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Extremal solutions for p-Laplacian fractional differential systems involving the Riemann-Liouville integral boundary conditions

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Abstract
In this paper, we investigate the existence of extremal solutions for fractional differential systems involving the p-Laplacian operator and Riemann-Liouville integral boundary conditions. We derive our results based on the monotone iterative technique, combined with the method of upper and lower solutions. An example is added to illustrate the main result.

MSC: 34B15

Keywords: upper and lower solutions; p-Laplacian operator; integral boundary conditions; fractional differential equations; monotone iterative technique

1 Introduction

In this paper, we study the existence of extremal solutions of the following fractional differential systems involving the p-Laplacian operator and Riemann-Liouville integral boundary conditions:

$$\begin{cases} -D^{\alpha}(\phi_p(-D^{\beta}x(t))) = f(t,x(t),D^{\beta}x(t)), \quad 0<t<1, \\ D^{\beta}x(0)=0, \qquad (\phi_p(-D^{\beta}x(0)))'=0, \\ D^{\gamma}(\phi_p(-D^{\beta}x(1))) = I^{\nu}(\phi_p(-D^{\beta}x(\eta))) = \int_0^{\eta}(\eta-s)^{\nu-1}\phi_p(-D^{\beta}x(s))\,ds, \\ x(0)=0, \qquad D^{\beta-1}x(1) = I^{\omega}g(\xi,x(\xi))+k = \frac{1}{\Gamma(\omega)}\int_0^{\xi}(\xi-s)^{\omega-1}g(s,x(s))\,ds+k, \end{cases} \tag{1.1}$$

where D^{α}, D^{β}, and D^{γ} are the standard Riemann-Liouville fractional derivatives, I^{ν} and I^{ω} are the Riemann-Liouville fractional integrals, and $0<\gamma<1<\beta<2<\alpha<3$, $\nu,\omega>0$, $0<\eta,\xi<1$, $k\in\mathbb{R}$, $f\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g\in C([0,1]\times\mathbb{R},\mathbb{R})$. The p-Laplacian operator is defined as $\phi_p(t)=|t|^{p-2}t$, $p>1$, and $(\phi_p)^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$.

The study of boundary value problems in the setting of fractional calculus has received a great attention in the last decade, and a variety of results concerning the of existence of solutions, based on various analytic techniques, can be found in the literature [1–10]. In particular, much effort has been made toward the study of the existence of solutions for fractional differential equations involving p-Laplacian operators; see [11–14]. Using the monotone iteration method, Ding [15] investigated a fractional boundary value problem with p-Laplacian operator

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}u(t))) = f(t, u(t), D^{\alpha}u(t)), & t \in (0,1], \\ t^{\frac{1-\beta}{p-1}} D^{\alpha}u(t)|_{t=0} = 0, & g(\widetilde{u}(0), \widetilde{u}(1)) = 0, \end{cases} \tag{1.2}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, and D^{α} is the standard Riemann-Liouville fractional derivative, and established the existence and uniqueness of extremal solutions for the BVP (1.2) under the condition that the nonlinear functions f and g are continuous and satisfy certain growth conditions. Zhang [16] considered the following nonlinear fractional integral boundary value problem:

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t), u(\theta(t)), & n < \alpha \leq n+1, n \geq 2, t \in [0,1], \\ u'(0) = u''(0) = u'''(0) = \cdots = u^{n}(0) = 0, \\ u(0) = \int_0^1 g(s, u(s))\,ds + \lambda, \end{cases} \tag{1.3}$$

where $\lambda \geq 0$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, and f and g are continuous functions. The authors constructed two well-defined monotone iterative sequences of upper and lower solutions and proved that they converge uniformly to the actual solution of problem (1.3). A numerical iterative scheme is also introduced to obtain an accurate approximate solution for the problem.

In this paper, we consider a kind of fractional differential equations involving p-Laplacian operators and nonlocal boundary conditions based on the Riemann-Liouville integral. To the best of our knowledge, little work has been conducted to deal with this kind of problem, and no work has been done concerning the maximal and minimal solutions of (1.1) by using the method of upper and lower solutions and the monotone iteration technique.

