Yin and Wen *Advances in Difference Equations* (2017) 2017:369
DOI 10.1186/s13662-017-1427-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

# Dynamical properties of a fractional reaction-diffusion trimolecular biochemical model with autocatalysis

Hongwei Yin[1,2*] and Xiaoqing Wen[1]

*Correspondence:
hongwei-yin@hotmail.com;
yinhongwei@ncu.edu.cn
[1]School of Sciences, Nanchang University, Nanchang, 330031, P.R. China
[2]Numerical Simulation and High-Performance Computing Laboratory, Nanchang University, Nanchang, 330031, P.R. China

**Abstract**

In this paper, a reaction-diffusion trimolecular biochemical model with autocatalysis and fractional-order derivative is proposed. We establish the existence and uniqueness of a positive solution to this system in a Besov space. Besides, for this system, we obtain stability, Hopf and Turing bifurcations and spatial patterns. These dynamic behaviors of this system are slightly different from those of its corresponding first-order system. The difference is illustrated by performing some numerical simulations, through which our main results are verified.

**Keywords:** Caputo's derivative; reaction-diffusion trimolecular model; Besov spaces; spatial patterns; Hopf and Turing bifurcations

## 1 Introduction

In this paper, we deal with a trimolecular autocatalytic biochemical model. The reaction mechanism is

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} U, \qquad B \xrightarrow{k_2} V, \qquad 2U + V \xrightarrow{k_3} 3U,$$

in which $A$, $B$, $U$ and $V$ are chemical reactants and products. Moreover, nonnegative constants $k_i$, $i = -1, 1, 2, 3$, represent the reaction rates. It is assumed that the first step of the reaction process is reversible, that the last two steps of the reaction process are irreversible and that two molecules of $U$ react with one molecule of $V$ to create an additional molecule of $U$. This autocatalytic reaction creates a positive feedback loop, a common component of the regulatory network [1]. $V$ is considered to be stable and does not decay on the relevant timescales of the system, whereas $U$ can decay back to $A$. Each of these components could in reality represent multiple molecules, but for the sake of simplicity we consider them as single entities. Further, we assume that $U$ and $V$ are diffusible in a reactor, disregarding convective phenomena and considering an isothermal process only. Then, the above scheme can be described by the following nonlinear reaction-diffusion system:

$$\begin{cases} \frac{\partial [U](x,\tau)}{\partial \tau} = d_U \Delta [U](x,\tau) + k_1[A] - k_{-1}[U](x,\tau) + k_3[U]^2(x,\tau)[V](x,\tau), \\ \frac{\partial [V](x,\tau)}{\partial \tau} = d_V \Delta [V](x,\tau) + k_2[B] - k_3[U]^2(x,\tau)[V](x,\tau), \end{cases} \tag{1.1}$$

where the mark $[\cdot]$ represents the density of some chemical component. $\Delta$ is the Laplacian operator, showing the molecules' diffusion, and $d_U$ and $d_V$ denote the Fickian diffusion coefficients of $[U]$ and $[V]$, respectively, which are assumed to be positive constants. To simplify system (1.1), we introduce the new dimensionless quantities

$$u = [U](k_3/k_{-1})^{1/2}, \qquad v = [V](k_3/k_{-1})^{1/2}, \qquad a = \frac{k_1[A]}{k_{-1}}(k_3/k_{-1})^{1/2}, \qquad t = \tau k_1,$$
$$b = \frac{k_2[B]}{k_{-1}}(k_3/k_{-1})^{1/2}, \qquad d_1 = d_U/k_{-1}, \qquad d_2 = d_V/k_{-1}.$$

Substituting these new variables into (1.1), we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + a - u + u^2 v, \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + b - u^2 v. \end{aligned} \tag{1.2a}$$

Here, we assume that these biochemical reactions are limited to a bounded sufficiently regular domain $\Omega \in \mathbb{R}^N$, where $N$ is a spatial dimension number such as $N = 1, 2$ or 3. Further, we assume that system (1.2a) is equipped with the Neumann boundary conditions,

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, \tag{1.2b}$$

where $\nu$ is the unit outward normal to $\partial\Omega$. Besides, we set initial conditions for model (1.2a),

$$u(x,0) = u_0(x) \geq 0, \qquad v(x,0) = v_0(x) \geq 0, \quad x \in \Omega. \tag{1.2c}$$

In fact, system (1.2a)-(1.2c) is called *Schnakenberg* model [2]. At present, model (1.2a)-(1.2c) has drawn some researchers' attention. In detail, Liu et al. obtained the Turing and Hopf bifurcations of system (1.2a)-(1.2c). Madzvamuse et al. in [3] applied theoretical analysis and numerical simulations to study a spatial pattern of the cross-diffusion form of (1.1). Based on this result, Gambino et al. in [4] used the Stuart-Landau equation to capture patterns of this model. Besides, Jacobo and Hudspeth in [5] utilized model (1.2a)-(1.2c) to investigate pattern formation of hair-bundle morphogenesis.

Model (1.2a)-(1.2c) is an integer-order system, that is, the first-order derivative and the second-order derivative with respect to the time variable $t$ and the spatial variable $x$, respectively. Wherein, the first-order derivative to the variable $t$ implies the transient change rate of these reactions. However, due to the complexity of biochemical reactions, chemical reaction processes are often affected by or depend on the history of chemical reactions. Thus, this phenomenon can be described by fractional-order differential equations.

In fact, fractional calculus is an old mathematical topic developed as a pure theoretical field of mathematics for more than three centuries. Fractional-order derivatives allow us to deal comfortably with memory effects in a dynamical system [6–8], and thus it can be successfully applied in some fields such as physics, control engineering, biochemical reaction, signal processing, optimal control, quantum mechanics [9, 10] and so on. At present, a large number of monographs and papers [11–21] are devoted to fractional dynamical systems. From this viewpoint, we introduce fractional-order derivative into model (1.2a)-(1.2c), which results in a completely different model. In fact, model (1.2a)-(1.2c) ultimately turns into

$$\frac{\partial^\alpha u}{\partial t^\alpha} = d_1\Delta u + a - u + u^2 v, \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.3a}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = d_2\Delta v + b - u^2 v, \quad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.3b}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, \tag{1.3c}$$

$$u(x,0) = u_0(x) \geq 0, \qquad v(x,0) = v_0(x) \geq 0, \quad x \in \Omega, \tag{1.3d}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the standard Caputo derivative, $\alpha \in (0,1]$. As far as our knowledge goes, few literature works researched the dynamical properties of the fractional-order model (1.3a)-(1.3d) such as the existence of solution, stability and spatial patterns. In this paper, we first prove the existence and uniqueness of a solution in Besov spaces for model (1.3a)-(1.3d) with low regularity initial data, which is a matter of interest in the mathematical analysis, as well as the existence of solution, stability and spatial patterns. Recently, the problem of initial data in Besov spaces has been widely considered. For instance, Zhai in [22, 23] studied the generalized Keller-Segel system of chemotaxis. For more works dealing with PDEs in Besov spaces, we refer the readers to [24–26].

The outline of this paper follows here. In Section 2, some necessary lemmas and definitions are introduced. In Section 3, the solution to the fractional-order PDEs model (1.3a)-(1.3d) is established in Besov spaces. In Section 4 we study the stability and Hopf bifurcation of system (1.3a)-(1.3d) and perform numerical simulations. In Section 5, we investigate the Turing bifurcation of system (1.3a)-(1.3d), and some numerical simulations are made to show spatial patterns. Finally, we end our study with some conclusions.

## 2 Preliminaries

**Definition 2.1** [12, 27] Caputo's derivative of order $q$ with the lower limit 0 for the function $h : [0,\infty) \to \mathbb{R}$ can be written as

$$D^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{q+1-n}}\, ds, \quad n-1 < q < n, n \in \mathbb{Z}^+.$$

The space $B^\sigma_{p,q,\mathcal{N}}$ denotes the Besov space in $\Omega$ with the Neumann boundary conditions. This space can be regarded as the real interpolation space $(L^p(\Omega), W^{2,p}_{\mathcal{N}})_{\sigma/2,q}$ for $W^{2,p}_{\mathcal{N}} = \{\varphi \in W^{2,p}(\Omega) : \partial_\nu \varphi = 0 \text{ on } \partial\Omega\}$. It is well known that the operators $-d_1\Delta + I$ and $-d_2\Delta$ are sectorial operators from $W^{2,p}_{\mathcal{N}}$ into $L^p(\Omega)$, which are the infinitesimal generators of analytic and positive semigroups respectively denoted by $G_1(t)$ and $G_2(t)$. Thus, the following property holds:

$$\bigl\| G_i(t)\phi \bigr\|_{B^\sigma_{p,q,\mathcal{N}}} \leq M_1 t^{-\sigma/2} \|\phi\|_{L^p}, \quad i = 1,2, \tag{2.1}$$

where $M_1 \geq 1$ and $\sigma \neq 1 + 1/p$. Moreover, if $\sigma \leq \sigma' < 2$ and $\sigma' \neq 1 + 1/p$, then

