) holds for $t \in (t_i, t_{i+1}]$, for any nonnegative integer i. This completes the proof of Theorem 3.1. □

Theorem 3.2 *Suppose that x is a nonnegative piecewise continuous function defined on $[t_0,\infty)$ with discontinuities of the first kind in the points t_i ($i = 1,2,\ldots$) and satisfies the integro-sum inequality*

$$x^p(t)\le a(t)+\int_{t_0}^{t}f(s)x^q(s)\,\mathrm{d}s+\sum_{j=1}^{L}\int_{t_0}^{t}b_j(s)\int_{t_0}^{s}g_j(\tau)x^p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

$$+\sum_{k=1}^{M}\int_{t_0}^{t}c_k(s)\int_{t_0}^{s}\theta_k(\tau)x^q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s+d(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i-0),\quad t\ge t_0, \tag{3.17}$$

where $0 \le t_0 < t_1 < t_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, $a(t) \ge 0$ *is defined on* $[t_0,\infty)$, $f \in C(\mathrm{R}_+,\mathrm{R}_+)$, $b_j, g_j \in C(\mathrm{R}_+,\mathrm{R}_+)$ $(j = 1,2,\ldots,L)$, $c_j, \theta_j \in C(\mathrm{R}_+,\mathrm{R}_+)$ $(j = 1,2,\ldots,M)$, $\beta_i \ge 0$ $(i = 1,2,\ldots)$, $p > 0$, $q > 0$, $p \ne q$, *and* $m > 0$ *are constants. If*

$$1+\frac{p-q}{p}r_i^{\frac{q-p}{p}}(t)\int_{t_{i-1}}^{t}l(s)\exp\left(\frac{q-p}{p}\int_{t_{i-1}}^{s}e(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s>0,\quad i=1,2,\ldots,$$

then, for $t \ge t_0$, the following estimates hold:

$$x(t)\le v_1(t),\quad t\in[t_0,t_1], \tag{3.18}$$

$$x(t)\le v_i(t),\quad t\in(t_{i-1},t_i],\ i=2,3,\ldots, \tag{3.19}$$

where

$$v_i(t) = r_i^{\frac{1}{p}}(t)\exp\left(\frac{1}{p}\int_{t_{i-1}}^{t} e(s)\,\mathrm{d}s\right)$$
$$\times\left\{1 + \frac{p-q}{p} r_i^{\frac{q-p}{p}}(t)\int_{t_{i-1}}^{t} l(s)\exp\left(\frac{q-p}{p}\int_{t_{i-1}}^{s} e(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}^{\frac{1}{p-q}},$$
$$i = 1, 2, \ldots,$$

$$e(t) = \sum_{j=1}^{L} b_j(t)\int_{t_0}^{t} g_j(\tau)\,\mathrm{d}\tau, \qquad l(t) = f(t) + \sum_{k=1}^{M} c_k(t)\int_{t_0}^{t}\theta_k(\tau)\,\mathrm{d}\tau,$$

$$r_1(t) = \max_{t_0\le\tau\le t}\left|a(\tau)\right|, \qquad h(t) = \max_{t_0\le\tau\le t}\left|d(\tau)\right|,$$

$$r_{i+1}(t) = r_i(t) + \int_{t_{i-1}}^{t_i} f(s)v_i^q(s)\,\mathrm{d}s + \sum_{j=1}^{L}\int_{t_{i-1}}^{t}\left(b_j(s)\int_{t_{i-1}}^{t_i} g_j(\tau)v_i^p(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s$$
$$+ \sum_{k=1}^{M}\int_{t_{i-1}}^{t}\left(c_k(s)\int_{t_{i-1}}^{t_i}\theta_k(\tau)v_i^q(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + h(t)\beta_i v_i^m(t_i - 0), \quad i = 1, 2, \ldots.$$

The proof is similar to that of Theorem 3.1, and we omit these details.