The rest of this paper is organized as follows. In Section 2, we give some useful preliminaries and lemmas. In Section 3, the main result and proof are given, and an example is presented to illustrate the main results.

2 Preliminaries

In this section, we deduce some preliminary results that will be used in the next section to attain existence results for the nonlinear system (1.1)

Lemma 2.1 *Assume that $h(t) \in C[0,1]$, $l \in \mathbb{R}$. Then the fractional value boundary problem*

$$\begin{cases} -D^{\alpha}v(t) = h(t), & 0 < t < 1, \\ v(0) = 0, \quad v'(0) = 0, \quad D^{\gamma}v(1) = l, \end{cases} \tag{2.1}$$

has a unique solution

$$v(t) = \int_0^1 G(t,s)h(s)\,ds + \frac{\Gamma(\alpha-\gamma)lt^{\alpha-1}}{\Gamma(\alpha)},$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases}$$

Proof We can transform the equation $-D^\alpha v(t) = h(t)$ to the equivalent integral equation

$$v(t) = -I^\alpha h(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3}.$$

Noting that $v(0) = 0$ and $v'(0) = 0$, we have $C_2 = C_3 = 0$. Consequently, we have the form

$$v(t) = -I^\alpha h(t) + C_1 t^{\alpha-1} \tag{2.2}$$

and

$$\begin{aligned} D^\gamma v(t) &= -D^\gamma I^\alpha h(t) + C_1 D^\gamma t^{\alpha-1} \\ &= -I^{\alpha-\gamma} h(t) + C_1 D^\gamma t^{\alpha-1} \\ &= -\frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} h(s)\,ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} t^{\alpha-1}. \end{aligned}$$

So,

$$D^\gamma v(1) = -\frac{1}{\Gamma(\alpha-\gamma)} \int_0^1 (1-s)^{\alpha-\gamma-1} h(s)\,ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}. \tag{2.3}$$

On the other hand, $D^\gamma v(1) = l$, and combining with (2.3), we obtain

$$C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} h(s)\,ds + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} l.$$

Therefore, the unique solution of problem (2.1) is

$$\begin{aligned} v(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} h(s)\,ds + \frac{\Gamma(\alpha-\gamma) l t^{\alpha-1}}{\Gamma(\alpha)} \\ &= \int_0^1 G(t,s) h(s)\,ds + \frac{\Gamma(\alpha-\gamma) l t^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

The proof is completed. □

Lemma 2.2 ([17]) *Assume that* $z(t) \in C[0,1]$, $\lambda, k \in \mathbb{R}$, *and* $\Gamma(\beta+\omega) \neq \lambda \xi^{\beta+\omega-1}$. *Then the fractional boundary value problem*

$$\begin{cases} -D^\beta x(t) = z(t), & 0 < t < 1, \\ x(0) = 0, & D^{\beta-1} x(1) = \lambda I^\omega x(\xi) + k, \end{cases} \tag{2.4}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s) z(s)\,ds + \frac{k\Gamma(\beta+\omega) t^{\beta-1}}{\Gamma(\beta)[\Gamma(\beta+\omega) - \lambda \xi^{\beta+\omega-1}]},$$

where

$$H(t,s) = \frac{1}{\Delta}\begin{cases} [\Gamma(\beta+\omega) - \lambda(\xi - s)^{\beta+\omega-1}]t^{\beta-1} \\ \quad - [\Gamma(\beta+\omega) - \lambda\xi^{\beta+\omega-1}](t-s)^{\beta-1}, & s \le t, s \le \xi, \\ \Gamma(\beta+\omega)t^{\beta-1} - \lambda(\xi - s)^{\beta+\omega-1}t^{\beta-1}, & t \le s \le \xi, \\ \Gamma(\beta+\omega)[t^{\beta-1} - (t-s)^{\beta-1}] + \lambda\xi^{\beta+\omega-1}(t-s)^{\beta-1}, & \xi \le s \le t, \\ \Gamma(\beta+\omega)t^{\beta-1}, & s \ge t, s \ge \xi, \end{cases}$$

and $\Delta = \Gamma(\beta)[\Gamma(\beta+\omega) - \lambda\eta^{\beta+\omega-1}]$.