$$\bigl\| G_i(t)\phi \bigr\|_{B^{\sigma'}_{p,q,\mathcal{N}}} \leq M_1 t^{\sigma/2-\sigma'/2} \|\phi\|_{B^\sigma_{p,q,\mathcal{N}}}, \quad i = 1,2, \tag{2.2}$$

see Theorem V.2.1.3 in [28]. Furthermore, if $1 < p < \infty$, $-\infty < s < \infty$ and $1 \le q_1 \le q_2 \le \infty$, we have

$$B^s_{p,q_1}(\Omega) \hookrightarrow B^s_{p,q_2}(\Omega). \tag{2.3}$$

Let $-\infty < s < \infty$, $1 < p < q < \infty$ and $s - N/p \ge -N/q$, then

$$B^s_{p,q}(\Omega) \hookrightarrow L^q(\Omega). \tag{2.4}$$

We are interested in mild solutions to (1.3a)-(1.3d), i.e., there exists a pair $(u, v) \in C((0,\tau); B^\sigma_{p,q,\mathcal{N}} \times B^\sigma_{p,q,\mathcal{N}})$ such that, for $t > 0$,

$$u(t) = T^{(1)}_\alpha(t)u_0 + \int_0^t S^{(1)}_\alpha(t-s)f_1\bigl(u(s), v(s)\bigr)\,ds, \tag{2.5a}$$

$$v(t) = T^{(2)}_\alpha(t)v_0 + \int_0^t S^{(2)}_\alpha(t-s)f_2\bigl(u(s), v(s)\bigr)\,ds, \tag{2.5b}$$

$$u(0) = u_0 \in B^\sigma_{p,q,\mathcal{N}}, \qquad v(0) = v_0 \in B^\sigma_{p,q,\mathcal{N}}, \tag{2.5c}$$

where

$$T^{(i)}_\alpha(t) = \int_0^\infty \zeta_\alpha(\theta)G_i\bigl(t^\alpha\theta\bigr)\,d\theta, \tag{2.6a}$$

$$S^{(i)}_\alpha(t) = \alpha\int_0^\infty \theta t^{\alpha-1}\zeta_\alpha(\theta)G_i\bigl(t^\alpha\theta\bigr)\,d\theta, \tag{2.6b}$$

$$\zeta_\alpha(\theta) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda\theta}E_{\alpha,1}(-\lambda)\,d\lambda, \tag{2.6c}$$

for $i = 1, 2$, and $\zeta_\alpha(\theta)$ is a probability density function defined on $[0,\infty)$ [29, 30], and $\Gamma$ is a contour starting and ending at $-\infty$. $E_{\alpha,1}(-\lambda)$ is a Mittag-Leffler function [27]. By the property of the semigroups $G_i(t)$ for $i = 1, 2$, the operators $T^{(i)}_\alpha(t)$ and $S^{(i)}_\alpha(t)$ for $i = 1, 2$ are positive ones.

Consider the fractional system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(x), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{2.7}$$

where $\alpha \in (0,1)$, $A \in \mathbb{R}^{n\times n}$, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $Df(0) = 0$. The following lemma was proved in [31].

**Lemma 2.2** *System* (2.7) *with origin as a hyperbolic equilibrium point is linearly stable if each eigenvalue* $\lambda$ *of* $A$, $|\arg(\lambda)| > \frac{\pi\alpha}{2}$; *system* (2.7) *is linearly unstable if* $|\arg(\lambda)| < \frac{\pi\alpha}{2}$ *for some eigenvalue* $\lambda$ *of* $A$.

Now, we consider a Hopf bifurcation of the fractional system with a parameter $\mu \in \mathbb{R}$ as follows:

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(\mu)x(t) + f(\mu, x), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{2.8}$$

where $\alpha \in (0,1)$, $A(\mu) \in \mathbb{R}^{n\times n}$, $f(\mu, x) \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$, $Df(\mu, 0) = 0$.

It is well known that for system (2.8) with the first-order derivative, Hopf bifurcation conditions are

$$\operatorname{Re}\big[\lambda(0)\big] = 0, \qquad \operatorname{Im}\big[\lambda(0)\big] \neq 0, \qquad \frac{d\{\operatorname{Re}[\lambda(0)]\}}{d\mu} \neq 0.$$

The conditions of fractional-order Hopf bifurcation differ from those of the first-order case and are found in [32, 33], that is,

$$\big|\arg\big[\lambda\big(\mu^*\big)\big]\big| = \frac{\alpha\pi}{2}, \qquad \frac{d\{[\lambda(\mu)]\}}{d\mu}\bigg|_{\mu=\mu^*} \neq 0. \tag{2.9}$$

## 3 Existence and uniqueness of solution to (1.3a)-(1.3d)

In this section, we will give out some necessary a priori estimates. First, we introduce some notations. Let $\sigma \neq 1 + 1/p$. We look for a mild solution in the closed ball

$$\Big\{(u,v)^T \in C\big([0,\tau] : B^\sigma_{p,q,\mathcal{N}} \times B^\sigma_{p,q,\mathcal{N}}\big) : \sup_{t\in[0,\tau]} \big(\big\|u(t) - u_0\big\|_{B^\sigma_{p,q,\mathcal{N}}} + \big\|v(t) - v_0\big\|_{B^\sigma_{p,q,\mathcal{N}}}\big) \leq R\Big\}.$$

Let us denote it by $\mathcal{B} = \mathcal{B}(\tau, R, u_0, v_0)$ for fixed $\tau > 0$ and $R > 0$. It is clear that $\mathcal{B}$ is a complete metric space. We set

$$\mathcal{R} = \max\big\{R + \|u_0\|_{B^\sigma_{p,q,\mathcal{N}}}, R + \|v_0\|_{B^\sigma_{p,q,\mathcal{N}}}\big\}.$$

Next, we have the following.

**Lemma 3.1** *Let $1 < p < \infty$, $\frac{2N}{3p} \leq \sigma < 2$, $\sigma \neq 1 + \frac{1}{p}$ and $1 \leq q \leq 3p$. For $u_0, v_0 \in B^\sigma_{p,q,\mathcal{N}}$ and $(u,v)^T \in \mathcal{B}$, we have*

$$\begin{aligned}&\bigg\|\int_0^t S^{(1)}_\alpha(t-s) f_1\big(u(s), v(s), s\big)\,ds\bigg\|_{B^\sigma_{p,q,\mathcal{N}}} \\ &\quad \leq \alpha M_1 E\big(\theta^{1-\frac{\sigma}{2}}\big)\big(|\Omega|^{1/p} + |\Omega|^{\frac{2}{3p}}\mathcal{R} + \mathcal{R}^3\big)\frac{2}{\alpha(2-\sigma)} t^{\alpha(1-\frac{\sigma}{2})}\end{aligned} \tag{3.1}$$

*and*

$$\begin{aligned}&\bigg\|\int_0^t S^{(2)}_\alpha(t-s) f_2\big(u(s), v(s), s\big)\,ds\bigg\|_{B^\sigma_{p,q,\mathcal{N}}} \\ &\quad \leq \alpha M_1 E\big(\theta^{1-\frac{\sigma}{2}}\big)\big(|\Omega|^{1/p} + \mathcal{R}^3\big)\frac{2}{\alpha(2-\sigma)} t^{\alpha(1-\frac{\sigma}{2})},\end{aligned} \tag{3.2}$$

*where the constant $E(\theta^{1-\frac{\sigma}{2}})$ is the expectation of the function $\theta^{1-\frac{\sigma}{2}}$ for the probability density function $\zeta_\alpha(\theta)$.*

*Proof* For $u, v \in L^{3p}(\Omega)$ and from (2.1), (2.3), (2.4) and (2.6a)-(2.6c), we have

$$\begin{aligned}&\bigg\|\int_0^t S^{(1)}_\alpha(t-s) f_1\big(u(s), v(s), s\big)\,ds\bigg\|_{B^\sigma_{p,q,\mathcal{N}}} \\ &\quad = \bigg\|\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta) G_1\big((t-s)^\alpha\theta\big) f_1\big(u(s), v(s), s\big)\,d\theta\,ds\bigg\|_{B^\sigma_{p,q,\mathcal{N}}}\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha M_1 \int_0^\infty \theta^{1-\frac{\sigma}{2}} \zeta_\alpha(\theta)\, d\theta \int_0^t (t-s)^{\alpha-1-\frac{\sigma\alpha}{2}} \big\| f_1\big(u(s), v(s), s\big)\big\|_{L^p}\, ds \\
&\leq \alpha E\big(\theta^{1-\frac{\sigma}{2}}\big) M_1 \int_0^t (t-s)^{-\frac{\sigma\alpha}{2}+\alpha-1} \big(|\Omega|^{1/p} + |\Omega|^{\frac{2}{3p}} \big\|u(s)\big\|_{L^{3p}} + \big\|u(s)\big\|_{L^{3p}}^2 \big\|v(s)\big\|_{L^{3p}}\big)\, ds \\
&\leq \alpha E\big(\theta^{1-\frac{\sigma}{2}}\big) M_1 \int_0^t (t-s)^{-\frac{\sigma\alpha}{2}+\alpha-1} \big(|\Omega|^{1/p} + |\Omega|^{\frac{2}{3p}} \big\|u(s)\big\|_{B^\sigma_{p,q,\mathcal{N}}} \\
&\quad + \big\|u(s)\big\|_{B^\sigma_{p,q,\mathcal{N}}}^2 \big\|v(s)\big\|_{B^\sigma_{p,q,\mathcal{N}}}\big)\, ds \\
&\leq \alpha M_1 E\big(\theta^{1-\frac{\sigma}{2}}\big)\big(|\Omega|^{1/p} + |\Omega|^{\frac{2}{3p}}\mathcal{R} + \mathcal{R}^3\big) \frac{2}{\alpha(2-\sigma)} t^{\alpha(1-\frac{\sigma}{2})}.
\end{aligned}
$$