Theorem 3.3 *Suppose that x is a nonnegative piecewise continuous function defined on $[t_0,\infty)$ with discontinuities of the first kind in the points t_i ($i = 1, 2, \ldots$) and satisfies the integro-sum inequality*:

$$x^p(t) \le a(t) + b(t)\int_{t_0}^{t}\left[f(s)x^p(s) + g(s)x^q(s) - h(s)x^r(s)\right]\mathrm{d}s$$
$$+ c(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i - 0), \quad t \ge t_0, \tag{3.20}$$

where $0 \le t_0 < t_1 < t_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, *$a(t)$ is defined on $[t_0,\infty)$ and $a(t_0) \ne 0$, $b(t) \ge 0$ and $c(t) \ge 0$ are defined on $[t_0,\infty)$, $f, g \in C(\mathrm{R}_+, \mathrm{R}_+)$, $h \in C(\mathrm{R}_+, (0, +\infty))$, $0 < q < p < r$, $\beta_i \ge 0$ ($i = 1, 2, \ldots$) and $m > 0$ are constants.*

Then, for any continuous functions $k_1(t) > 0$ and $k_2(t) \ge 0$ on $[t_0,\infty)$ satisfying $k(t) = k_1(t) - k_2(t) \ge 0$, the following estimates hold:

$$x(t) \le v_1(t), \quad t \in [t_0, t_1], \tag{3.21}$$

$$x(t) \le v_i(t), \quad t \in (t_{i-1}, t_i], i = 2, 3, \ldots, \tag{3.22}$$

where

$$v_i(t) = r_i^{\frac{1}{p}}(t)\exp\left\{\frac{1}{p}e(t)\int_{t_{i-1}}^{t}\left[f(s) + k(s)\right]\mathrm{d}s\right\}, \quad i = 1, 2, \ldots, \tag{3.23}$$

$$d(t) = \max_{t_0\le\tau\le t}\left|a(\tau)\right|, \qquad e(t) = \max_{t_0\le\tau\le t}\left|b(\tau)\right|, \qquad l(t) = \max_{t_0\le\tau\le t}\left|c(\tau)\right|, \tag{3.24}$$

$$r_1(t) = d(t) + e(t)w(t),$$
$$w(t) = \int_{t_0}^{t}\left[\theta_1\big(p, q, g(s), k_1(s)\big) + \theta_2\big(p, r, h(s), k_2(s)\big)\right]\mathrm{d}s, \tag{3.25}$$

$$\theta_1\big(p,q,g(s),k_1(s)\big) = \frac{p-q}{q}\left(\frac{q}{p}\right)^{\frac{p}{p-q}} g^{\frac{p}{p-q}}(s)k_1^{\frac{-q}{p-q}}(s), \tag{3.26}$$

$$\theta_2\big(p,r,h(s),k_2(s)\big) = \frac{r-p}{r}\left(\frac{p}{r}\right)^{\frac{p}{r-p}} h^{\frac{-p}{r-p}}(s)k_2^{\frac{r}{r-p}}(s), \tag{3.27}$$

$$r_{i+1}(t) = r_i(t) + e(t)\int_{t_{i-1}}^{t_i}\big[f(s)+k(s)\big]v_i^p(s)\,\mathrm{d}s + l(t)\beta_i v_i^m(t_i-0), \quad i=1,2,\ldots. \tag{3.28}$$

Proof From (3.20) and (3.24), we obtain, for $t \in I_0 = [t_0, t_1]$,

$$x^p(t) \le d(t) + e(t)\int_{t_0}^t \big[f(s)x^p(s) + g(s)x^q(s) - h(s)x^r(s)\big]\,\mathrm{d}s. \tag{3.29}$$