Lemma 2.3 ([17]) *Suppose that* $\lambda \ge 0$ *and* $\Gamma(\beta+\omega) > \lambda\xi^{\beta+\omega-1}$. *Then the functions* $G(t,s)$ *and* $H(t,s)$ *are continuous, and* $0 \le G(t,s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}$ *and* $0 \le H(t,s) \le \frac{\Gamma(\beta+\omega)}{\Gamma(\beta)[\Gamma(\beta+\omega)-\lambda\xi^{\beta+\omega-1}]}(1+t^{\beta-1})$ *for* $(t,s) \in [0,1] \times [0,1]$.

Lemma 2.4 *If* $v(t) \in C[0,1]$ *satisfies*

$$\begin{cases} -D^{\alpha}v(t) \ge 0, & 0 < t < 1, \\ v(0) = 0, \quad v'(0) = 0, \quad D^{\gamma}v(1) \ge 0, \end{cases} \tag{2.5}$$

then $v(t) \ge 0$ *for* $t \in [0,1]$.

Proof By Lemma 2.1 we know that (2.1) has a unique solution

$$v(t) = \int_0^1 G(t,s)h(s)\,ds + \frac{\Gamma(\alpha-\gamma)lt^{\alpha-1}}{\Gamma(\alpha)}.$$

In view of Lemma 2.3, we have $G(t,s) \ge 0$ for $t,s \in [0,1]$. Let $h(t) \ge 0$ and $l \ge 0$. Then we obtain (2.5) and $v(t) \ge 0$ for $t \in [0,1]$. □

Lemma 2.5 *Let* $\lambda \ge 0$ *and* $\Gamma(\beta+\omega) > \lambda\xi^{\beta+\omega-1}$. *If* $x(t) \in C[0,1]$ *satisfies*

$$\begin{cases} -D^{\beta}x(t) \ge 0, & 0 < t < 1, \\ x(0) = 0, \quad D^{\beta-1}x(1) \ge \lambda I^{\omega}x(\xi), \end{cases} \tag{2.6}$$

then $x(t) \ge 0$ *for* $t \in [0,1]$.

Proof We can easily prove Lemma 2.5 similarly to Lemma 2.4. □

Lemma 2.6 *Assume that* $h(t) \in C[0,1]$ *and* $\Gamma(\beta+\omega) \ne \lambda\xi^{\beta+\omega-1}$. *Then the following boundary value problem*

$$\begin{cases} -D^{\alpha}(\phi_p(-D^{\beta}x(t))) = h(t), & 0 < t < 1, \\ D^{\beta}x(0) = 0, \quad (\phi_p(-D^{\beta}x(0)))' = 0, \quad D^{\gamma}(\phi_p(-D^{\beta}x(1))) = l, \\ x(0) = 0, \quad D^{\beta-1}x(1) = \lambda I^{\omega}x(\xi) + k, \end{cases} \tag{2.7}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s)\phi_q\left(\int_0^1 G(s,\tau)h(\tau)\,d\tau + \frac{\Gamma(\alpha-\gamma)ls^{\alpha-1}}{\Gamma(\alpha)}\right)ds + \frac{k\Gamma(\beta+\omega)t^{\beta-1}}{\Gamma(\beta)[\Gamma(\beta+\omega)-\lambda\xi^{\beta+\omega-1}]}.$$

Proof Let $\phi_p(-D^\beta x(t)) = v(t)$ and consider the boundary value problem

$$\begin{cases} -D^\alpha v(t) = h(t), & 0<t<1, \\ v(0)=0, \quad v'(0)=0, \quad D^\gamma v(1) = l. \end{cases} \tag{2.8}$$

By Lemma 2.1 we obtain

$$v(t) = \int_0^1 G(t,s)h(s)\,ds + \frac{\Gamma(\alpha-\gamma)lt^{\alpha-1}}{\Gamma(\alpha)}.$$

Noting that $\phi_p(-D^\beta x(t)) = v(t)$ and $-D^\beta x(t) = \phi_q(v(t))$, we get that the boundary problem (2.7) is equivalent to the following problem:

$$\begin{cases} -D^\beta x(t) = \phi_q(v(t)), & 0<t<1, \\ x(0)=0, \quad D^{\beta-1}x(1) = \lambda I^\omega x(\xi) + k. \end{cases} \tag{2.9}$$

