Liu and Jia *Advances in Difference Equations* (2018) 2018:28
https://doi.org/10.1186/s13662-017-1446-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The method of lower and upper solutions for the general boundary value problems of fractional differential equations with p-Laplacian

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Abstract

We present here a new method of lower and upper solutions for a general boundary value problem of fractional differential equations with p-Laplacian operators. By using this approach, some new results on the existence of positive solutions for the equations with multiple types of nonlinear integral boundary conditions are established. Finally, some examples are presented to illustrate the wide range of potential applications of our main results.

Keywords: fractional differential equations; p-Laplacian operators; functional; general boundary value problem; method of lower and upper solutions; positive solution

1 Introduction

In this paper, we study the fractional differential equation with p-Laplacian operators

$$
{}^{C}D_{0^+}^{\alpha}\left(\varphi_p\left({}^{C}D_{0^+}^{\beta}u(t)\right)\right)=f\left(t,u(t),{}^{C}D_{0^+}^{\beta}u(t)\right),\quad t\in(0,1),\tag{1.1}
$$

with the general boundary conditions

$$
\begin{cases}
u'(0)=(\varphi_p({}^{C}D_{0^+}^{\beta}u(0)))'=0,\\
u(1)=T_1[u(t)],\\
{}^{C}D_{0^+}^{\beta}u(1)=T_2[u(t)],
\end{cases}\tag{1.2}
$$

where $1<\alpha,\beta\leq 2$, ${}^{C}D_{0^+}^{\alpha}$ and ${}^{C}D_{0^+}^{\beta}$ are the Caputo fractional derivatives. $p>1$, φ_p is the p-Laplacian operator, which is given by $\varphi_p(x)=|x|^{p-2}x$. Obviously, φ_p is continuous, increasing, invertible, and its inverse operator is $\varphi_p^{-1}=\varphi_q$, where $q>1$ is a constant such that $\frac{1}{p}+\frac{1}{q}=1$. $T_1[u(t)]$ and $T_2[u(t)]$ are two functionals, which could be $T_j[u(t)]=\sum_{i=1}^{m}g_j(\xi_i,u(\xi_i),{}^{C}D_{0^+}^{\beta}u(\xi_i))$, $T_j[u(t)]=\int_0^1 g_j(s,u(s),{}^{C}D_{0^+}^{\beta}u(s))\,\mathrm{d}s$, $T_j[u(t)]=\int_0^1 g_j(s,u(s),{}^{C}D_{0^+}^{\beta}u(s))\,\mathrm{d}\Lambda_j(s)$, $j=1,2$, or the other cases.

In recent years, the theory of fractional differential equations has become an important investigation area, see [1–5]. Many important results for certain boundary value conditions related to the fractional differential equations had been obtained, for example, two-point boundary value problem, multi-point boundary value problem, integral boundary value problem and so on, see [6–16].

In [16], the authors discuss the two-point boundary value problem of the systems of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)), & t \in (0, T], \\ D^{\alpha}v(t) = g(t, v(t), u(t)), & t \in (0, T], \\ t^{1-\alpha}u(t)|_{t=0} = x_0, & t^{1-\alpha}v(t)|_{t=0} = y_0, \end{cases}$$

where $0 < T < \infty$, D^{α} is the Riemann-Liouville fractional derivative of order $0 < \alpha \leq 1$. By using the monotone iterative technique, some existence results of solutions are established.

On the other hand, the turbulent flow in a porous medium is a fundamental mechanics phenomenon. For studying this kind of problems, the models of the p-Laplacian equation are introduced, see [17]. Many important results for the boundary value problems of fractional p-Laplacian equations have been obtained, see [18–28] and the references therein.

In [25], Wang and Xiang studied the four-point boundary value problem of the fractional p-Laplacian equations

$$\begin{cases} D_{0^+}^{\gamma}(\varphi_p(D_{0^+}^{\alpha}u(t))) = f(t, u(t)), & t \in (0, 1), \\ u(0) = 0, & u'(1) = au(\xi), \\ D_{0^+}^{\alpha}u(0) = 0, & D_{0^+}^{\alpha}u(1) = bD_{0^+}^{\alpha}u(\eta), \end{cases}$$

where $1 < \alpha, \gamma \leq 1$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\gamma}$ are Riemann-Liouville fractional derivatives. By using the method of lower and upper solutions, the existence results of at least one nonnegative solution of the boundary value problem are established.

In [27], Mahmudov and Unul studied the following integral boundary value problem of fractional p-Laplacian equation:

$$\begin{cases} D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha}u(t))) = f(t, u(t), D_{0^+}^{\gamma}u(t)), & t \in (0, 1), \\ u(0) + \mu_1 u(1) = \sigma_1 \int_0^1 g(s, u(s))\,\mathrm{d}s, \\ u(1) + \mu_2 u'(1) = \sigma_2 \int_0^1 h(s, u(s))\,\mathrm{d}s, \\ D_{0^+}^{\alpha}u(0) = 0, & D_{0^+}^{\alpha}u(1) = \upsilon D_{0^+}^{\alpha}u(\eta), \end{cases}$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are Caputo fractional derivatives with $1 < \alpha, \beta \leq 2$. By the fixed point theorems, the existence and uniqueness results of the solutions are established.