According to the properties of the probability density function $\zeta_\alpha$, we conclude that $E(\theta^{1-\frac{\sigma}{2}})$ exists. Thus, we obtain (3.1). Analogously, we can get (3.2). □

**Lemma 3.2** *Let $1 < p < \infty$, $\frac{2N}{3p} \leq \sigma < 2$, $\sigma \neq 1 + \frac{1}{p}$ and $1 \leq q \leq 3p$. For $u_0, v_0 \in B^\sigma_{p,q,\mathcal{N}}$ and $(u_i, v_i) \in \mathcal{B}$, $i = 1, 2$, we have*

$$
\begin{aligned}
&\bigg\| \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta) G_1\big[(t-s)^\alpha\theta\big] \\
&\qquad \times \big(u_2(s) - u_1(s) + u_1^2(s)v_1(s) - u_2^2(s)v_2(s)\big)\, d\theta\, ds \bigg\|_{B^\sigma_{p,q,\mathcal{N}}} \\
&\quad \leq M_2 t^{\alpha(1-\frac{\sigma}{2})} \sup_{t\in[0,\tau]} \big(\big\|u_1(t) - u_2(t)\big\|_{B^\sigma_{p,q,\mathcal{N}}} + \big\|v_1(t) - v_2(t)\big\|_{B^\sigma_{p,q,\mathcal{N}}}\big), \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
&\bigg\| \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta) G_2\big[(t-s)^\alpha\theta\big] \big(u_2^2(s)v_2(s) - u_1^2(s)v_1(s)\big)\, d\theta\, ds \bigg\|_{B^\sigma_{p,q,\mathcal{N}}} \\
&\quad \leq M_3 t^{\alpha(1-\frac{\sigma}{2})} \sup_{t\in[0,\tau]} \big(\big\|u_1(t) - u_2(t)\big\|_{B^\sigma_{p,q,\mathcal{N}}} + \big\|v_1(t) - v_2(t)\big\|_{B^\sigma_{p,q,\mathcal{N}}}\big), \qquad (3.4)
\end{aligned}
$$

*where*

$$
\begin{aligned}
M_2 &= \frac{M_1}{\alpha(2-\sigma)} E\big(\theta^{1-\frac{\alpha}{2}}\big) \max\big\{|\Omega|^{\frac{2}{3p}} + \mathcal{R}^2, 2\mathcal{R}^2\big\}, \\
M_3 &= \frac{2M_1\mathcal{R}^2}{\alpha(2-\sigma)} E\big(\theta^{1-\frac{\alpha}{2}}\big).
\end{aligned}
$$

*Proof* If $u_i, v_i \in L^{3p}(\Omega)$, $i = 1, 2$, we have

$$
\begin{aligned}
&\big\|u_2 - u_1 + u_1^2 v_1 - u_2^2 v_2\big\|_{L^p} \\
&\quad \leq \|u_1 - u_2\|_{L^p} + \big\|u_1^2(v_1 - v_2)\big\|_{L^p} + \big\|v_2\big(u_1^2 - u_2^2\big)\big\|_{L^p} \\
&\quad \leq \|\Omega\|^{\frac{2}{3p}} \|u_1 - u_2\|_{L^{3p}} + \|u_1\|_{L^{3p}}^2 \|v_1 - v_2\|_{L^{3p}} + \|v_2\|_{L^{3p}} \|u_1 - u_2\|_{L^{3p}} \|u_1 + u_2\|_{L^{3p}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\bigg\| \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta) G_1\big[(t-s)^\alpha\theta\big] \\
&\qquad \times \big(u_2(s) - u_1(s) + u_1^2(s)v_1(s) - u_2^2(s)v_2(s)\big)\, d\theta\, ds \bigg\|_{B^\sigma_{p,q,\mathcal{N}}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) \\
&\quad \times \left\| G_1\big[(t-s)^\alpha \theta\big]\big(u_2(s) - u_1(s) + u_1^2(s)v_1(s) - u_2^2(s)v_2(s)\big)\right\|_{B^\sigma_{p,q,\mathcal{N}}} \,d\theta\,ds \\
&\le M_1 E\big(\theta^{1-\frac{\alpha}{2}}\big) \int_0^t (t-s)^{\alpha-1-\frac{\sigma\alpha}{2}} \left\| u_2 - u_1 + u_1^2 v_1 - u_2^2 v_2 \right\|_{L^p} ds \\
&\le M_1 E\big(\theta^{1-\frac{\alpha}{2}}\big) \int_0^t (t-s)^{\alpha-1-\frac{\sigma\alpha}{2}} \big[ |\Omega|^{\frac{2}{3p}} \|u_1 - u_2\|_{L^{3p}} + \|u_1\|^2_{L^{3p}} \|v_1 - v_2\|_{L^{3p}} \\
&\quad + \|v_2\|_{L^{3p}} \|u_1 - u_2\|_{L^{3p}} \|u_1 + u_2\|_{L^{3p}} \big] ds \\
&\le M_1 E\big(\theta^{1-\frac{\alpha}{2}}\big) \int_0^t (t-s)^{\alpha-1-\frac{\sigma\alpha}{2}} \big[ |\Omega|^{\frac{2}{3p}} \|u_1 - u_2\|_{B^\sigma_{p,q,\mathcal{N}}} \\
&\quad + \|u_1\|^2_{B^\sigma_{p,q,\mathcal{N}}} \|v_1 - v_2\|_{B^\sigma_{p,q,\mathcal{N}}} \\
&\quad + \|v_2\|_{B^\sigma_{p,q,\mathcal{N}}} \|u_1 - u_2\|_{B^\sigma_{p,q,\mathcal{N}}} \|u_1 + u_2\|_{B^\sigma_{p,q,\mathcal{N}}} \big] ds \\
&\le M_3 t^{\alpha(1-\frac{\sigma}{2})} \sup_{t\in[0,\tau]} \big( \left\|u_1(t) - u_2(t)\right\|_{B^\sigma_{p,q,\mathcal{N}}} + \left\|v_1(t) - v_2(t)\right\|_{B^\sigma_{p,q,\mathcal{N}}} \big).
\end{aligned}
$$

Thus, we get inequality (3.3). Using the same method, we can get inequality (3.4). □

Note that $\zeta_\alpha(\theta)$ is one-sided probability density function and that $G_i(t)$, $i = 1, 2$, are strong continuous contraction semigroups with respect to the variable $t$. By Lemma 3.1 we can take $\tau_1 > 0$ small enough such that

$$
\left\| T^{(1)}_\alpha(t)u_0 - u_0 \right\|_{B^\sigma_{p,q,\mathcal{N}}} = \left\| \int_0^\infty \zeta_\alpha(\theta)\big[G_1\big(t^\alpha\theta\big) - I\big]u_0 \,d\theta \right\|_{B^\sigma_{p,q,\mathcal{N}}} \le \frac{R}{4}, \tag{3.5}
$$

$$
\left\| T^{(2)}_\alpha(t)u_0 - u_0 \right\|_{B^\sigma_{p,q,\mathcal{N}}} = \left\| \int_0^\infty \zeta_\alpha(\theta)\big[G_2\big(t^\alpha\theta\big) - I\big]u_0 \,d\theta \right\|_{B^\sigma_{p,q,\mathcal{N}}} \le \frac{R}{4}, \tag{3.6}
$$

and

$$
\alpha M_1 E\big(\theta^{1-\frac{\sigma}{2}}\big)\big(|\Omega|^{1/p} + |\Omega|^{\frac{2}{3p}}\mathcal{R} + \mathcal{R}^3\big)\frac{2}{\alpha(2-\sigma)}\tau^{\alpha(1-\frac{\sigma}{2})} \le \frac{R}{4}. \tag{3.7}
$$

We now define maps $P_i : \mathcal{B} \to \mathcal{B}$ for $i = 1, 2$, where

$$
P_1(u,v)(t) = T^{(1)}_\alpha(t)u_0 + \int_0^t S^{(1)}_\alpha(t-s)f_1\big(u(s), v(s)\big)\,ds
$$

and

$$
P_2(u,v)(t) = T^{(2)}_\alpha(t)u_0 + \int_0^t S^{(2)}_\alpha(t-s)f_2\big(u(s), v(s)\big)\,ds.
$$

**Theorem 3.3** *Let $1 < p < \infty$, $\frac{2N}{3p} \le \sigma < 2$, $\sigma \neq 1 + \frac{1}{p}$ and $1 \le q \le 3p$. Then, given $(u_0, v_0)^T \in B^\sigma_{p,q,\mathcal{N}}$, there exists a constant $\tau > 0$ such that problem* (1.3a)-(1.3d) *has a unique locally mild positive solution $(u,v)^T : [0,\tau] \to B^\sigma_{p,q,\mathcal{N}} \times B^\sigma_{p,q,\mathcal{N}}$.*