By Lemma 2.2 the solution of (2.9) can be written as

$$x(t) = \int_0^1 H(t,s)\phi_q\bigl(v(s)\bigr)\,ds + \frac{k\Gamma(\beta+\omega)t^{\beta-1}}{\Gamma(\beta)[\Gamma(\beta+\omega)-\lambda\xi^{\beta+\omega-1}]}. \tag{2.10}$$

Combining with (2.8) and (2.9), we assert that the boundary problem (2.7) has a unique solution

$$x(t) = \int_0^1 H(t,s)\phi_q\left(\int_0^1 G(s,\tau)h(\tau)\,d\tau + \frac{\Gamma(\alpha-\gamma)ls^{\alpha-1}}{\Gamma(\alpha)}\right)ds + \frac{k\Gamma(\beta+\omega)t^{\beta-1}}{\Gamma(\beta)[\Gamma(\beta+\omega)-\lambda\xi^{\beta+\omega-1}]}.$$

□

3 Main results

Let $E = \{x : x \in C[0,1], D^\beta x(t) \in C[0,1]\}$ be endowed with the norm $\|x\|_\beta = \|x\| + \|D^\beta x\|$, where $\|x\| = \max_{0\le t\le 1}|x(t)|$ and $\|D^\beta x\| = \max_{0\le t\le 1}|D^\beta x(t)|$. Then $(E, \|\cdot\|_\beta)$ is a Banach space.

Definition 3.1 We say that $x(t) \in E$ is a lower solution of problem (1.1), if

$$\begin{cases} -D^\alpha(\phi_p(-D^\beta x(t))) \le f(t,x(t),D^\beta x(t)), & 0<t<1, \\ D^\beta x(0)=0, \quad (\phi_p(-D^\beta x(0)))'=0, \quad D^\gamma(\phi_p(-D^\beta x(1))) \le I^\nu(\phi_p(-D^\beta x(\eta))), \\ x(0)=0, \quad D^{\beta-1}x(1) \le I^\omega g(\xi,x(\xi)) + k, \end{cases}$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

We further need the following assumptions.

(H_1) $x_0, y_0 \in E$ are lower and upper solutions of problem (1.1), respectively, and $x_0(t) \leq y_0(t)$ and $D^\beta y_0(t) \leq D^\beta x_0(t)$ for $t \in [0,1]$.

(H_2) The function $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies

$$f(t, y, D^\beta y) \geq f(t, x, D^\beta x)$$

if $x_0(t) \leq x(t) \leq y(t) \leq y_0(t)$ and $D^\beta y_0(t) \leq D^\beta y(t) \leq D^\beta x(t) \leq D^\beta x_0(t)$ for $t \in [0,1]$.

(H_3) There exists a constant $\lambda \geq 0$ such that $\Gamma(\beta + \omega) > \lambda \xi^{\beta+\omega-1}$ and

$$g(t, y) - g(t, x) \geq \lambda(y - x)$$

if $x_0(t) \leq x(t) \leq y(t) \leq y_0(t)$ for $t \in [0,1]$.

Theorem 3.1 *Suppose that* (H_1)-(H_3) *hold. Then boundary value problem* (1.1) *has an extremal solution* $x^*, y^* \in [x_0, y_0]$. *Moreover,*

$$x_0(t) \leq x^*(t) \leq y^*(t) \leq y_0(t)$$

and

$$D^\beta y_0(t) \leq D^\beta y^*(t) \leq D^\beta x^*(t) \leq D^\beta x_0(t) \quad \text{for } t \in [0,1].$$