The purpose of this paper is to establish a method of lower and upper solutions for the general boundary value problems of fractional p-Laplacian equations and prove the existence of positive solutions for some specific nonlinear integral boundary value problems of the fractional p-Laplacian equations. Our paper is organized as the following parts. In Section 2, we give some basic definitions and lemmas to prove our main results. In Section 3, we establish the lower and upper solutions method for the general boundary value problem (1.1)-(1.2). In Section 4, by using the lower and upper solutions method obtained in Section 3, the existence of positive solutions for fractional p-Laplacian equation (1.1) with the following nonlinear boundary conditions are obtained:

$$\begin{cases} u'(0) = (\varphi_p({}^CD^{\beta}_{0^+}u(0)))' = 0, \\ u(1) = \int_0^1 g_1(s,u(s),{}^CD^{\beta}_{0^+}u(s))\,\mathrm{d}s, \\ {}^CD^{\beta}_{0^+}u(1) = \int_0^1 g_2(s,u(s),{}^CD^{\beta}_{0^+}u(s))\,\mathrm{d}s, \end{cases} \tag{1.3}$$

$$\begin{cases} u'(0) = (\varphi_p({}^CD^{\beta}_{0^+}u(0)))' = 0, \\ r_1u(1) - r_2u(\xi) = \int_0^1 g_1(s,u(s),{}^CD^{\beta}_{0^+}u(s))\,\mathrm{d}s, \\ m_1{}^CD^{\beta}_{0^+}u(1) + m_2{}^CD^{\beta}_{0^+}u(\eta) = \int_0^1 g_2(s,u(s),{}^CD^{\beta}_{0^+}u(s))\,\mathrm{d}s, \end{cases} \tag{1.4}$$

and

$$\begin{cases} u'(0) = (\varphi_p({}^CD^{\beta}_{0^+}u(0)))' = 0, \\ u(1) = \int_0^1 g_1(s,u(s),{}^CD^{\beta}_{0^+}u(s))\,\mathrm{d}\Lambda_1(s), \\ {}^CD^{\beta}_{0^+}u(1) = \int_0^1 g_2(s,u(s),{}^CD^{\beta}_{0^+}u(s))\,\mathrm{d}\Lambda_2(s), \end{cases} \tag{1.5}$$

respectively. In Section 5, as applications, some examples are presented to illustrate our main results.

2 Preliminary definitions and lemmas

For the convenience of reading, in this section, we provide the background knowledge on the fractional calculus and fractional differential equations.

Definition 2.1 (see [1, 2]) The Riemann-Liouville fractional integral of order $\gamma > 0$ of a function $y : [0, +\infty) \to \mathbb{R}$ is defined by

$$I^{\gamma}_{0^+}y(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}y(s)\,\mathrm{d}s,$$

and the Caputo derivative is given by

$${}^{\mathcal{C}}D^{\gamma}_{0^+}y(t) = D^{\gamma}_{0^+}y(t) - \sum_{k=0}^{n-1}\frac{y^{(n)}(0)}{\Gamma(k-\gamma+1)}t^{k-\gamma},$$

where

$$D^{\gamma}_{0^+}y(t) = \frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{y(s)}{(t-s)^{\gamma-n+1}}\,\mathrm{d}s$$

is the standard Riemann-Liouville fractional derivative of order $\gamma > 0$ of a function $y : [0, +\infty) \to \mathbb{R}$, n is an integer with $n - 1 < \gamma < n$, provided the right-hand integral converges.

Lemma 2.1 (see [1, 2]) *For $\gamma > 0$, the general solution of fractional differential equation ${}^{C}D_{0^+}^{\gamma}y(t) = 0$ is given by*

$$y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_j \in \mathbb{R}$, $j = 0, 1, 2, \ldots, n-1$, and n is an integer with $n-1 < \gamma < n$.

Lemma 2.2 *For any given function $h \in C[0,1]$ and real numbers $a, b \in \mathbb{R}$, the following boundary value problem of fractional differential equations*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}(\varphi_p({}^{C}D_{0^+}^{\beta}u(t))) = h(t), & t \in (0,1), \\ u'(0) = (\varphi_p({}^{C}D_{0^+}^{\beta}u(0)))' = 0, \\ u(1) = b, \qquad {}^{C}D_{0^+}^{\beta}u(1) = a \end{cases} \tag{2.1}$$

has a unique solution $u = u(t)$, which is given by

$$u(t) = b - \int_0^1 G_\beta(t,s)\varphi_q\left(\varphi_p(a) - \int_0^1 G_\alpha(s,\tau)h(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \tag{2.2}$$

and

$${}^{C}D_{0^+}^{\beta}u(t) = \varphi_q\left(\varphi_p(a) - \int_0^1 G_\alpha(t,s)h(s)\,\mathrm{d}s\right), \tag{2.3}$$

where

$$G_\alpha(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ (1-s)^{\alpha-1}, & 0 \le t < s \le 1, \end{cases} \tag{2.4}$$

and

$$G_\beta(t,s) = \frac{1}{\Gamma(\beta)}\begin{cases} (1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ (1-s)^{\beta-1}, & 0 \le t < s \le 1. \end{cases} \tag{2.5}$$

Proof Let $\varphi_p({}^{C}D_{0^+}^{\beta}u(t)) = v(t)$, we can easily show that boundary value problem (2.1) can be decomposed into the following coupled boundary value problems:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}v(t) = h(t), & t \in (0,1), \\ v'(0) = 0, \qquad v(1) = \varphi_p(a), \end{cases} \tag{2.6}$$

and

$$\begin{cases} {}^{C}D_{0^+}^{\beta}u(t) = \varphi_q(v(t)), & t \in (0,1), \\ u'(0) = 0, \qquad u(1) = b. \end{cases} \tag{2.7}$$

It follows from Lemma 2.1 that the general solution of fractional differential equation ${}^{C}D_{0^+}^{\alpha} v(t) = h(t)$ is given by

$$\begin{aligned} v(t) &= I_{0^+}^{\alpha} h(t) + c_0 + c_1 t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s + c_0 + c_1 t, \quad c_j \in \mathbb{R}, j = 0, 1. \end{aligned}$$

The boundary condition $v'(0) = 0$ implies that $c_1 = 0$, and by the boundary condition $v(1) = \varphi_p(a)$, we can obtain that

$$c_0 = \varphi_p(a) - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\,\mathrm{d}s.$$