*Proof* The operators $P_i(u,v):(0,\tau]\to B^{\sigma}_{p,q,\mathcal{N}}$ for $i=1,2$ are continuous maps for $\frac{2N}{3p}\le\sigma<2$ and $\sigma\neq 1+\frac{1}{p}$. In fact, for $0<t_1<t_2<\tau$ and $(u,v)^T\in B^{\sigma}_{p,q,\mathcal{N}}\times B^{\sigma}_{p,q,\mathcal{N}}$, we have

$$\begin{aligned}
&\big\|P_1(u,v)(t_2)-P_1(u,v)(t_1)\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\\
&\quad\le\int_0^{\infty}\zeta_\alpha(\theta)\big\|\big[G_1\big(t_2^{\alpha}\theta\big)-G_1\big(t_1^{\alpha}\theta\big)\big]u_0\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\,d\theta\\
&\qquad+\int_{t_1}^{t_2}\big\|S_\alpha^{(1)}(t_2-s)f_1\big(u(s),v(s)\big)\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\,ds\\
&\qquad+\int_0^{t_1}\big\|\big[S_\alpha^{(1)}(t_2-s)-S_\alpha^{(1)}(t_1-s)\big]f_1\big(u(s),v(s)\big)\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\,ds\\
&\quad\le M_1\int_0^{\infty}\zeta_\alpha(\theta)\big\|\big(G\big[\big(t_2^{\alpha}-t_1^{\alpha}\big)\theta\big]-I\big)u_0\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\,d\theta\\
&\qquad+\alpha M_1E\big(\theta^{1-\frac{\sigma}{2}}\big)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1-\frac{\sigma\alpha}{2}}\big\|f_1\big(u(s),v(s)\big)\big\|_{L^p}\,ds\\
&\qquad+\alpha\int_0^{t_1}\int_0^{\infty}\theta\zeta_\alpha(\theta)\big\|\big[(t_2-s)^{\alpha-1}G_1\big((t_2-s)^{\alpha}\theta\big)\\
&\qquad-(t_1-s)^{\alpha-1}G_1\big((t_1-s)^{\alpha}\theta\big)\big]f_1\big(u(s),v(s)\big)\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\,d\theta\,ds.
\end{aligned}$$

Since the operator $(t-s)^{\alpha-1}G_1((t-s)^{\alpha}\theta)$ is continuous with respect to the parameter $t>s$ in the sense of the norm $\mathcal{L}(B^{\sigma}_{p,q,\mathcal{N}})$, we can conclude

$$\big\|P_1(u,v)(t_2)-P_1(u,v)(t_1)\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\to 0\quad\text{as } t_2\to t_1^{+}.$$

In the same way, we can show that if $0<t_1<t_2<\tau$, then

$$\big\|P_i(u,v)(t_2)-P_i(u,v)(t_1)\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}\to 0\quad\text{as } t_2\to t_1^{+}, i=1,2.$$

Thus, for $0\le\sigma<2$, $P_i(u,v):(0,\tau]\to B^{\sigma}_{p,q,\mathcal{N}}$ is a continuous map for $i=1,2$.

Next, we will show $P=(P_1,P_2):\mathcal{B}\to\mathcal{B}$. In fact, it follows from estimates (3.5) and (3.1) that

$$\begin{aligned}
\big\|P_1(u,v)(t)-u_0\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}&\le\big\|T_\alpha^{(1)}u_0-u_0\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}+\bigg\|\int_0^t S_\alpha^{(1)}(t-s)f_1\big(u(s),v(s)\big)\,ds\bigg\|_{B^{\sigma}_{p,q,\mathcal{N}}}\\
&\le\frac{R}{2},
\end{aligned}$$

and from (3.6) and (3.2) we have the following estimate:

$$\begin{aligned}
\big\|P_2(u,v)(t)-v_0\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}&\le\big\|T_\alpha^{(2)}u_0-u_0\big\|_{B^{\sigma}_{p,q,\mathcal{N}}}+\bigg\|\int_0^t S_\alpha^{(2)}(t-s)f_2\big(u(s),v(s)\big)\,ds\bigg\|_{B^{\sigma}_{p,q,\mathcal{N}}}\\
&\le\frac{R}{2}.
\end{aligned}$$

Now, we will prove that the maps $P_i:\mathcal{B}\to\mathcal{B}$, $i=1,2$, are contractive ones in $\mathcal{B}$. In fact, for $(u_1,v_1)^T,(u_2,v_2)^T\in B^{\sigma}_{p,q,\mathcal{N}}$ with the same initial value $(u_0,v_0)^T$. According to Lemma 3.2, we take $\tau_2$ such that $M_2\tau_2^{\alpha(1-\frac{\sigma}{2})} < \frac{1}{2}$. Let $\tau = \min\{\tau_1, \tau_2\}$ and then we have

$$\begin{aligned}
&\|P_1(u_1,v_1)(t) - P_1(u_2,v_2)(t)\|_{B^\sigma_{p,q,\mathcal{N}}} + \|P_2(u_1,v_1)(t) - P_2(u_2,v_2)(t)\|_{B^\sigma_{p,q,\mathcal{N}}} \\
&\quad\le \left\| \int_0^t S_\alpha^{(1)}(t-s)\big(u_2 - u_1 + u_1^2 v_1 - u_2^2 v_2\big) \right\|_{B^\sigma_{p,q,\mathcal{N}}} + \left\| \int_0^t S_\alpha^{(2)}(t-s)\big(u_2^2 v_2 - u_1^2 v_1\big) \right\|_{B^\sigma_{p,q,\mathcal{N}}} \\
&\quad\le \frac{\alpha}{2} \sup_{t\in[0,\tau]} \big( \|u_1(t) - u_2(t)\|_{B^\sigma_{p,q,\mathcal{N}}} + \|v_1(t) - v_2(t)\|_{B^\sigma_{p,q,\mathcal{N}}} \big).
\end{aligned}$$

Then, by the Banach fixed point theorem, there exists a unique fixed point $(u,v) \in \mathcal{B}$, that is, the unique local mild solution of (2.5a)-(2.5c) in $\mathcal{B}$. By virtue of the positive property of the operators $T_\alpha^{(i)}$ and $S_\alpha^{(i)}$ for $i = 1, 2$, this result is obtained. □

According to Theorem 3.3, there exists small enough $\tau > 0$ such that problem (1.3a)-(1.3d) has a unique local mild solution defined in $\mathcal{B}(\tau, R, u_0, v_0)$. This solution is bounded, i.e., $\|u\|_{C([0,\tau]:B^\sigma_{p,q,\mathcal{N}})}, \|v\|_{C([0,\tau]:B^\sigma_{p,q,\mathcal{N}})} \le \mathcal{R} = \max\{R + \|u_0\|_{B^\sigma_{p,q,\mathcal{N}}}, R + \|v_0\|_{B^\sigma_{p,q,\mathcal{N}}}\}$. The mild solution of (1.3a)-(1.3d) at $t = \tau$ exists, which is denoted by $(u(\tau), v(\tau))^T$. We take $(u(\tau), v(\tau))^T$ as another initial value of (1.3a)-(1.3d). Repeating the above discussion and by Theorem 3.3, we know that under the conditions in Theorem 3.3 the problem of (1.3a)-(1.3d) with the initial value $(u(0), v(0))^T = (u(\tau), v(\tau))^T$ has a unique mild solution (denoted by $(u_1(t), v_1(t))$) defined on the interval $[\tau, \tau_1]$, and this solution is bounded, that is, there exist two positive constants $\mathcal{R}_1$, $R_1$ such that $\|u_1\|_{C([\tau,\tau_1]:B^\sigma_{p,q,\mathcal{N}})}, \|v_1\|_{C([\tau,\tau_1]:B^\sigma_{p,q,\mathcal{N}})} \le \mathcal{R}_1 = \max\{R_1 + \|u(\tau)\|_{B^\sigma_{p,q,\mathcal{N}}}, R_1 + \|v(\tau)\|_{B^\sigma_{p,q,\mathcal{N}}}\}$. Repeating this process over and over, a mild solution of (1.3a)-(1.3d) is ultimately established on a maximum interval $(0, T_{\max})$. So, we have the following result.