Proof For $n = 0, 1, 2 \ldots$, we define

$$\begin{cases} -D^\alpha(\phi_p(-D^\beta x_{n+1}(t))) = f(t, x_n(t), D^\beta x_n(t)), \quad 0 < t < 1, \\ D^\beta x_{n+1}(0) = 0, \qquad (\phi_p(-D^\beta x_{n+1}(0)))' = 0, \\ D^\gamma(\phi_p(-D^\beta x_{n+1}(1))) = I^\nu(\phi_p(-D^\beta x_n(\eta))), \\ x_{n+1}(0) = 0, \qquad D^{\beta-1} x_{n+1}(1) = I^\omega\{g(\xi, x_n(\xi)) + \lambda[x_{n+1}(\xi) - x_n(\xi)]\} + k, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -D^\alpha(\phi_p(-D^\beta y_{n+1}(t))) = f(t, y_n(t), D^\beta y_n(t)), \quad 0 < t < 1, \\ D^\beta y_{n+1}(0) = 0, \qquad (\phi_p(-D^\beta y_{n+1}(0)))' = 0, \\ D^\gamma(\phi_p(-D^\beta y_{n+1}(1))) = I^\nu(\phi_p(-D^\beta y_n(\eta))), \\ y_{n+1}(0) = 0, \qquad D^{\beta-1} y_{n+1}(1) = I^\omega\{g(\xi, y_n(\xi)) + \lambda[y_{n+1}(\xi) - y_n(\xi)]\} + k. \end{cases} \tag{3.2}$$

By Lemma 2.6 we know that (3.1) and (3.2) have unique solutions

$$\begin{aligned} x_{n+1}(t) = {} & \int_0^1 H(t,s)\phi_q\biggl(\int_0^1 G(s,\tau) f(\tau, x_n(\tau), D^\beta x_n(\tau))\,d\tau \\ & + \frac{\Gamma(\alpha-\gamma)s^{\alpha-1} I^\nu(\phi_p(-D^\beta x_n(\eta)))}{\Gamma(\alpha)}\biggr)\,ds \\ & + \frac{\Gamma(\beta+\omega)t^{\beta-1}\{I^\omega[g(\xi, x_n(\xi)) - \lambda x_n(\xi)] + k\}}{\Gamma(\beta)[\Gamma(\beta+\omega) - \lambda\xi^{\beta+\omega-1}]} \end{aligned}$$

and

$$y_{n+1}(t) = \int_0^1 H(t,s)\phi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau, y_n(\tau), D^\beta y_n(\tau)\bigr)\,d\tau \\ + \frac{\Gamma(\alpha-\gamma)s^{\alpha-1}I^\nu(\phi_p(-D^\beta y_n(\eta)))}{\Gamma(\alpha)}\biggr)\,ds \\ + \frac{\Gamma(\beta+\omega)t^{\beta-1}\{I^\omega[g(\xi,y_n(\xi)) - \lambda y_n(\xi)] + k\}}{\Gamma(\beta)[\Gamma(\beta+\omega) - \lambda\xi^{\beta+\omega-1}]}.$$

First, we show that $x_0(t) \le x_1(t) \le y_1(t) \le y_0(t)$, and $D^\beta y_0 \le D^\beta y_1(t) \le D^\beta x_1(t) \le D^\beta x_0(t)$, $t \in [0,1]$. Let $\varepsilon(t) = \phi_p(-D^\beta x_1(t)) - \phi_p(-D^\beta x_0(t))$. From (3.1) and (H_1) we obtain

$$\begin{cases} -D^\alpha\varepsilon(t) = -D^\alpha(\phi_p(-D^\beta x_1(t))) + D^\alpha(\phi_p(-D^\beta x_0(t))) \\ \qquad \ge f(t,x_0(t),D^\beta x_0(t)) - f(t,x_0(t),D^\beta x_0(t)) = 0, \\ \varepsilon(0) = 0, \qquad \varepsilon'(0) = 0, \\ D^\gamma\varepsilon(1) = D^\gamma(\phi_p(-D^\beta x_1(t))) - D^\gamma(\phi_p(-D^\beta x_0(t))) \\ \qquad \ge I^\omega(\phi_p(-D^\beta x_0(\eta))) - I^\omega(\phi_p(-D^\beta x_0(\eta))) = 0. \end{cases}$$