So, boundary value problem (2.6) has a unique solution, which is given by

$$\begin{aligned} v(t) &= \varphi_p(a) - \frac{1}{\Gamma(\alpha)} \left(\int_0^1 (1-s)^{\alpha-1} h(s)\,\mathrm{d}s - \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s \right) \\ &= \varphi_p(a) - \int_0^1 G_\alpha(t,s) h(s)\,\mathrm{d}s. \end{aligned} \tag{2.8}$$

In the same way, we can get that the unique solution of boundary value problem (2.7) is given by

$$\begin{aligned} u(t) &= b + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varphi_q\big(v(s)\big)\,\mathrm{d}s - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \varphi_q\big(v(s)\big)\,\mathrm{d}s \\ &= b - \int_0^1 G_\beta(t,s) \varphi_q\big(v(s)\big)\,\mathrm{d}s. \end{aligned} \tag{2.9}$$

Therefore, boundary value problem (2.1) has a unique solution $u = u(t)$ which is given by (2.2), and ${}^{C}D_{0^+}^{\beta} u(t)$ is given by (2.3). □

From (2.4) and (2.5), it is obvious that $G_\alpha(t,s)$ and $G_\beta(t,s)$ satisfy the following lemma.

Lemma 2.3 *The functions $G_\alpha(t,s)$ and $G_\beta(t,s)$ are continuous and $G_\alpha(t,s) \geq 0$, $G_\beta(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$.*

3 The method of lower and upper solutions for the general nonlinear boundary value problem

In this section, we present a new method of lower and upper solutions for the general boundary value problem (1.1)-(1.2) and prove the existence of positive solutions for the problem.

Definition 3.1 We say a function $x = x(t)$ is a positive solution of boundary value problem (1.1)-(1.2) if and only if $x(t) \geq 0$, $t \in [0,1]$, and $x = x(t)$ satisfies equation (1.1) and conditions (1.2).

We denote by $AC^1[0,1]$ the space of functions which are absolutely continuous on $[0,1]$ (see [1, 2]).

Definition 3.2 Let $x \in AC^1[0,1]$, and we say that $x = x(t)$ is a lower solution of boundary value problem (1.1)-(1.2) if

$$\begin{cases} {}^C D_{0^+}^{\alpha}(\varphi_p({}^C D_{0^+}^{\beta} x(t))) \leq f(t, x(t), {}^C D_{0^+}^{\beta} x(t)), \quad t \in (0,1), \\ x'(0) = (\varphi_p({}^C D_{0^+}^{\beta} x(0)))' = 0, \\ x(1) \leq T_1[x(t)], \\ {}^C D_{0^+}^{\beta} x(1) \geq T_2[x(t)]. \end{cases} \tag{3.1}$$

Let $y \in AC^1[0,1]$, and we say that $y = y(t)$ is an upper solution of boundary value problem (1.1)-(1.2) if

$$\begin{cases} {}^C D_{0^+}^{\alpha}(\varphi_p({}^C D_{0^+}^{\beta} y(t))) \geq f(t, y(t), {}^C D_{0^+}^{\beta} y(t)), \quad t \in (0,1), \\ y'(0) = (\varphi_p({}^C D_{0^+}^{\beta} y(0)))' = 0, \\ y(1) \geq T_1[y(t)], \\ {}^C D_{0^+}^{\beta} y(1) \leq T_2[y(t)]. \end{cases} \tag{3.2}$$

We denote that $E = C^{\beta}[0,1] := \{u : u \in C[0,1], {}^C D_{0^+}^{\beta} u \in C[0,1]\}$ and endowed with the norm $\|u\|_{\beta} = \|u\|_{\infty} + \|{}^C D_{0^+}^{\beta} u\|_{\infty}$, where $\|u\|_{\infty} = \max_{0 \leq t \leq 1} |u(t)|$ and $\|{}^C D_{0^+}^{\beta} u\|_{\infty} = \max_{0 \leq t \leq 1} |{}^C D_{0^+}^{\beta} u(t)|$. Then $(E, \|\cdot\|_{\beta})$ is a Banach space. We denote that

$$P = \left\{u : u \in E, u(t) \geq 0, {}^C D_{0^+}^{\beta} u(t) \leq 0, t \in [0,1]\right\}.$$

It is obvious that P is a normal cone on E. We denote $x \preceq y$ if and only if $y - x \in P$ for $x, y \in E$.

Definition 3.3 Let P be a cone on a Banach space, a functional $T = T[u(t)]$ is called increasing on P if and only if $T[x(t)] \leq T[y(t)]$ for any $x \preceq y \in P$. And it is called decreasing on P if and only if $T[x(t)] \geq T[y(t)]$ for any $x \preceq y \in P$.

We assume the following conditions hold:

(H1) $f \in C([0,1] \times [0,+\infty) \times (-\infty,0])$, $0 \leq f(t, w_1, z_1) \leq f(t, w_2, z_2)$ for any $t \in [0,1]$ and $0 \leq w_1 \leq w_2$, $0 \geq z_1 \geq z_2 \in \mathbb{R}$.

(H2) The functional T_1 is continuous nonnegative increasing on P, and T_2 is continuous nonpositive decreasing on P.

Theorem 3.1 *Assume that* (H1) *and* (H2) *hold, boundary value problem* (1.1)-(1.2) *has a lower solution* $x_0 \in P$ *and an upper solution* $y_0 \in P$ *with* $x_0 \preceq y_0$. *Then the general boundary value problem* (1.1)-(1.2) *has positive solutions* $x^*, y^* \in P$. *Furthermore,*

$$x_0(t) \leq x^*(t) \leq y^*(t) \leq y_0(t)$$

and

$${}^C D_{0^+}^{\beta} y_0(t) \leq {}^C D_{0^+}^{\beta} y^*(t) \leq {}^C D_{0^+}^{\beta} x^*(t) \leq {}^C D_{0^+}^{\beta} x_0(t) \leq 0, \quad t \in [0,1].$$

In order to prove Theorem 3.1, we provide the following two lemmas and the corresponding proofs.