**Theorem 3.4** *Let $1 < p < \infty$, $\frac{2N}{3p} \le \sigma < 2$, $\sigma \ne 1 + \frac{1}{p}$ and $1 \le q \le 3p$. Then, given $(u_0, v_0)^T \in B^\sigma_{p,q,\mathcal{N}} \times B^\sigma_{p,q,\mathcal{N}}$, problem* (1.3a)-(1.3d) *has a unique mild positive solution $(u,v)^T : [0, T_{\max}) \to B^\sigma_{p,q,\mathcal{N}} \times B^\sigma_{p,q,\mathcal{N}}$.*

## 4 Stability and Hopf bifurcation

In this section, we study the stability and Hopf bifurcation of a spatially homogeneous equilibrium point for (1.3a)-(1.3d). System (1.3a)-(1.3d) has a unique homogeneous steady state $E_0 = (u^*, v^*)$

$$u^* = a + b, \qquad v^* = \frac{b}{(a+b)^2},$$

which satisfies

$$f_1\big(u^*, v^*\big) = 0, \qquad f_2\big(u^*, v^*\big) = 0.$$

Linearizing model (1.3a)-(1.3d) at $E_0$ yields

$$\frac{\partial^\alpha w}{\partial t^\alpha} = D\Delta w + J(E_0)w,$$

where

$$w=(u,v)^T,\qquad D=\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},\qquad J(E_0)=\begin{bmatrix}\frac{2b}{a+b}-1 & (a+b)^2\\ -\frac{2b}{a+b} & -(a+b)^2\end{bmatrix}.$$

Let $\{\mu_k,\varphi_k\}_{k=1}^{\infty}$ be an eigenpair of the operator $-\Delta$ on $\Omega$ with the Neumann boundary condition, where $0=\mu_1<\mu_2<\cdots$. $E(\mu_k)$ is the eigenspace corresponding to $\mu_k$ in $C^1(\overline{\Omega})$, and $\varphi_{kj}$, $j=1,2,\ldots,\dim E(\mu_k)$, is an orthonormal basis of $E(\mu_k)$. Let

$$X=\left\{(u,v)^T\in\left[C^2(\Omega)\cap C^1(\overline{\Omega})\right]^2\Big|\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0\right\}\tag{4.1}$$

and $X_{kj}=\{\mathbf{c}\varphi_{kj}|\mathbf{c}\in\mathbb{R}^2\}$. Consider the following decomposition:

$$X=\bigoplus_{k=1}^{\infty}X_k,\tag{4.2}$$

where $X_k=\bigoplus_{j=1}^{\dim E(\mu_k)}X_{kj}$ and $X_{kj}$ is the eigenspace corresponding to $\mu_k$.

For each $k\geq 1$, $X_k$ is invariant under the operator $L=D\Delta+J(E_0)$, and $\lambda$ is an eigenvalue of $L$ on $X_k$ if and only if it is an eigenvalue of the matrix $-\mu_kD+J(E_0)$. Denote

$$A(\mu_k)=-\mu_kD+J(E_0).\tag{4.3}$$

The characteristic equation of $A(\mu_k)$ is

$$\lambda^2-\operatorname{tr}A(\mu_k)\lambda+\det A(\mu_k)=0,\tag{4.4}$$

where

$$\operatorname{tr}A(\mu_k)=-(d_1+d_2)\mu_k+\frac{b-a}{a+b}-(a+b)^2,\tag{4.5a}$$

$$\det A(\mu_k)=d_1d_2\mu_k^2+\frac{(a+b)^3d_1+(a-b)d_2}{a+b}\mu_k+(a+b)^2.\tag{4.5b}$$

If the following inequalities hold,

$$\begin{cases}(a+b)^3+a-b>0,\\ d_1(a+b)^3+d_2(a-b)>0,\end{cases}$$

i.e.,

$$(a+b)^3>\max\left\{b-a,\frac{d_2(b-a)}{d_1}\right\},\tag{4.6}$$

then the roots of (4.4) are negative for $k\in\mathbb{N}$, so the homogeneous steady state $E_0$ is stable. Recently, some literature works [32, 34, 35] have shown that time fractional-order derivatives can induce stability of a steady state for a fractional-order system although the corresponding characteristic roots at this steady state have positive real parts. To find another parameter region of the stability for (1.3a)-(1.3d), one can check the parameters satisfying the following inequalities:

$$\begin{cases} (a+b)^3 + a - b < 0, \\ d_1(a+b)^3 + d_2(a-b) > 0, \end{cases}$$

i.e.,

$$\frac{d_2(b-a)}{d_1} < (a+b)^3 < b-a. \tag{4.7}$$

Under conditions (4.7), the characteristic roots of (4.4) for $k = 1$ obviously are complex with a positive real part. For this case, the corresponding first-order derivative system of (1.3a)-(1.3d) is unstable at $E_0$. However, $E_0$ might be stable for the fractional-order system (1.3a)-(1.3d). Denote the characteristic roots of (4.4) by

$$\lambda_m(\mu_k) = P_m(\mu_k) + Q_m(\mu_k)i, \quad \text{for } m = 1, 2,$$

where $P_m$ and $Q_m$ belong to $\mathbb{R}$ and $i^2 = -1$. Let

$$\mu^* = -\frac{(a+b)^3 + a - b}{(a+b)(d_1+d_2)},$$

and there exists a positive constant $K$ such that $\mu_K \leq \mu^*$ and $\mu_{K+1} > \mu^*$. According to Lemma 2.2, $E_0$ is stable if and only if the tangent of the characteristic roots

$$\tan^2\bigl[\bigl|\arg\bigl(\lambda_m(\mu_k)\bigr)\bigr|\bigr] = \left(\frac{Q_m(\mu_k)}{P_m(\mu_k)}\right)^2 = \frac{4\det A(\mu_k)}{[\operatorname{tr} A(\mu_k)]^2} - 1 > \tan^2\left(\frac{\alpha\pi}{2}\right), \tag{4.8}$$

for $k = 1, \ldots, K$. Under condition (4.7), formula (4.5a) ((4.5b)) decreases (increases) with respect to $\mu_k$ for $k = 1, \ldots, K$. Thus, in order to check the stability of $E_0$, we only need to verify the condition

$$\tan^2\bigl[\bigl|\arg\bigl(\lambda_m(\mu_1)\bigr)\bigr|\bigr] > \tan^2\left(\frac{\alpha\pi}{2}\right).$$

For the case of $k = 1$, after calculating (4.8), we obtain

$$\tan^2\bigl[\bigl|\arg\bigl(\lambda_m(\mu_1)\bigr)\bigr|\bigr] = \frac{4\det A(\mu_1)}{[\operatorname{tr} A(\mu_1)]^2} - 1 = \frac{4(a+b)^4}{((a+b)^3 + a - b)^2} - 1 > \tan^2\left(\frac{\alpha\pi}{2}\right), \tag{4.9}$$

i.e.,

$$\frac{4(a+b)^4}{((a+b)^3 + a - b)^2} > \tan^2\left(\frac{\alpha\pi}{2}\right) + 1. \tag{4.10}$$

From the above discussion, we can get the following result about the stability for system (1.3a)-(1.3d).

**Theorem 4.1** *For system* (1.3a)-(1.3d), *the spatially homogeneous equilibrium point* $E_0$ *is stable if any of the following two conditions is satisfied*:

(i) $(a+b)^3 > \max\{b-a, \frac{d_2(b-a)}{d_1}\}$,

(ii) $\frac{d_2(b-a)}{d_1} < (a+b)^3 < b-a$ *and* $\frac{4(a+b)^4}{((a+b)^3+a-b)^2} > \tan^2(\frac{\alpha\pi}{2}) + 1$.

For the corresponding first-order system of (1.3a)-(1.3d), $E_0$ is stable if and only if condition (i) in Theorem 4.1 holds. However, for the fractional-order system (1.3a)-(1.3d), $E_0$ is still stable under conditions (ii) in Theorem 4.1, except for condition (i), because the fractional-order derivative can induce the stability. In addition, condition (ii) in Theorem 4.1 seems very complicated, but its parameter set is not empty. For example, take $d_1 = 0.02$ and $d_2 = 0.01$, and we plot Figure 1 to illustrate the parameter region of stability for (1.3a)-(1.3d). In this figure, the parameters in region IV satisfy conditions (ii) in Theorem 4.1. $E_0$ is stable for system (1.3a)-(1.3d) with the parameters in regions IV, V and VI. For other parameters (such as regions I, II and III), $E_0$ is unstable and around it spatially homogeneous periodic orbits arise, that is, a Hopf bifurcation happens.

To show the stability and spatially homogeneous periodic orbits of (1.3a)-(1.3d), we will perform numerical simulations for system (1.3a)-(1.3d) in a one-dimensional space with parameters satisfying stable conditions. To this end, we discretize the space and the time of the problem because the dynamical behavior of system (1.3a)-(1.3d) cannot be investigated by using analytical methods or normal forms. We will transform it from an infinite-dimensional (continuous) to a finite-dimensional (discrete) form. System (1.3a)-(1.3d) is solved in a discrete domain with $M$ lattice sites. The step length between the lattice points is defined by the lattice constant $\Delta h = 0.25$. In this discrete system, the Laplacian operator describing diffusion is calculated by using a three central difference scheme. The time evolution is also discrete, that is, the time goes by steps of $\Delta t = 0.02$, and is solved by an Adams-type predictor-corrector method for a fractional-order equation. For simplicity, in this section, we take the spatial region $\Omega = (0, 10)$. In Figure 2, we choose the parameters $a = 0.08$ and $b = 0.3$ located in region I in Figure 1, and plot the spatially homogeneous periodic orbits. Besides, we also plot the spatially homogeneous periodic orbits of the corresponding time first-order model of (1.3a)-(1.3d) with the same parameters, see Figure 3. However, these two figures indicate that there exists some difference in the spatially homogeneous periodic orbits coming from system (1.3a)-(1.3d) and its corresponding first-order system, respectively. Compared with the first-order system, the amplitude of the periodic orbit of the fractional-order system (1.3a)-(1.3d) is smaller. This shows that the amplitude of the periodic orbit of system (1.3a)-(1.3d) is affected by the fractional-order derivative.