In view of Lemma 2.4, we have $\phi_p(-D^\beta x_1(t)) \ge \phi_p(-D^\beta x_0(t))$, $t \in [0,1]$, since $\phi_p(x)$ is nondecreasing, and thus

$$D^\beta x_1(t) \le D^\beta x_0(t). \tag{3.3}$$

Let $v(t) = x_1(t) - x_0(t)$, it follows from (3.1), (3.3) and (H_3) that

$$\begin{cases} -D^\beta v(t) = -D^\beta x_1(t) + D^\beta x_0(t) \ge 0, \quad t \in [0,1], \\ v(0) = 0, \\ D^{\beta-1}v(1) \ge I^\omega\{g(\xi,x_0(\xi)) + \lambda[x_1(\xi) - x_0(\xi)]\} - I^\omega g(\xi,x_0(\xi)) \ge \lambda I^\omega v(\xi). \end{cases}$$

According to Lemma 2.5, we have $x_1(t) \ge x_0(t)$ for $t \in [0,1]$.

Using similar reasoning, we can show that $y_0(t) \ge y_1(t)$ and $D^\beta y_0(t) \le D^\beta y_1(t)$. Now, we let $w(t) = \phi_p(-D^\beta y_1(t)) - \phi_p(-D^\beta x_1(t))$. From (H_1) and (H_2) we have

$$\begin{cases} -D^\alpha w(t) = -D^\alpha\phi_p(-D^\beta y_1(t)) + D^\alpha\phi_p(-D^\beta x_1(t)) \\ \qquad = f(t,y_0(t),D^\beta y_0(t)) - f(t,x_0(t),D^\beta x_0(t)) \ge 0, \\ w(0) = 0, \qquad w'(0) = 0, \\ D^\gamma w(1) = D^\gamma(\phi_p(-D^\beta y_1(t))) - D^\gamma(\phi_p(-D^\beta x_1(t))) \\ \qquad = I^\nu(\phi_p(-D^\beta y_0(\eta))) - I^\nu(\phi_p(-D^\beta x_0(\eta))) \ge 0. \end{cases}$$

In view of Lemma 2.4, we have $w(t) \ge 0$ for $t \in [0,1]$. Thus we have $\phi_p(-D^\beta y_1(t)) \ge \phi_p(-D^\beta x_1(t))$, that is, $D^\beta y_1(t) \le D^\beta x_1(t)$, since $\phi_p(x)$ is nondecreasing. Therefore $D^\beta y_0(t) \le D^\beta y_1(t) \le D^\beta x_1(t) \le D^\beta x_0(t)$ for $t \in [0,1]$. Let $\delta(t) = y_1(t) - x_1(t)$. From (H_3) we get

$$-D^\beta\delta(t) = -D^\beta y_1(t) + D^\beta x_1(t) \ge 0.$$

Also, $\delta(0) = 0$, and

$$\begin{aligned} D^{\beta-1}\delta(1) &= I^{\omega}\{g(\xi, y_0(\xi)) + \lambda[y_1(\xi) - y_0(\xi)]\} - I^{\omega}\{g(\xi, x_0(\xi)) + \lambda[x_1(\xi) - x_0(\xi)]\} \\ &= I^{\omega}\{g(\xi, y_0(\xi)) - g(\xi, x_0(\xi)) + \lambda[y_1(\xi) - y_0(\xi)] - \lambda[x_1(\xi) - x_0(\xi)]\} \\ &\geq I^{\omega}\{\lambda[y_0(\xi) - x_0(\xi)] + \lambda[y_1(\xi) - y_0(\xi)] - \lambda[x_1(\xi) - x_0(\xi)]\} \\ &= \lambda I^{\omega}\delta(\xi). \end{aligned}$$

Moreover, we get $y_1(t) \geq x_1(t)$ from Lemma 2.5. Hence, we have the relation $x_0(t) \leq x_1(t) \leq y_1(t) \leq y_0(t)$ for $t \in [0,1]$.