Lemma 3.2 *Assume that conditions* (H1) *and* (H2) *hold, and there exists* $x_k \in P$, *a nonnegative lower solution of boundary value problem* (1.1)-(1.2). *Then the following boundary value problem of fractional p-Laplacian equation*

$$\begin{cases} {}^C D^{\alpha}_{0^+}(\varphi_p({}^C D^{\beta}_{0^+} x_{k+1}(t))) = f(t, x_k(t), {}^C D^{\beta}_{0^+} x_k(t)), \quad t \in (0,1), \\ x'_{k+1}(0) = (\varphi_p({}^C D^{\beta}_{0^+} x_{k+1}(0)))' = 0, \\ x_{k+1}(1) = T_1[x_k(t)], \\ {}^C D^{\beta}_{0^+} x_{k+1}(1) = T_2[x_k(t)] \end{cases} \tag{3.3}$$

has a unique solution $x_{k+1} = x_{k+1}(t)$ *which is a nonnegative lower solution of boundary value problem* (1.1)-(1.2), *and* $x_k \preceq x_{k+1}$.

Proof In view of Lemma 2.2, for the given $x_k \in P$, boundary value problem (3.3) has a unique solution $x_{k+1} = x_{k+1}(t)$ which is given by

$$\begin{aligned} x_{k+1}(t) = T_1\bigl[x_k(t)\bigr] - \int_0^1 G_\beta(t,s)\varphi_q\biggl(&\varphi_p\bigl(T_2\bigl[x_k(s)\bigr]\bigr) \\ &- \int_0^1 G_\alpha(s,\tau) f\bigl(\tau, x_k(\tau), {}^C D^{\beta}_{0^+} x_k(\tau)\bigr)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s, \end{aligned} \tag{3.4}$$

and

$${}^C D^{\beta}_{0^+} x_{k+1}(t) = \varphi_q\biggl(\varphi_p\bigl(T_2\bigl[x_k(t)\bigr]\bigr) - \int_0^1 G_\alpha(t,s) f\bigl(s, x_k(s), {}^C D^{\beta}_{0^+} x_k(s)\bigr)\,\mathrm{d}s\biggr). \tag{3.5}$$

From Lemma 2.3, conditions (H1) and (H2), we easily get that $x_{k+1}(t) \geq 0$ and ${}^C D^{\beta}_{0^+} x_{k+1}(t) \leq 0$, which implies $x_{k+1} \in P$.

Next, we will prove that $x_k \preceq x_{k+1}$ and $x_{k+1} = x_{k+1}(t)$ is a lower solution of boundary value problem (1.1)-(1.2).

Since x_k is a lower solution of boundary value problem (1.1)-(1.2), then

$$\begin{cases} {}^C D^{\alpha}_{0^+}(\varphi_p({}^C D^{\beta}_{0^+} x_k(t))) \leq f(t, x_k(t), {}^C D^{\beta}_{0^+} x_k(t)), \quad t \in (0,1), \\ x'_k(0) = (\varphi_p({}^C D^{\beta}_{0^+} x_k(0)))' = 0, \\ x_k(1) \leq T_1[x_k(t)], \\ {}^C D^{\beta}_{0^+} x_k(1) \geq T_2[x_k(t)]. \end{cases} \tag{3.6}$$

By (3.3) and (3.6), we can get that

$$\begin{cases} {}^C D^{\alpha}_{0^+}(\varphi_p({}^C D^{\beta}_{0^+} x_{k+1}(t)) - \varphi_p({}^C D^{\beta}_{0^+} x_k(t))) \geq 0, \quad t \in (0,1), \\ x'_{k+1}(0) - x'_k(0) = (\varphi_p({}^C D^{\beta}_{0^+} x_{k+1}(0)))' - (\varphi_p({}^C D^{\beta}_{0^+} x_k(0)))' = 0, \\ x_{k+1}(1) - x_k(1) \geq 0, \\ {}^C D^{\beta}_{0^+} x_{k+1}(1) - {}^C D^{\beta}_{0^+} x_k(1) \leq 0. \end{cases} \tag{3.7}$$

Denote

$$\varphi_p\big({}^C D_{0^+}^{\beta} x_{k+1}(t)\big) - \varphi_p\big({}^C D_{0^+}^{\beta} x_k(t)\big) := v(t).$$

Then

$$\big(\varphi_p\big({}^C D_{0^+}^{\beta} x_{k+1}(0)\big)\big)' - \big(\varphi_p\big({}^C D_{0^+}^{\beta} x_k(0)\big)\big)' = v'(0) = 0.$$

And since

$${}^C D_{0^+}^{\beta} x_{k+1}(1) - {}^C D_{0^+}^{\beta} x_k(1) \leq 0,$$

we get that

$$v(1) = \varphi_p\big({}^C D_{0^+}^{\beta} x_{k+1}(1)\big) - \varphi_p\big({}^C D_{0^+}^{\beta} x_k(1)\big) \leq 0.$$

Denote

$${}^C D_{0^+}^{\alpha}\big(\varphi_p\big({}^C D_{0^+}^{\beta} x_{k+1}(t)\big) - \varphi_p\big({}^C D_{0^+}^{\beta} x_k(t)\big)\big) := h_{k+1}(t)$$

and

$$v(1) = \varphi_p\big({}^C D_{0^+}^{\beta} x_{k+1}(1)\big) - \varphi_p\big({}^C D_{0^+}^{\beta} x_k(1)\big) := a_{k+1},$$

then we obtain the following boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{\alpha} v(t) = h_{k+1}(t) \geq 0, \quad t \in (0,1), \\ v'(0) = 0, \\ v(1) = a_{k+1} \leq 0. \end{cases} \tag{3.8}$$