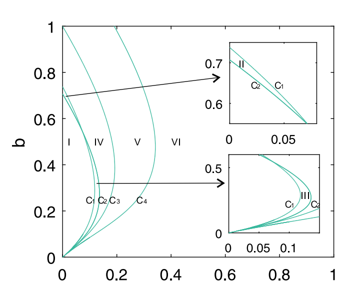



**Figure 1 The parameter region of stability and instability for (1.3a)-(1.3d) with $\alpha = 0.8$.** The curves $C_1$, $C_2$, $C_3$ and $C_4$ divide the plane of the parameters $a$, $b$ into six regions (respectively denoted by I, II, III, IV, V and VI), where the curves $C_1$ and $C_4$ come from the equation $\frac{4(a+b)^4}{((a+b)^3+a-b)^2} = \tan^2(\frac{2\pi}{5}) + 1$, $C_2$ from $(a+b)^3 = 0.5(b-a)$ and $C_3$ from $(a+b)^3 = b-a$. Wherein, $d_1 = 0.02$ and $d_2 = 0.01$.

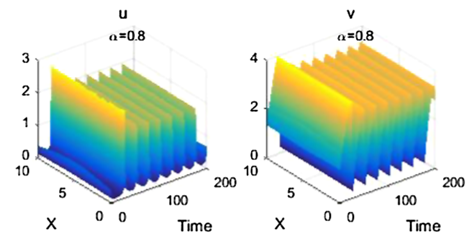



**Figure 2 The spatially homogeneous periodic orbit of (1.3a)-(1.3d) with the parameters *a* = 0.08 and *b* = 0.3 in region I.** Other parameters are as in Figure 1.

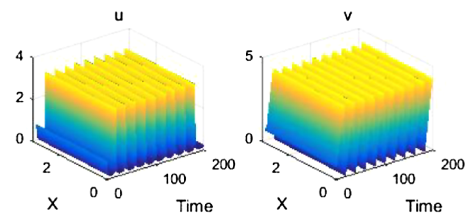



**Figure 3 The spatially homogeneous periodic orbit of the corresponding time first-order model of (1.3a)-(1.3d).** The other parameters are taken as in Figure 2.

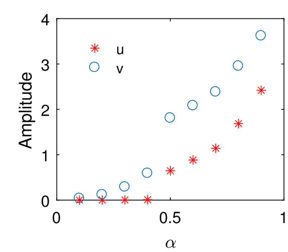



**Figure 4 The amplitude of the spatially homogeneous periodic orbit vs the fractional-order derivative for system (1.3a)-(1.3d).** Other parameters are as in Figure 2.

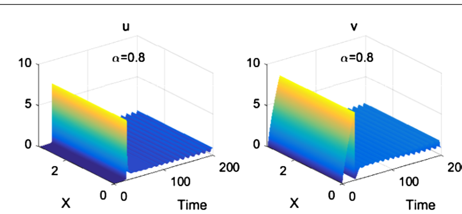



**Figure 5 The spatially homogeneous periodic orbit of (1.3a)-(1.3d) with the parameters *a* = 0.042 and *b* = 0.626 in region II.** Other parameters are as in Figure 1.

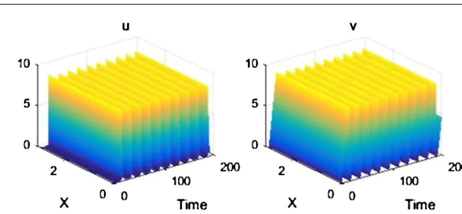



**Figure 6 The spatially homogeneous periodic orbit of (1.3a)-(1.3d) with the first-order derivative and the parameters *a* = 0.042 and *b* = 0.626 in region II.** Other parameters are as in Figure 1.

To discover the relationship between the amplitude and the fractional-order derivative, we plot Figure 4. In this figure, the amplitudes of the periodic orbits of $u$ and $v$ increase with the fractional order $\alpha$. We take $a = 0.042$ and $b = 0.626$ in region II in Figure 1 and respectively plot the spatially homogeneous periodic orbits of system (1.3a)-(1.3d) and its corresponding first-order system, see Figures 5 and 6. The amplitude of the spatially homogeneous periodic orbits of system (1.3a)-(1.3d) are still smaller in comparison with

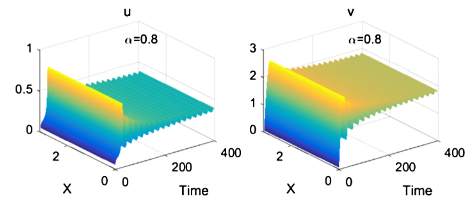



**Figure 7 The spatially homogeneous periodic orbit of (1.3a)-(1.3d) with the parameters $a = 0.12$ and $b = 0.26$ in region III.** Other parameters are as in Figure 1.

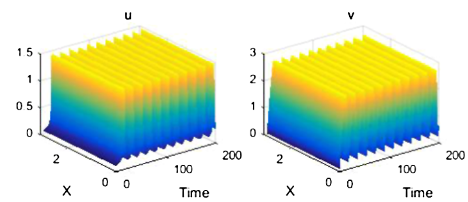



**Figure 8 The spatially homogeneous periodic orbit of (1.3a)-(1.3d) with the first-order derivative and the parameters $a = 0.12$ and $b = 0.26$ in region III.** Other parameters are as in Figure 1.

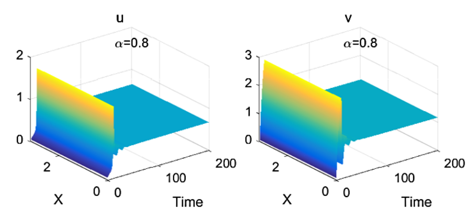



**Figure 9 The stability of the spatially homogeneous steady state for (1.3a)-(1.3d) with the first-order derivative and the parameters $a = 0.14$ and $b = 0.54$ in region IV.** Other parameters are as in Figure 1.

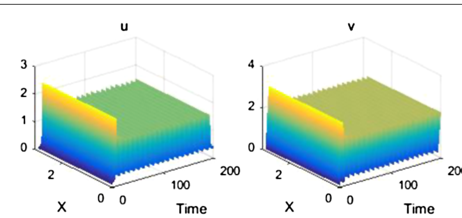



**Figure 10 The spatially homogeneous periodic orbit of (1.3a)-(1.3d) with the first-order derivative and the parameters $a = 0.14$ and $b = 0.54$ in region IV.** Other parameters are as in Figure 1.

the corresponding first-order system. The similar results appear when the parameters are chosen in region III in Figure 1, see Figures 7 and 8.

Next, we will focus on some difference of the stability between the fractional-order system (1.3a)-(1.3d) and its corresponding first-order system. First, in region IV in Figure 1 we choose parameters randomly, for example, $a = 0.14$ and $b = 0.54$. For these given parameters, the spatially homogeneous steady state is $E_0 = (0.68, 1.678)$. System (1.3a)-(1.3d) presents the stability of $E_0$, see Figure 9, but its corresponding first-order system presents the spatially homogeneous periodic orbits, see Figure 10. This phenomenon implies that the time fractional-order derivative can induce the stability of system (1.3a)-(1.3d) or that the fractional order can expand the parameter region of stability compared with its corresponding first-order system. When the parameters $a$ and $b$ are chosen from region V or VI in Figure 1, the spatially homogeneous steady state $E_0$ is stable for system (1.3a)-(1.3d) and its corresponding first-order system, see Figures 11 and 12. Here, we only illustrate for the parameters in region V and do not show any figure with the parameters chosen in region VI because of the similar results as shown in Figures 11 and 12.

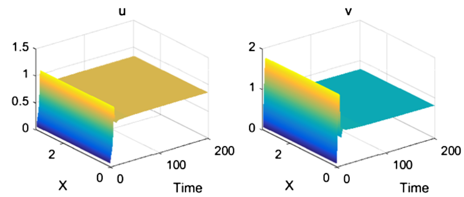



**Figure 11 The stability of the spatially homogeneous steady state for (1.3a)-(1.3d).** The parameters $a = 0.26$ and $b = 0.6$ in region V. Other parameters are as in Figure 1.

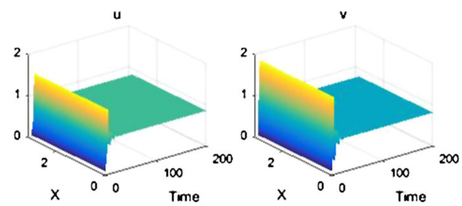



**Figure 12 The stability of the spatially homogeneous steady state for (1.3a)-(1.3d) with the first-order derivative and the parameters $a = 0.26$ and $b = 0.6$ in region V.** Other parameters are as in Figure 1.