In the following, we show that $x_1(t)$ and $y_1(t)$ are lower and upper solutions of problem (1.1), respectively. From (3.1)-(3.2) and (H_1)-(H_3) we get

$$-D^{\alpha}(\phi_p(-D^{\beta}x_1(t))) = f(t, x_0(t), D^{\beta}x_0(t)) \leq f(t, x_1(t), D^{\beta}x_1(t)),$$

Also,

$$\begin{aligned} &D^{\beta}x_1(0) = 0, \qquad (\phi_p(-D^{\beta}x_0(0)))' = 0, \\ &D^{\gamma}(\phi_p(-D^{\beta}x_1(1))) = I^{\nu}(\phi_p(-D^{\beta}x_0(\eta))) \leq I^{\nu}(\phi_p(-D^{\beta}x_1(\eta))), \end{aligned}$$

and $x_1(0) = 0$,

$$\begin{aligned} D^{\beta-1}x_1(1) &= I^{\omega}\{g(\xi, x_0(\xi)) - g(\xi, x_1(\xi)) + g(\xi, x_1(\xi)) + \lambda[x_1(\xi) - x_0(\xi)]\} + k \\ &\leq I^{\omega}\{\lambda[x_0(\xi) - x_1(\xi)] + g(\xi, x_1(\xi)) + \lambda[x_1(\xi) - x_0(\xi)]\} + k \\ &= I^{\beta}g(\xi, x_1(\xi)) + k. \end{aligned}$$

This proves that $x_1(t)$ is a lower solution of problem (1.1). Similarly, we can obtain that $y_1(t)$ is an upper solution of (1.1).

Using mathematical induction, we see that

$$x_0(t) \leq x_1(t) \leq \cdots \leq x_n(t) \leq \cdots \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t)$$

and

$$D^{\beta}y_0(t) \leq D^{\beta}y_1(t) \leq \cdots \leq D^{\beta}y_n(t) \leq \cdots \leq D^{\beta}x_n(t) \leq \cdots \leq D^{\beta}x_1(t) \leq D^{\beta}x_0(t)$$

for $t \in [0,1]$ and $n = 1, 2, 3, \ldots$.

Since the sequence $\{x_n(t)\}$ is nondecreasing and bounded from above, the sequence $\{y_n(t)\}$ is nonincreasing and bounded from below. By standard argument we know that the sequences $\{x_n(t)\}$ and $\{y_n(t)\}$ uniformly converge to their limit functions $x^*(t)$ and $y^*(t)$, respectively, that is,

$$\lim_{n\to\infty} x_n(t) = x^*(t), \qquad \lim_{n\to\infty} y_n(t) = x^*(t), \quad \forall t \in [0,1],$$

and

$$\lim_{n\to\infty} D^{\beta}x_n(t) = D^{\alpha}x^*(t), \qquad \lim_{n\to\infty} D^{\beta}y_n(t) = D^{\alpha}x^*(t), \quad \forall t \in [0,1].$$

Moreover, from (3.1) and (3.2) we can obtain that $x^*(t)$ and $y^*(t)$ are solutions of problem (1.1).

Finally, we prove that $x^*(t)$ and $y^*(t)$ are the minimal and maximal solutions of problem (1.1), respectively. Let $x(t) \in [x_0, y_0]$ be any solution of problem (1.1). We suppose that, for some n, $x_n(t) \le x(t) \le y_n(t)$ and $D^{\beta}y_n(t) \le D^{\beta}x(t) \le D^{\beta}x_n(t)$ for $t \in [0,1]$. Let $p(t) = (\phi_p(-D^{\beta}x(t))) - \phi_p(-D^{\beta}x_{n+1}(t))$ and $q(t) = (\phi_p(-D^{\beta}y_{n+1}(t))) - \phi_p(-D^{\beta}x(t))$. Then by assumption (H_2) we see that

$$\begin{cases} -D^{\alpha}p(t) = f(t,x(t),D^{\beta}x(t)) - f(t,x_n(t),D^{\beta}x_n(t)) \ge 0, \\ p(0) = 0, \qquad p'(0) = 0, \\ D^{\gamma}p(1) = I^{\nu}(\phi_p(-D^{\beta}x(\eta))) - I^{\nu}(\phi_p(-D^{\beta}x_n(\eta))) \ge 0, \end{cases}$$

and

$$\begin{cases} -D^{\alpha}q(t) \ge 0, \\ q(0) = 0, \qquad q'(0) = 0, \\ D^{\gamma}q(1) \ge 0. \end{cases}$$