By (2.8) and Lemma 2.3, $\varphi_p({}^C D_{0^+}^{\beta} x_{k+1}(t)) - \varphi_p({}^C D_{0^+}^{\beta} x_k(t)) = v(t) = a_{k+1} - \int_0^1 G_\alpha(t,s) \times h_{k+1}(s)\,\mathrm{d}s \leq 0$, $t \in [0,1]$. According to the monotonicity of p-Laplacian operator φ_p, we have

$${}^C D_{0^+}^{\beta} x_{k+1}(t) - {}^C D_{0^+}^{\beta} x_k(t) = {}^C D_{0^+}^{\beta}\big(x_{k+1}(t) - x_k(t)\big) \leq 0.$$

Then we obtain the following boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{\beta}(x_{k+1}(t) - x_k(t)) := \delta_{k+1}(t) \leq 0, \quad t \in (0,1), \\ x'_{k+1}(0) - x'_k(0) = 0, \\ x_{k+1}(1) - x_k(1) := b_{k+1} \geq 0. \end{cases} \tag{3.9}$$

So that, $x_{k+1}(t) - x_k(t) = b_{k+1} - \int_0^1 G_\beta(t,s)\delta_{k+1}(s)\,\mathrm{d}s \geq 0$ and ${}^C D_{0^+}^{\beta}(x_{k+1}(t) - x_k(t)) = \delta_{k+1}(t) \leq 0$, which implies $x_k \preceq x_{k+1}$.

From conditions (H1) and (H2), we get

$$\begin{cases} {}^CD_{0^+}^{\alpha}(\varphi_p({}^CD_{0^+}^{\beta}x_{k+1}(t))) = f(t, x_k(t), {}^CD_{0^+}^{\beta}x_k(t)) \leq f(t, x_{k+1}(t), {}^CD_{0^+}^{\beta}x_{k+1}(t)), \\ x'_{k+1}(0) = (\varphi_p({}^CD_{0^+}^{\beta}x_{k+1}(0)))' = 0, \\ x_{k+1}(1) = T_1[x_k(t)] \leq T_1[x_{k+1}(t)], \\ {}^CD_{0^+}^{\beta}x_{k+1}(1) = T_2[x_k(t)] \geq T_2[x_{k+1}(t)], \end{cases} \tag{3.10}$$

which implies that $x = x_{k+1}(t)$ is a lower solution of boundary value problem (1.1)-(1.2). □

Similar to Lemma 3.2, we can get the following lemma.

Lemma 3.3 *Assume that conditions* (H1) *and* (H2) *hold, and $y_k \in P$ is an upper solution of boundary value problem* (1.1)-(1.2). *Then the following boundary value problem*

$$\begin{cases} {}^CD_{0^+}^{\alpha}(\varphi_p({}^CD_{0^+}^{\beta}y_{k+1}(t))) = f(t, y_k(t), {}^CD_{0^+}^{\beta}y_k(t)), \quad t \in (0,1), \\ y'_{k+1}(0) = (\varphi_p({}^CD_{0^+}^{\beta}y_{k+1}(0)))' = 0, \\ y_{k+1}(1) = T_1[y_k(t)], \\ {}^CD_{0^+}^{\beta}y_{k+1}(1) = T_2[y_k(t)] \end{cases} \tag{3.11}$$

has a unique solution $y_{k+1} = y_{k+1}(t)$ which is a nonnegative upper solution of boundary value problem (1.1)-(1.2), *$y_{k+1} \in P$ and $y_{k+1} \preceq y_k$.*

Proof of Theorem 3.1 Starting from the initial functions $x_0, y_0 \in P$, we define iterative sequences $\{x_k\}$ and $\{y_k\}$ by (3.3) and (3.11), respectively.

From Lemma 3.2 and Lemma 3.3, $x = x_k(t)$, $k = 0, 1, 2, \ldots$, are lower solutions of boundary value problem (1.1)-(1.2), and $x_k \preceq x_{k+1}$ such that $\{x_k\} \subset P$ is monotonically increasing. Moreover, $y = y_k(t)$, $k = 0, 1, 2, \ldots$, are upper solutions, and $y_{k+1} \preceq y_k$ such that $\{y_k\} \subset P$ is monotonically decreasing.

Since $x_k \preceq y_k$, then $x_k(t) \leq y_k(t)$ and ${}^CD_{0^+}^{\beta}x_k(t) \geq {}^CD_{0^+}^{\beta}y_k(t)$, and from (H1), (H2), we have that

$$f\big(t, x_k(t), {}^CD_{0^+}^{\beta}x_k(t)\big) \leq f\big(t, y_k(t), {}^CD_{0^+}^{\beta}y_k(t)\big),$$
$$T_1\big[x_k(t)\big] \leq T_1\big[y_k(t)\big], \quad \text{and} \quad T_2\big[x_k(t)\big] \geq T_2\big[y_k(t)\big].$$

By (3.3) and (3.11), we get

$$\begin{cases} {}^CD_{0^+}^{\alpha}(\varphi_p({}^CD_{0^+}^{\beta}y_{k+1}(t)) - \varphi_p({}^CD_{0^+}^{\beta}x_{k+1}(t))) \geq 0, \quad t \in (0,1), \\ y'_{k+1}(0) - x'_{k+1}(0) = (\varphi_p({}^CD_{0^+}^{\beta}y_{k+1}(0)))' - (\varphi_p({}^CD_{0^+}^{\beta}x_{k+1}(0)))' = 0, \\ y_{k+1}(1) - x_{k+1}(1) \geq 0, \qquad {}^CD_{0^+}^{\beta}y_{k+1}(1) - {}^CD_{0^+}^{\beta}x_{k+1}(1) \leq 0. \end{cases} \tag{3.12}$$

We can show that $x_{k+1} \preceq y_{k+1}$ in the same way as above.