From Lemma 2.1, (3.24)-(3.27) and (3.29), we have

$$\begin{aligned} x^p(t) &\le d(t) + e(t)\int_{t_0}^t \big[\big[f(s)+k(s)\big]x^p(s) + \theta_1\big(p,q,g(s),k_1(s)\big) + \theta_2\big(p,r,h(s),k_2(s)\big)\big]\,\mathrm{d}s \\ &= d(t) + e(t)\int_{t_0}^t \big[\theta_1\big(p,q,g(s),k_1(s)\big) + \theta_2\big(p,r,h(s),k_2(s)\big)\big]\,\mathrm{d}s \\ &\quad + e(t)\int_{t_0}^t \big[f(s)+k(s)\big]x^p(s)\,\mathrm{d}s \\ &= d(t) + e(t)w(t) + e(t)\int_{t_0}^t \big[f(s)+k(s)\big]x^p(s)\,\mathrm{d}s \\ &= r_1(t) + e(t)\int_{t_0}^t \big[f(s)+k(s)\big]x^p(s)\,\mathrm{d}s, \end{aligned} \tag{3.30}$$

$r_1(t)$ and $e(t)$ are non-decreasing on $[t_0,\infty)$. Take any fixed $T \in [t_0,t_1]$, and for arbitrary $t \in [t_0,T]$, we have

$$x^p(t) \le r_1(T) + e(T)\int_{t_0}^t \big[f(s)+k(s)\big]x^p(s)\,\mathrm{d}s. \tag{3.31}$$

Let $u(t) = x^p(t)$. Inequality (3.31) is equivalent to

$$u(t) \le r_1(T) + e(T)\int_{t_0}^t \big[f(s)+k(s)\big]u(s)\,\mathrm{d}s. \tag{3.32}$$

Define a function $V(t)$ by the right-hand side of (3.32). Then $V(t)$ is positive and

$$V(t_0) = r_1(T), \qquad u(t) \le V(t), \tag{3.33}$$

$$V'(t) = e(T)\big[f(t)+k(t)\big]u(t) \le e(T)\big[f(t)+k(t)\big]V(t), \quad t \in [t_0,T].$$

We have

$$\begin{aligned} V(t) &\le V(t_0)\exp\left\{e(T)\int_{t_0}^t \big[f(s)+k(s)\big]\,\mathrm{d}s\right\} \\ &= r_1(T)\exp\left\{e(T)\int_{t_0}^t \big[f(s)+k(s)\big]\,\mathrm{d}s\right\}, \end{aligned} \tag{3.34}$$

and then, from (3.33), (3.34) and the assumption $u(t) = x^p(t)$, we get

$$x(t) \le r_1^{\frac{1}{p}}(T) \exp\left\{ \frac{1}{p} e(T) \int_{t_0}^{t} [f(s) + k(s)] \,\mathrm{d}s \right\}.$$

Since the above inequality is true for any $t \in [t_0, T]$, we obtain

$$x(T) \le r_1^{\frac{1}{p}}(T) \exp\left\{ \frac{1}{p} e(T) \int_{t_0}^{T} [f(s) + k(s)] \,\mathrm{d}s \right\}.$$

Replacing T by t yields

$$x(t) \le r_1^{\frac{1}{p}}(t) \exp\left\{ \frac{1}{p} e(t) \int_{t_0}^{t} [f(s) + k(s)] \,\mathrm{d}s \right\} = v_1(t), \quad t \in I_0 = [t_0, t_1]. \tag{3.35}$$

This means that (3.21) is true for $t \in [t_0, t_1]$.

For $t \in I_1 = (t_1, t_2]$, from Lemma 2.1 and (3.20), (2.24)-(2.27) and (3.35), we obtain