Using Lemma 2.4, we have

$$D^{\beta}y_{n+1}(t) \le D^{\beta}x(t) \le D^{\beta}x_{n+1}(t). \tag{3.4}$$

Let $m(t) = x(t) - x_{n+1}(t)$ and $n(t) = y_{n+1}(t) - x(t)$. By assumption (H_3) and (3.4) we get

$$\begin{cases} -D^{\beta}m(t) = -D^{\beta}x(t) + D^{\beta}x_{n+1}(t) \ge 0, \quad t \in [0,1], \\ m(0) = 0, \\ D^{\beta-1}m(1) = I^{\omega}\{g(\xi,x(\xi)) - g(\xi,x_n(\xi)) - \lambda[x_{n+1}(\xi) - x_n(\xi)]\} \ge \lambda I^{\omega}m(\xi), \end{cases}$$

and

$$\begin{cases} -D^{\beta}n(t) \ge 0, \quad t \in [0,1], \\ n(0) = 0, \\ D^{\beta-1}n(1) \ge \lambda I^{\omega}n(\xi). \end{cases}$$

These and Lemma 2.5 imply that $x_{n+1}(t) \le x(t) \le y_{n+1}(t)$, $t \in [0,1]$, so by induction $x^*(t) \le x(t) \le y^*(t)$ and $D^{\beta}y^*(t) \le D^{\beta}x(t) \le D^{\beta}x^*(t)$, $t \in [0,1]$, as $n \to \infty$. The proof is complete. □

Example Consider the following fractional boundary value problem:

$$\begin{cases} -D^{\frac{5}{2}}(\phi_4(-D^{\frac{3}{2}}x(t))) = \frac{1}{10}tx - \frac{1}{(t+3)^2}D^{\frac{3}{2}}x(t) + \frac{1}{7}t^{\frac{1}{2}}, \quad 0 < t < 1, \\ D^{\frac{3}{2}}x(0) = 0, \qquad (\phi_4(-D^{\frac{3}{2}}x(0)))' = 0, \\ D^{\frac{1}{4}}(\phi_p(-D^{\frac{3}{2}}x(1))) = I^{\frac{5}{2}}(\phi_4(-D^{\frac{3}{2}}x(\frac{1}{2}))), \\ x(0) = 0, \qquad D^{\frac{1}{2}}x(1) = I^{\frac{3}{2}}g(\frac{1}{4}, x(\frac{1}{4})) + 0.1 = \frac{1}{\Gamma(\frac{3}{2})}\int_0^{\frac{1}{4}}(\frac{1}{4} - s)^{\frac{1}{2}}(s+1)x(s)\,ds + 0.1, \end{cases} \tag{3.5}$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $\gamma = \frac{1}{4}$, $\nu = \frac{5}{2}$, $\omega = \frac{3}{2}$, $\eta = \frac{1}{2}$, $\xi = \frac{1}{4}$, $k = 0.1$, $p = 4$, and

$$\begin{cases} f(t, x, D^{\beta}x) = \frac{1}{10}tx - \frac{1}{(t+3)^2}D^{\frac{3}{2}}x(t), \\ g(t, x) = (t+1)x. \end{cases}$$

Take $x_0(t) = 0$ and $y_0(t) = t^{\frac{1}{2}} - \frac{\sqrt{\Pi}}{4}t^2 + \frac{2}{15\sqrt{\Pi}}t^{\frac{5}{2}}$. Then $-1 \le -t^{\frac{1}{2}} + \frac{1}{4}t = D^{\beta}y_0(t) \le D^{\beta}x_0(t) = 0$. It is not difficult to verify that x_0, y_0 are lower and upper solutions of problem (3.5), respectively. Moreover,

$$g(t, y) - g(t, x) = (t+1)(y - x) \ge (y - x),$$

where $x_0(t) \le x \le y \le y_0(t)$.

For $\lambda = 1$, we have

$$\Gamma(\beta + \omega) = \Gamma\left(\frac{3}{2} + \frac{3}{2}\right) = 2 > \lambda\xi^{\beta+\omega-1} = 1 \cdot \left(\frac{1}{4}\right)^2.$$

All conditions (H_1), (H_2), and (H_3) are satisfied. Therefore by Theorem 3.1 the boundary value problem (3.5) has extremal solutions in $[x_0(t), y_0(t)]$.

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Authors' contributions
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