Therefore,

$$x_0 \preceq x_1 \preceq \cdots \preceq x_k \preceq \cdots \preceq \cdots \preceq y_k \preceq \cdots \preceq y_1 \preceq y_0.$$

Since P is a normal cone on E, the sequences $\{x_k\}$ and $\{y_k\}$ are uniformly bounded. Because G_α, G_β, φ_p, φ_q and f are continuous, we can easily get that $\{x_k\}$ and $\{y_k\}$ are equicontinuous. Hence, $\{x_k\}$ and $\{y_k\}$ are relatively compact. Then there exist x^* and y^* such that

$$\lim_{k\to\infty} x_k = x^*, \qquad \lim_{k\to\infty} {}^C D_{0^+}^{\beta} x_k(t) = {}^C D_{0^+}^{\beta} x^*(t), \tag{3.13}$$

and

$$\lim_{k\to\infty} y_k = y^*, \qquad \lim_{k\to\infty} {}^C D_{0^+}^{\beta} y_k(t) = {}^C D_{0^+}^{\beta} y^*(t), \tag{3.14}$$

which imply that x^* is a lower solution, y^* is an upper solution of boundary value problem (1.1)-(1.2), and $x^* \preceq y^* \in P$.

In the following, we prove that both x^* and y^* are solutions of boundary value problem (1.1)-(1.2).

From (3.4), (3.13), and by the continuity of $\varphi_p, f, G_\alpha, G_\beta$ and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} x^*(t) = {} & T_1\bigl[x^*(t)\bigr] - \int_0^1 G_\beta(t,s)\varphi_q\biggl(\varphi_p\bigl(T_2\bigl[x^*(s)\bigr]\bigr) \\ & - \int_0^1 G_\alpha(s,\tau) f\bigl(\tau, x^*(\tau), {}^C D_{0^+}^{\beta} x^*(\tau)\bigr)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s. \end{aligned} \tag{3.15}$$

In view of Lemma 2.2, x^* is a solution of boundary value problem (1.1)-(1.2).

In the same way, we can show that y^* is a solution of boundary value problem (1.1)-(1.2), too.

Furthermore, $x_0(t) \le x^*(t) \le y^*(t) \le y_0(t)$, ${}^C D_{0^+}^{\beta} y_0(t) \le {}^C D_{0^+}^{\beta} y^*(t) \le {}^C D_{0^+}^{\beta} x^*(t) \le {}^C D_{0^+}^{\beta} x_0(t) \le 0$. □

4 The existence of positive solutions for some nonlinear boundary value problems

In this section, we deal with fractional p-Laplacian equations (1.1) with nonlinear integral boundary value conditions (1.3), (1.4) and (1.5).

(H3) Assume that $g_i \in C([0,1] \times [0,+\infty) \times (-\infty,0])$ $(i = 1,2)$, $0 \le g_1(t,w_1,z_1) \le g_1(t,w_2,z_2)$ and $0 \ge g_2(t,w_1,z_1) \ge g_2(t,w_2,z_2)$ for any $t \in [0,1]$ and $0 \le w_1 \le w_2$, $0 \ge z_1 \ge z_2 \in \mathbb{R}$.

Theorem 4.1 *Assume that* (H1) *and* (H3) *hold, boundary value problem* (1.1)-(1.3) *has a nonnegative lower solution x_0 and an upper solution y_0 such that $x_0, y_0 \in P$ and $x_0 \preceq y_0$. Then boundary value problem* (1.1)-(1.3) *has positive solutions x^*, y^*, both x^* and $y^* \in P$. Furthermore,*

$$x_0(t) \le x^*(t) \le y^*(t) \le y_0(t)$$

and

$${}^C D_{0^+}^{\beta} y_0(t) \le {}^C D_{0^+}^{\beta} y^*(t) \le {}^C D_{0^+}^{\beta} x^*(t) \le {}^C D_{0^+}^{\beta} x_0(t) \le 0, \quad t \in [0,1].$$

Proof Boundary value problem (1.1)-(1.3) is a special case of the general boundary value problem (1.1)-(1.2) when $T_i[u(t)] = \int_0^1 g_i(s, u(s), {}^CD^{\beta}_{0^+}u(s))\,\mathrm{d}s$, $i = 1, 2$. By assumption (H3), T_1 is a continuous nonnegative increasing functional on P, and T_2 is a continuous nonpositive decreasing functional on P. Then assumption (H2) holds. By using Theorem 3.1, the results in this theorem can be obtained. □

Theorem 4.2 *Assume that* (H1) *and* (H3) *hold, the constants* $r_1, m_1 > 0$, $r_2, m_2 \geq 0$, *and boundary value problem* (1.1)-(1.4) *has a nonnegative lower solution* $x_0 \in P$ *and an upper solution* $y_0 \in P$ *with* $x_0 \preceq y_0$. *Then boundary value problem* (1.1)-(1.4) *has positive solutions* $x^*, y^* \in P$. *Furthermore,*

$$x_0(t) \leq x^*(t) \leq y^*(t) \leq y_0(t)$$

and

$${}^CD^{\beta}_{0^+}y_0(t) \leq {}^CD^{\beta}_{0^+}y^*(t) \leq {}^CD^{\beta}_{0^+}x^*(t) \leq {}^CD^{\beta}_{0^+}x_0(t) \leq 0, \quad t \in [0, 1].$$

Proof Boundary value problem (1.1)-(1.4) is a special case of the general boundary value problem (1.1)-(1.2) when

$$T_1\big[u(t)\big] = \frac{1}{r_1}\left(r_2 u(\xi) + \int_0^1 g_1\big(s, u(s), {}^CD^{\beta}_{0^+}u(s)\big)\,\mathrm{d}s\right)$$

and

$$T_2\big[u(t)\big] = \frac{1}{m_1}\left(-m_2\,{}^CD^{\beta}_{0^+}u(\eta) + \int_0^1 g_2\big(s, u(s), {}^CD^{\beta}_{0^+}u(s)\big)\,\mathrm{d}s\right).$$