$$\begin{aligned} x^p(t) &\le d(t) + e(t) \int_{t_0}^{t} [f(s)x^p(s) + g(s)x^q(s) - h(s)x^r(s)] \,\mathrm{d}s + l(t)\beta_1 x^m(t_1 - 0) \\ &\le d(t) + e(t) \int_{t_0}^{t} \big[[f(s) + k(s)]x^p(s) + \theta_1(p, q, g(s), k_1(s)) + \theta_2(p, r, h(s), k_2(s))\big] \,\mathrm{d}s \\ &\quad + l(t)\beta_1 v_1^m(t_1 - 0) \\ &= d(t) + e(t) \int_{t_0}^{t} [\theta_1(p, q, g(s), k_1(s)) + \theta_2(p, r, h(s), k_2(s))] \,\mathrm{d}s + l(t)\beta_1 v_1^m(t_1 - 0) \\ &\quad + e(t) \int_{t_0}^{t} [f(s) + k(s)]x^p(s) \,\mathrm{d}s \\ &\le d(t) + e(t)w(t) + l(t)\beta_1 v_1^m(t_1 - 0) + e(t) \int_{t_0}^{t_1} [f(s) + k(s)]v_1^p(s) \,\mathrm{d}s \\ &\quad + e(t) \int_{t_1}^{t} [f(s) + k(s)]x^p(s) \,\mathrm{d}s \\ &= r_1(t) + l(t)\beta_1 v_1^m(t_1 - 0) \\ &\quad + e(t) \int_{t_0}^{t_1} [f(s) + k(s)]v_1^p(s) \,\mathrm{d}s + e(t) \int_{t_1}^{t} [f(s) + k(s)]x^p(s) \,\mathrm{d}s \\ &= r_2(t) + e(t) \int_{t_1}^{t} [f(s) + k(s)]x^p(s) \,\mathrm{d}s. \end{aligned} \tag{3.36}$$

Inequality (3.36) is the same as (3.30) if we replace $r_1(t)$ and t_0 with $r_2(t)$ and t_1 in (3.36), respectively. Thus, by (3.35) and (3.36), we get, for $t \in I_1 = (t_1, t_2]$,

$$x(t) \le r_2^{\frac{1}{p}}(t) \exp\left\{ \frac{1}{p} e(t) \int_{t_1}^{t} [f(s) + k(s)] \,\mathrm{d}s \right\} = v_2(t).$$

Suppose that

$$\begin{aligned} x(t) &\le r_i^{\frac{1}{p}}(t) \exp\left\{ \frac{1}{p} e(t) \int_{t_{i-1}}^{t} [f(s) + k(s)] \,\mathrm{d}s \right\} \\ &= v_i(t) \quad \text{holds for } t \in I_{i-1} = (t_{i-1}, t_i], i = 2, 3, \ldots. \end{aligned} \tag{3.37}$$

Then, for $t \in I_i = (t_i, t_{i+1}]$, from Lemma 2.1 and (3.20), (3.24)-(3.27) and (3.37), we have

$$\begin{aligned} x^p(t) &\le d(t) + e(t)\int_{t_0}^{t}\big[f(s)x^p(s) + g(s)x^q(s) - h(s)x^r(s)\big]\,ds + l(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i - 0) \\ &\le d(t) + e(t)\int_{t_0}^{t}\big[\big[f(s)+k(s)\big]x^p(s) + \theta_1\big(p,q,g(s),k_1(s)\big) + \theta_2\big(p,r,h(s),k_2(s)\big)\big]\,ds \\ &\quad + l(t)\sum_{t_0<t_i<t}\beta_i v_i^m(t_i - 0) \\ &= d(t) + e(t)w(t) + e(t)\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}}\big[f(s)+k(s)\big]x^p(s)\,ds + e(t)\int_{t_i}^{t}\big[f(s)+k(s)\big]x^p(s)\,ds \\ &\quad + l(t)\sum_{t_0<t_i<t}\beta_i v_i^m(t_i - 0) \\ &\le r_1(t) + e(t)\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}}\big[f(s)+k(s)\big]v_{k+1}^p(s)\,ds + e(t)\int_{t_i}^{t}\big[f(s)+k(s)\big]x^p(s)\,ds \\ &\quad + l(t)\sum_{t_0<t_i<t}\beta_i v_i^m(t_i - 0) \\ &\le r_{i+1}(t) + e(t)\int_{t_i}^{t}\big[f(s)+k(s)\big]x^p(s)\,ds. \end{aligned} \tag{3.38}$$