## 5 Turing pattern

In the above section, we have obtained the stability and Hopf bifurcation of system (1.3a)-(1.3d). However, these results focus on the spatial homogeneity of the dynamical behaviors of system (1.3a)-(1.3d). The spatial heterogeneity (i.e., Turing pattern) is interesting and significant for a reaction-diffusion system, which breaks the homogeneous states because the Turing bifurcation occurs. In this section, we continue to investigate the Turing instability of system (1.3a)-(1.3d). In particular, we will find difference of spatial patterns between system (1.3a)-(1.3d) and its corresponding first-order system. Turing patterns require two conditions. First, a nontrivial homogeneous steady state exists and is stable for spatially homogeneous perturbations. This condition is obtained in Theorem 4.1. Second, the stable steady state is unstable to at least one type of spatially heterogeneous perturbations. The second condition defines the condition for Turing instability, which ensures that local perturbations on the stable homogeneous steady state gradually expand globally.

Now, adding the heterogeneous disturbance term into the steady state $E_0$ yields

$$(u,v)^T = \left(u^*,v^*\right)^T + \epsilon (u_k,v_k)^T e^{\lambda t + ik\bar{r}} + c.c + O\left(\epsilon^2\right), \tag{5.1}$$

where $\bar{r}$ is the disturbance growth rate of $t$ moment, $i$ is the imaginary unit, $k$ represents wave number, $\bar{r} = (X,Y)$ is a two-dimensional factor in the complex conjugate plane. After inserting the above equation into system (1.3a)-(1.3d) and keeping the first degree term of $\epsilon$, we obtain the characteristic equation as follows:

$$\lambda^2 - B(k)\lambda + \det D(k) = 0, \tag{5.2}$$

where

$$B(k) = -(d_1 + d_2)k^2 + \frac{b-a}{a+b} - (a+b)^2, \tag{5.3a}$$

$$D(k) = d_1 d_2 k^4 + \frac{(a+b)^3 d_1 + (a-b)d_2}{a+b} k^2 + (a+b)^2. \tag{5.3b}$$

In order to produce the Turing pattern, the nontrivial homogeneous steady state $E_0$ must be stable under spatially homogeneous perturbations. Note the condition of stability for a fractional-order system (1.3a)-(1.3d). We can find the following conditions of stability under spatially homogeneous perturbations:

$$(a+b)^3 + a - b > 0 \tag{5.4a}$$

or

$$\begin{cases} (a+b)^3 + a - b < 0, \\ \frac{4(a+b)^4}{((a+b)^3+a-b)^2} > \tan^2(\frac{\alpha\pi}{2}) + 1. \end{cases} \tag{5.4b}$$

Further, according to the second condition of forming Turing pattern, the spatially homogeneous steady state $E_0$ destabilizes under some spatially heterogeneous perturbations. Thus, for some wave number $k$, equation (5.3b) must be less than zero. Consequently, we have

$$\begin{cases} (a+b)^3 d_1 < d_2(b-a), \\ ((a+b)^3 d_1 + (a-b)d_2)^2 > 4d_1 d_2 (a+b)^4. \end{cases} \tag{5.5}$$

Here, according to inequalities (5.4a), (5.4b), we plot a parameter $a-b$ diagram to show the stability and instability of $E_0$ under spatially homogeneous perturbations, see Figure 13(A). In this figure, if the parameters are chosen in regions II, III or IV, then $E_0$ is stable under spatially homogeneous perturbations for the fractional-order system (1.3a)-(1.3d). However, for the corresponding first-order system of (1.3a)-(1.3d), the steady state $E_0$ becomes unstable due to the parameters in region II. Besides, by inequalities (5.4a), (5.4b) and (5.5) we present Figure 13(B), which is also a parameter $a-b$ diagram in which Turing bifurcations can occur. In this figure, take $d_1 = 0.01$, $d_2 = 0.25$ and $\alpha = 0.8$, and the parameter $a-b$ diagram is divided into three segments: the homogeneous steady state region consisting of regions 3 and 4, pure Turing instability consisting of regions 2, 5 and 6 and Hopf-Turing instability consisting of only one region 1.

Next, we will perform numerical simulations of system (1.3a)-(1.3d) in a two-dimensional space with parameters satisfying Turing conditions to obtain Turing patterns. To this end, we should discretize the space and the time of the problem because the dynamical behavior of system (1.3a)-(1.3d) cannot be investigated by using analytical methods or normal forms. We will transform it from an infinite-dimensional (continuous) to a

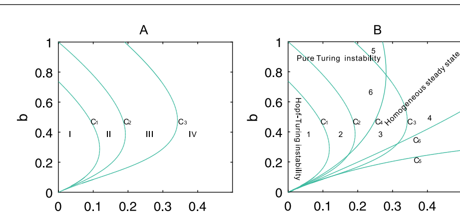



**Figure 13 The parameter $a-b$ diagram of stability and bifurcation. (A)** Parameter region of stability and instability for $E_0$ under spatially homogeneous perturbations. The curves $C_1$, $C_2$ and $C_3$ divide the parameter $a-b$ plane into four regions (respectively denoted by I, II, III and IV), where the curves $C_1$ and $C_3$ come from the equation $\frac{4(a+b)^4}{((a+b)^3+a-b)^2} = \tan^2(\frac{2\pi}{5}) + 1$ and $C_2$ from $(a+b)^3 = b-a$. **(B)** Parameter $a-b$ diagram showing three parameter regions, homogeneous steady state region (consisting of regions 3 and 4), pure Turing instability (consisting of regions 2, 5 and 6) and Hopf-Turing instability (only one region 1). The curves $C_4$ and $C_5$ come from the equation $((a+b)^3 d_1 + (a-b)d_2)^2 = 4d_1 d_2 (a+b)^4$, the curve $C_6$ from $d_1(a+b)^3 = d_2(b-a)$. Set $\boldsymbol{\alpha} = 0.8$, $d_1 = 0.01$ and $d_2 = 0.25$.

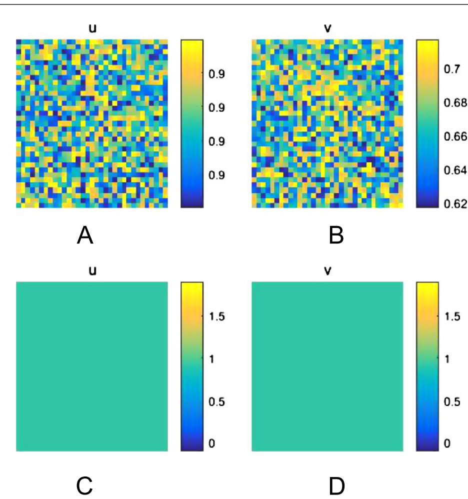
finite-dimensional (discrete) form. System (1.3a)-(1.3d) is solved in a discrete domain with $M \times N$ lattice sites. The spacing between the lattice points is defined by the lattice constant $\Delta h = 0.25$. In this discrete system, the Laplacian operator describing diffusion is calculated by using a five central difference scheme. The time evolution is also discrete, that is, the time goes by steps of $\Delta t = 0.02$, and is solved by an Adams-type predictor-corrector method for fractional-order equation. In order to avoid numerical artefacts, we checked the sensitivity of the results to the choice of the time and space steps, and their values have been chosen sufficiently small. Both numerical schemes are standard, hence we do not describe them here.

Set the parameters as follows: $a = 0.4$, $b = 0.5$, $d_1 = 0.01$, $d_2 = 0.25$ and $\alpha = 0.8$. Then these parameters are located in region 4 of $a - b$ diagram 13(B), and we have the positive equilibrium $(u^*, v^*) = (0.9, 0.617)$. The initial density distributions are random spatial distribution of the species near $E_0$, which is more general from biological point of view. We plot the evolution of the spatial patterns of prey and predator at 0 and 10,000 iterations, see Figure 14. This figure shows that the spatially homogeneous steady state $E_0$ is stable.

Take the above parameters, but let $a = 0.2$ and $b = 0.8$, whose position is located in region 6 in Figure 13(B). System (1.3a)-(1.3d) produces spatial patterns, see Figure 15. In Figure 15, (A) and (B) respectively show initial states of prey and predator. (C) and (D) show the pure Turing patterns of prey and predator, respectively. Meanwhile, we also simulate the corresponding first-order system of (1.3a)-(1.3d) with the same parameters, and the same patterns form as in Figure 15, which here is not displayed.

For the parameter chosen in region 2 of the parameter $a - b$ diagram 13(B), spatial patterns of (1.3a)-(1.3d) and its corresponding first-order system, however, are completely different, see Figure 16 with $a = 0.1$ and $b = 0.7$. We find that prey's and predator's patterns of system (1.3a)-(1.3d) are spots, and that strip patterns appear for the corresponding first-order system of (1.3a)-(1.3d). The reason is that for the same parameters $a$ and $b$, system (1.3a)-(1.3d) produces pure Turing patterns, but its corresponding first-order system does Turing-Hopf patterns. Finally, we numerically simulate system (1.3a)-(1.3d) and its corresponding first-order system with the parameters $a = 0.05$ and $b = 0.5$ chosen in region 1 in the parameter $a - b$ diagram 13B, and find that these two systems form the same patterns, see Figure 17. The reason is that these two kinds of systems have the same reaction field.

**Figure 14 Snapshots of contour pictures of the time evolution of the prey and predator in a 2D spatial domain with $a = 0.4$, $b = 0.5$, $d_1 = 0.01$, $d_2 = 0.25$ and $\alpha = 0.8$.** Wherein prey **(A)** and predator **(B)** for 0 iteration, and prey **(C)** and predator **(D)** for 10,000 iterations.