By (H1) and (H3), all the conditions in Theorem 4.1 hold. In view of Theorem 4.1, the results can be obtained. □

(H4) Assume that $\Lambda_i(t)$ are increasing bounded variation functions,
$g_i \in C([0, 1] \times [0, +\infty) \times (-\infty, 0])$ $(i = 1, 2)$, $0 \leq g_1(t, w_1, z_1) \leq g_1(t, w_2, z_2)$ and
$0 \geq g_2(t, w_1, z_1) \geq g_2(t, w_2, z_2)$ for any $t \in [0, 1]$ and $0 \leq w_1 \leq w_2$, $0 \geq z_1 \geq z_2 \in \mathbb{R}$.

In the same way as Theorem 4.1, we can get the following results.

Theorem 4.3 *Assume that* (H1) *and* (H4) *hold, boundary value problem* (1.1)-(1.5) *has a nonnegative lower solution* x_0 *and an upper solution* y_0 *such that* $x_0, y_0 \in P$ *and* $x_0 \preceq y_0$. *Then boundary value problem* (1.1)-(1.5) *has positive solutions* $x^*, y^* \in P$. *Furthermore,*

$$x_0(t) \leq x^*(t) \leq y^*(t) \leq y_0(t)$$

and

$${}^CD^{\beta}_{0^+}y_0(t) \leq {}^CD^{\beta}_{0^+}y^*(t) \leq {}^CD^{\beta}_{0^+}x^*(t) \leq {}^CD^{\beta}_{0^+}x_0(t) \leq 0, \quad t \in [0, 1].$$

5 Illustration

Assume that $f(t,w,z) = \frac{1}{10\Gamma(\frac{1}{3})}t^{\frac{1}{3}}e^{w-z-10}$, $g_1(t,w,z) = f(t,w,z) = \frac{1}{10\Gamma(\frac{1}{3})}t^{\frac{1}{3}}e^{w-z-10}$, $g_2(t,w,z) = -f(t,w,z) = \frac{-1}{10\Gamma(\frac{1}{3})}t^{\frac{1}{3}}e^{w-z-10}$. We can easily check that the functions f, g_1 and g_2 satisfy conditions (H1) and (H3).

Let $\alpha = \frac{5}{3}$, $\beta = \frac{5}{4}$, $p = \frac{3}{2}$, We consider the following nonlinear fractional p-Laplacian equation:

$$\begin{aligned} {}^{C}D_{0^+}^{\frac{5}{3}}\left(\varphi_{\frac{3}{2}}\left({}^{C}D_{0^+}^{\frac{5}{4}}u(t)\right)\right) &= f\left(t,u(t),{}^{C}D_{0^+}^{\frac{5}{4}}u(t)\right) \\ &= \frac{1}{10\Gamma(\frac{1}{3})}t^{\frac{1}{3}}e^{u(t)-{}^{C}D_{0^+}^{\frac{5}{4}}u(t)-10}, \quad t\in(0,1), \end{aligned} \tag{5.1}$$

with the nonlinear integral boundary conditions

$$\begin{cases} u'(0) = (\varphi_{\frac{3}{2}}({}^{C}D_{0^+}^{\frac{5}{4}}u(0)))' = 0, \\ u(1) = \int_0^1 g_1(s,u(s),{}^{C}D_{0^+}^{\frac{5}{4}}u(s))\,\mathrm{d}s = \int_0^1 \frac{1}{10\Gamma(\frac{1}{3})}s^{\frac{1}{3}}e^{u(s)-{}^{C}D_{0^+}^{\frac{5}{4}}u(s)-10}\,\mathrm{d}s, \\ {}^{C}D_{0^+}^{\frac{5}{4}}u(1) = \int_0^1 g_2(s,u(s),{}^{C}D_{0^+}^{\frac{5}{4}}u(s))\,\mathrm{d}s = -\int_0^1 \frac{1}{10\Gamma(\frac{1}{3})}s^{\frac{1}{3}}e^{u(s)-{}^{C}D_{0^+}^{\frac{5}{4}}u(s)-10}\,\mathrm{d}s, \end{cases} \tag{5.2}$$

which is a special case of boundary value problem (1.1)-(1.2).

Let $x_0 = 0$, we can easily check that $x_0 = x_0(t) \equiv 0$ is a lower solution of boundary value problem (5.1)-(5.2).

Let

$$y_0(t) = 10 - \frac{16t^{\frac{5}{4}}(13{,}923 - 3808t^2 + 512t^4)}{69{,}615\Gamma(\frac{5}{4})}.$$

In the following, we check that $y_0 = y_0(t)$ is an upper solution of boundary value problem (5.1)-(5.2).