Inequality (3.38) is the same as (3.30) if we replace $r_1(t)$ and t_0 with $r_{i+1}(t)$ and t_i in (3.38), respectively. Thus, by (3.35) and (3.38), we have, for $t \in I_i = (t_i, t_{i+1}]$,

$$x(t) \le r_{i+1}^{\frac{1}{p}}(t)\exp\left\{\frac{1}{p}e(t)\int_{t_i}^{t}\big[f(s)+k(s)\big]\,ds\right\}.$$

By induction, we know that (3.30) holds for $t \in (t_i, t_{i+1}]$, for any nonnegative integer i. This completes the proof of Theorem 3.3. □

Theorem 3.4 *Suppose that x is a nonnegative piecewise continuous function defined on $[t_0, \infty)$ with discontinuities of the first kind in the points t_i $(i = 1, 2, \ldots)$ and satisfies the integro-sum inequality*

$$\begin{aligned} x^p(t) &\le a(t) + \int_{t_0}^{t} f(s)x^p(s)\,ds + \sum_{j=1}^{L}\int_{t_0}^{t} g_j(s)x^{q_j}(s)\,ds - \sum_{j=1}^{L}\int_{t_0}^{t} h_j(s)x^{r_j}(s)\,ds \\ &\quad + c(t)\sum_{t_0<t_i<t}\beta_i x^m(t_i - 0), \quad t \ge t_0, \end{aligned}$$

where $0 \le t_0 < t_1 < t_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, $a(t)$ is defined on $[t_0, \infty)$ and $a(t_0) \ne 0$, $b(t) \ge 0$ and $c(t) \ge 0$ are defined on $[t_0, \infty)$, $f, g \in C(\mathrm{R}_+, \mathrm{R}_+)$, $h \in C(\mathrm{R}_+, (0, +\infty))$, $0 < q_j < p < r_j$ $(j = 1, 2, \ldots, L)$, $\beta_i \ge 0$, $i = 1, 2, \ldots$, and $m > 0$ are constants.

Then, for any continuous functions $k_1(t) > 0$ and $k_2(t) \ge 0$ on $[t_0, \infty)$ satisfying $k(t) = k_1(t) - k_2(t) \ge 0$, the following estimates hold:

$$x(t) \le v_1(t), \quad t \in [t_0, t_1],$$

$$x(t) \le v_i(t), \quad t \in (t_{i-1}, t_i], i = 2, 3, \ldots,$$

where

$$v_i(t) = r_i^{\frac{1}{p}}(t)\exp\left\{\frac{1}{p}\int_{t_{i-1}}^{t}\big[f(s)+Lk(s)\big]\,\mathrm{d}s\right\}, \quad i=1,2,\ldots,$$

$$d(t) = \max_{t_0\le\tau\le t}\big|a(\tau)\big|, \qquad l(t) = \max_{t_0\le\tau\le t}\big|c(\tau)\big|, \qquad r_1(t) = d(t)+w(t),$$

$$w(t) = \sum_{j=1}^{L}\int_{t_0}^{t}\big[\theta_j\big(p,q_j,g_j(s),k_1(s)\big)+\widetilde{\theta}_j\big(p,r_j,h_j(s),k_2(s)\big)\big]\,\mathrm{d}s,$$

$$\theta_j\big(p,q_j,g_j(s),k_1(s)\big) = \frac{p-q_j}{q_j}\left(\frac{q_j}{p}\right)^{\frac{p}{p-q_j}} g_j^{\frac{p}{p-q_j}}(s)k_1^{\frac{-q}{p-q_j}}(s), \quad j=1,2,\ldots,L,$$

$$\widetilde{\theta}_j\big(p,r_j,h_j(s),k_2(s)\big) = \frac{r_j-p}{r_j}\left(\frac{p}{r_j}\right)^{\frac{p}{r_j-p}} h_j^{\frac{-p}{r_j-p}}(s)k_2^{\frac{r_j}{r_j-p}}(s), \quad j=1,2,\ldots,L,$$

$$r_{i+1}(t) = r_i(t)+\int_{t_{i-1}}^{t_i}\big[f(s)+Lk(s)\big]v_i^p(s)\,\mathrm{d}s + l(t)\beta_i v_i^m(t_i-0), \quad i=1,2,\ldots.$$

The proof is similar to that of Theorem 3.3, and we omit these details.