We can easily get that

$$y_0'(t) = -\frac{16t^{\frac{5}{4}}(-7616t + 2048t^3)}{69{,}615\Gamma(\frac{5}{4}))} - \frac{4t^{\frac{1}{4}}(13{,}923 - 3808t^2 + 512t^4)}{13{,}923\Gamma(\frac{5}{4})},$$

$${}^{C}D_{0^+}^{\frac{5}{4}}y_0(t) = -\left(2-t^2\right)^2, \qquad \varphi_{\frac{3}{2}}\left({}^{C}D_{0^+}^{\frac{5}{4}}y_0(t)\right) = t^2 - 2, \qquad \left(\varphi_{\frac{3}{2}}\left({}^{C}D_{0^+}^{\frac{5}{4}}y_0(t)\right)\right)' = 2t,$$

and

$${}^{C}D_{0^+}^{\frac{5}{3}}\left(\varphi_{\frac{3}{2}}\left({}^{C}D_{0^+}^{\frac{5}{4}}y_0(t)\right)\right) = {}^{C}D_{0^+}^{\frac{5}{3}}\left(t^2-2\right) = \frac{6}{\Gamma(\frac{1}{3})}t^{\frac{1}{3}}.$$

Then

$$y_0'(0) = 0, \qquad \left(\varphi_{\frac{3}{2}}\left({}^{C}D_{0^+}^{\frac{5}{4}}y_0(0)\right)\right)' = 0,$$

and

$$y_0(1) = 10 - \frac{170{,}032}{69{,}615\Gamma(\frac{5}{4})} \approx 7.30532, \qquad {}^{C}D_{0^+}^{\frac{5}{4}}y_0(1) = -1.$$

We can easily check that

$$
\begin{aligned}
0 &\le f\bigl(t, y_0(t), {}^{C}D_{0^+}^{\frac{5}{4}} y_0(t)\bigr) \\
&= \frac{1}{10\Gamma(\frac{1}{3})} t^{\frac{1}{3}} \exp\biggl(-\frac{16t^{\frac{5}{4}}(13{,}923-3808t^2+512t^4)}{69{,}615\Gamma(\frac{5}{4})} + \bigl(-2+t^2\bigr)^2\biggr) < 1; \\
0 &\le g_1\bigl(t, y_0(t), {}^{C}D_{0^+}^{\frac{5}{4}} y_0(t)\bigr) \\
&= \frac{1}{10\Gamma(\frac{1}{3})} t^{\frac{1}{3}} \exp\biggl(-\frac{16t^{\frac{5}{4}}(13{,}923-3808t^2+512t^4)}{69{,}615\Gamma(\frac{5}{4})} + \bigl(-2+t^2\bigr)^2\biggr) < 1; \\
0 &\ge g_2\bigl(t, y_0(t), {}^{C}D_{0^+}^{\frac{5}{4}} y_0(t)\bigr) \\
&= \frac{-1}{10\Gamma(\frac{1}{3})} t^{\frac{1}{3}} \exp\biggl(-\frac{16t^{\frac{5}{4}}(13{,}923-3808t^2+512t^4)}{69{,}615\Gamma(\frac{5}{4})} + \bigl(-2+t^2\bigr)^2\biggr) > -1.
\end{aligned}
$$

Then

$$
\begin{aligned}
&{}^{C}D_{0^+}^{\frac{5}{3}}\bigl({}^{C}D_{0^+}^{\frac{5}{4}} y_0(t)\bigr) = \frac{6}{\Gamma(\frac{1}{3})} t^{\frac{1}{3}} \ge f\bigl(t, y_0(t), {}^{C}D_{0^+}^{\frac{5}{4}} y_0(t)\bigr), \quad t \in (0,1), \\
&y_0'(0) = 0, \qquad \bigl(\varphi_{\frac{3}{2}}\bigl({}^{C}D_{0^+}^{\frac{5}{4}} y_0(0)\bigr)\bigr)' = 0, \\
&y_0(1) = 10 - \frac{170{,}032}{69{,}615\Gamma(\frac{5}{4})} \approx 7.30532 > 1 \\
&\qquad > \int_0^1 g_1\bigl(s, y_0(s), {}^{C}D_{0^+}^{\frac{5}{4}} y_0(s)\bigr)\,\mathrm{d}s \\
&\qquad = \int_0^1 \frac{1}{10\Gamma(\frac{1}{3})} s^{\frac{1}{3}} e^{y_0(s) - {}^{C}D_{0^+}^{\frac{5}{4}} y_0(s) - 10}\,\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{C}D_{0^+}^{\frac{5}{4}} y_0(1) &= -1 \\
&< \int_0^1 g_2\bigl(s, y_0(s), {}^{C}D_{0^+}^{\frac{5}{4}} y_0(s)\bigr)\,\mathrm{d}s \\
&= -\int_0^1 \frac{1}{10\Gamma(\frac{1}{3})} s^{\frac{1}{3}} e^{y_0(s) - {}^{C}D_{0^+}^{\frac{5}{4}} y_0(s) - 10}\,\mathrm{d}s.
\end{aligned}
$$

Therefore, $y_0(t) = 10 - \frac{16t^{\frac{5}{4}}(13{,}923-3808t^2+512t^4)}{69{,}615\Gamma(\frac{5}{4})}$ is an upper solution of boundary value problem (5.1)-(5.2).

Based on the above discussion, we can get that $x_0 \preceq y_0$. So all the conditions in Theorem 3.1 hold. According to Theorem 3.1, boundary value problem (1.3) has the maximum lower solution x^* and the minimal upper solution y^*, both x^* and y^* are solutions of the boundary value problem. Furthermore,

$$
0 \le x^*(t) \le y^*(t) \le 10 - \frac{16t^{\frac{5}{4}}(13{,}923-3808t^2+512t^4)}{69{,}615\Gamma(\frac{5}{4})} = y_0(t),
$$

and

$$-1 \le -\big(2-t^2\big)^2 = {}^C D_{0^+}^{\beta} y_0(t) \le {}^C D_{0^+}^{\beta} y^*(t) \le {}^C D_{0^+}^{\beta} x^*(t) \le {}^C D_{0^+}^{\beta} x_0(t) = 0, \quad t \in [0,1].$$

Remark Since $x_0 = x_0(t) \equiv 0$ is a lower solution of boundary value problem (1.3) but not a solution of the problem. Therefore, the solutions $x^* = x^*(t)$ and $y^* = y^*(t)$ are the nontrivial solutions of boundary value problem (1.3).

Acknowledgements
This work is supported by the National Natural Science Foundation of China (Grant No. 11171220).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 October 2017 Accepted: 11 December 2017 Published online: 22 January 2018

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