4 Application

In this section, we will apply the results which we have established above to the estimates of solutions of certain impulsive differential equations.

Example 4.1 Consider the following impulsive differential equation:

$$\begin{cases} \frac{\mathrm{d}x^{p}(t)}{\mathrm{d}t} = F(t,x(t),\int_{t_0}^{t}G(s,t,x(s))\,\mathrm{d}s), & t\ne t_i,\\ \triangle x|_{t=t_i} = d(t)\beta_i x^m(t_i-0), \\ x(t_0)=x_0, \end{cases} \tag{4.1}$$

where $p>0$, $m>0$ are constants, the functions $d(t)\ge 0$, $t\in[t_0,\infty)$, $F\in C(\mathrm{R}\times\mathrm{R}\times\mathrm{R},\mathrm{R}_+)$ and $G\in C(\mathrm{R}\times\mathrm{R}\times\mathrm{R},\mathrm{R}_+)$ satisfy the following conditions:

$$\big|F(t,u,v)\big| \le f(t)|u|^q+|v|, \tag{4.2}$$

$$\big|G(s,t,w)\big| \le \sum_{j=1}^{L}b_j(t)g_j(s)|w|^p + \sum_{k=1}^{M}c_k(t)\theta_k(s)|w|^q, \tag{4.3}$$

where $q>0$ ($q\ne p$) is a constant, and $f(t)$, $g_j(t)$, $b_j(t)$ ($j=1,2,\ldots,L$), $c_j(t)$, $\theta_j(t)$ ($j=1,2,\ldots,M$) are defined as in Theorem 3.2. If

$$1+\frac{p-q}{p}r_i^{\frac{q-p}{p}}(t)\int_{t_{i-1}}^{t}l(s)\exp\left(\frac{q-p}{p}\int_{t_{i-1}}^{s}e(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s>0, \quad i=1,2,\ldots,$$

then for $t\ge t_0$, every solution $x(t)$ of Eq. (4.1) satisfies the following estimates:

$$\big|x(t)\big| \le v_1(t), \quad t\in[t_0,t_1], \tag{4.4}$$

$$\big|x(t)\big| \le v_i(t), \quad t\in(t_{i-1},t_i], i=2,3,\ldots, \tag{4.5}$$

where $l(t)$, $e(t)$, $r_i(t)$ and $v_i(t)$ ($i=1,2,\ldots$) are defined as in Theorem 3.2.

Proof The solution $x(t)$ of Eq. (4.1) satisfies the following equivalent equation:

$$x^p(t) = x_0^p + \int_{t_0}^{t} F\left(\tau, x(\tau), \int_{t_0}^{\tau} G\big(s,\tau,x(s)\big)\,ds\right) d\tau + d(t) \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0).$$

From conditions (4.2) and (4.3), it is easy to have

$$\begin{aligned} \big|x(t)\big|^p &\le |x_0|^p + \int_{t_0}^{t} \left|F\left(\tau, x(\tau), \int_{t_0}^{\tau} G\big(s,\tau,x(s)\big)\,ds\right)\right| d\tau \\ &\quad + c(t) \sum_{t_0<t_i<t} \beta_i \big|x(t_i - 0)\big|^m \\ &\le |x_0|^p + \int_{t_0}^{t} f(\tau)\big|x(\tau)\big|^q\,d\tau + \sum_{j=1}^{L} \int_{t_0}^{t} b_j(\tau) \int_{t_0}^{\tau} g_j(s)\big|x(s)\big|^p\,ds\,d\tau \\ &\quad + \sum_{k=1}^{M} \int_{t_0}^{t} c_k(\tau) \int_{t_0}^{\tau} \theta_k(s)\big|x(s)\big|^q\,ds\,d\tau + d(t) \sum_{t_0<t_i<t} \beta_i \big|x(t_i - 0)\big|^m, \quad t \ge t_0. \end{aligned}$$

By using Theorem 3.2, we easily obtain estimates (4.4) and (4.5) of solutions of Eq. (4.1). □

Example 4.2 Consider the following impulsive differential equation:

$$\begin{cases} \frac{dx(t)}{dt} = f(t)x(t) + g(t)x^{\frac{1}{3}}(t) - h(t)x^2(t), \quad t \ne t_i, \\ \triangle x|_{t=t_i} = a(t)\beta_i x^3(t_i - 0), \\ x(t_0) = x_0, \end{cases} \tag{4.6}$$

where $0 \le t_0 < t_1 < t_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, $f,g \in C(\mathrm{R}_+, \mathrm{R}_+)$, $h \in C(\mathrm{R}_+, (0,+\infty))$, $a(t) \ge 0$ is defined on $[t_0,\infty)$ and $\beta_i \ge 0$ $(i = 1,2,\ldots)$ are constants. Then, for any continuous functions $k_1(t) > 0$ and $k_2(t) \ge 0$ on $[t_0,\infty)$ satisfying $k(t) = k_1(t) - k_2(t) \ge 0$, the following estimates hold:

$$\big|x(t)\big| \le v_1(t), \quad t \in [t_0, t_1], \tag{4.7}$$

$$\big|x(t)\big| \le v_i(t), \quad t \in (t_{i-1}, t_i], i = 2,3,\ldots, \tag{4.8}$$

where

$$v_i(t) = r_i(t) \exp\left\{\int_{t_{i-1}}^{t} \big[f(s) + k(s)\big]\,ds\right\}, \quad i = 1,2,\ldots, \tag{4.9}$$

$$r_1(t) = |x_0| + w(t), \qquad w(t) = \int_{t_0}^{t} \left[\frac{2}{3\sqrt{3}} g^{\frac{3}{2}}(s) k_1^{-\frac{1}{2}}(s) + \frac{1}{4} h^{-1}(s) k_2^2(s)\right] ds, \tag{4.10}$$

$$r_{i+1}(t) = r_i(t) + \int_{t_{i-1}}^{t_i} \big[f(s) + k(s)\big] v_i(s)\,ds + l(t)\beta_i v_i^3(t_i - 0), \quad i = 1,2,\ldots, \tag{4.11}$$

$$\text{and} \quad l(t) = \max_{t_0 \le \tau \le t} \big|a(t)\big|. \tag{4.12}$$

Proof The solution $x(t)$ of Eq. (4.6) satisfies the following equivalent equation:

$$x(t) = x_0 + \int_{t_0}^{t} \bigl(f(s)x(s) + g(s)x^{\frac{1}{3}}(s) - h(s)x^2(s)\bigr)\,\mathrm{d}s + a(t)\sum_{t_0<t_i<t} \beta_i x^3(t_i - 0).$$

From the assumptions of f, g and h, it follows

$$\begin{aligned} \bigl|x(t)\bigr| \le{}& |x_0| + \int_{t_0}^{t} \bigl(f(s)\bigl|x(s)\bigr| + g(s)\bigl|x(s)\bigr|^{\frac{1}{3}} - h(s)\bigl|x(s)\bigr|^2\bigr)\,\mathrm{d}s \\ &+ a(t)\sum_{t_0<t_i<t} \beta_i \bigl|x(t_i - 0)\bigr|^3, \quad t \ge t_0. \end{aligned}$$

By using Theorem 3.3, we easily obtain estimates (4.7) and (4.8) of solutions of Eq. (4.6). □

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Competing interests
The author declares that he has no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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