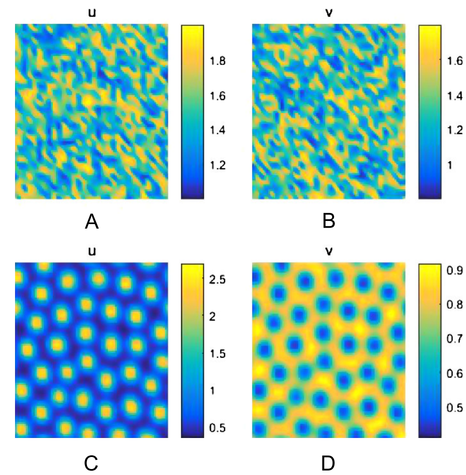



**Figure 15 Snapshots of contour pictures of the time evolution of the prey and predator in a 2D spatial domain with $a = 0.2$, $b = 0.8$, $d_1 = 0.01$, $d_2 = 0.25$ and $\alpha = 0.8$.** Wherein prey **(A)** and predator **(B)** for 0 iteration, and prey **(C)** and predator **(D)** for 10,000 iterations.

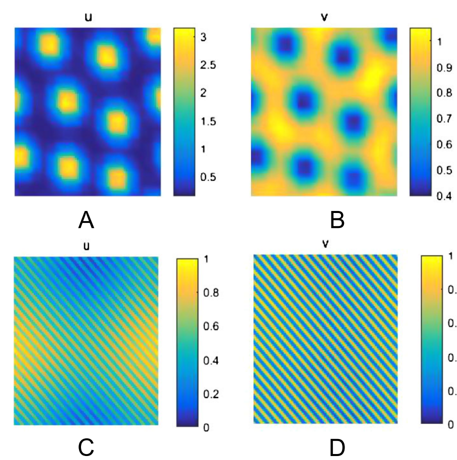



**Figure 16 Spatial patterns of prey and predator.** **(A)** and **(B)** are prey's and predator's patterns of system (1.3a)-(1.3d), but **(C)** and **(D)** are their patterns of the corresponding first-order system of (1.3a)-(1.3d). Wherein $a = 0.1$ and $b = 0.7$.

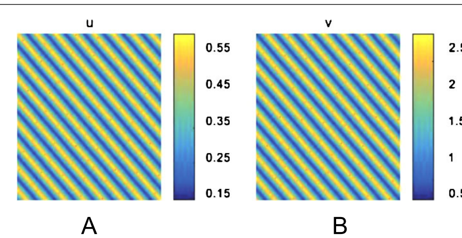



**Figure 17 Spatial patterns of prey and predator, (A) and (B) are prey's and predator's patterns coming from the system (1.3a)-(1.3d). Wherein $a = 0.05$ and $b = 0.5$.**

## 6 Conclusion and discussion

In this paper, for system (1.3a)-(1.3d), which is a fractional-order partial differential equation model, we have discussed the existence and uniqueness of the positive solution in a Besov space. Besides, we have obtained stability, Hopf bifurcation, Turing bifurcation and spatial patterns of this model. From theoretical analysis and numerical simulations, we find some difference between system (1.3a)-(1.3d) and its corresponding first-order system on the stability, Hopf bifurcation and spatial patterns. In detail, the fractional-order derivative enlarges the parameter region of the stability in comparison with its corresponding first-order system. Besides, these two systems with the same parameters can form different spatial patterns (see Figure 16). These results, as far as our knowledge goes, are completely new. So far, there are few literature works to study fractional-order partial differential equations. This kind of equations are still open, such as Hopf bifurcation direction, normal form and their application, which are worth being investigated in the future.

**Acknowledgements**
We thank for helpful comments of referees. This work is supported by grants 61563033, 11563005 and 11563005 from the National Natural Science Foundation of China, and 20161BAB201010 from the Natural Science Foundation of Jiangxi Province.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors read and approved the final manuscript.

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 July 2017 Accepted: 16 November 2017 Published online: 25 November 2017

**References**

1. Mellman, I, Misteli, T: Computational Cell Biology. Springer, Berlin (2002)
2. Liu, P, Shi, J, Wang, Y, Feng, X: Bifurcation analysis of reaction-diffusion Schnakenberg model. J. Math. Chem. **51**, 2001-2019 (2013)
3. Madzvamuse, A, Ndakwo, HS, Barreira, R: Cross-diffusion-driven instability for reaction-diffusion systems: analysis and simulations. J. Math. Biol. **70**, 709-743 (2015)
4. Gambino, G, Lupo, S, Sammartino, M: Effects of cross-diffusion on Turing patterns in a reaction-diffusion Schnakenberg model (2015). arXiv:1501.04890
5. Jacobo, A, Hudspeth, AJ: Reaction-diffusion model of hair-bundle morphogenesis. Proc. Natl. Acad. Sci. **111**, 15444-15449 (2014)
6. Du, M, Wang, Z, Hu, H: Measuring memory with the order of fractional derivative. Sci. Rep. **3**, 3431 (2013)
7. Maraaba, T, Baleanu, D, Jarad, F: Existence and uniqueness theorem for a class of delay differential equations with left and right Caputo fractional derivatives. J. Math. Phys. **49**, 083507 (2008)
8. Babakhani, A, Baleanu, D, Agarwal, RP: The existence and uniqueness of solutions for a class of nonlinear fractional differential equations with infinite delay. Abstr. Appl. Anal. **2013**, 592964 (2013)
9. Bhrawy, AH, Zaky, MA: Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations. Comput. Math. Appl. **73**, 1100-1117 (2017)
10. Zaky, MA, Machado, JAT: On the formulation and numerical simulation of distributed-order fractional optimal control problems. Commun. Nonlinear Sci. Numer. Simul. **52**, 177-189 (2017)
11. Klages, R, Radons, G, Sokolov, IM: Anomalous Transport: Foundations and Applications. Wiley, New York (2008)
12. Kilbas, AA, Srivastava, HM, Trujilio, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
13. Wang, J, Zhou, Y: Mittag-Leffler-Ulam stabilities of fractional evolution equations. Appl. Math. Lett. **25**, 723-728 (2012)
14. Angstmann, CN, Henry, BI, McGann, AV: A fractional order recovery SIR model from a stochastic process. Bull. Math. Biol. **78**, 468-499 (2016)
15. Chang, Y-K, Zhang, R, N'Guerekata, GM: Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations. Comput. Math. Appl. **64**, 3160-3170 (2012)
16. Bostan, M: Strongly anisotropic diffusion problems; asymptotic analysis. J. Differ. Equ. **256**, 1043-1092 (2014)
17. Wang, J, Zhou, Y: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262-272 (2011)
18. Sakthivel, R, Revathi, P, Ren, Y: Existence of solutions for nonlinear fractional stochastic differential equations. Nonlinear Anal., Theory Methods Appl. **81**, 70-86 (2013)
19. Guo, G, Chen, B, Zhao, X, Zhao, F, Wang, Q: First passage time distribution of a modified fractional diffusion equation in the semi-infinite interval. Phys. A, Stat. Mech. Appl. **433**, 279-290 (2015)
20. Tchier, F, Inc, M, Korpinar, ZS, Baleanu, D: Solutions of the time fractional reaction-diffusion equations with residual power series method. Adv. Mech. Eng. **8**, 1-10 (2016)
21. Bhrawy, AH, Baleanu, D, Mallawi, F: A new numerical technique for solving fractional sub-diffusion and reaction sub-diffusion equations with a non-linear source term. Therm. Sci. **19**, 25-34 (2015)
22. Zhai, Z: Well-posedness for two types of generalized Keller-Segel system of chemotaxis in critical Besov spaces. Commun. Pure Appl. Anal. **10**, 287-308 (2011)
23. Zhai, Z: Global well-posedness for nonlocal fractional Keller-Segel systems in critical Besov spaces. Nonlinear Anal., Theory Methods Appl. **72**, 3173-3189 (2010)
24. Prömel, DJ, Trabs, M: Rough differential equations driven by signals in Besov spaces. J. Differ. Equ. **260**, 5202-5249 (2016)

25. Yan, K, Qiao, Z, Zhang, Y: Blow-up phenomena for an integrable two-component Camassa-Holm system with cubic nonlinearity and peakon solutions. J. Differ. Equ. **259**, 6644-6671 (2015)
26. Zhao, J, Liu, Q, Cui, S: Existence of solutions for the Debye-Hückel system with low regularity initial data. Acta Appl. Math. **125**, 1-10 (2013)
27. Morel, JM, Takens, F, Teissier, B: The Analysis of Fractional Differential Equations. Springer, Berlin (2004)
28. Amann, H: Linear and Quasilinear Parabolic Problems: Volume I: Abstract Linear Theory, vol. 1. Springer, Media (1995)
29. El-Borai, MM: Some probability densities and fundamental solutions of fractional evolution equations. Chaos Solitons Fractals **14**, 433-440 (2002)
30. Cao, J, Yang, Q, Huang, Z: Optimal mild solutions and weighted pseudo-almost periodic classical solutions of fractional integro-differential equations. Nonlinear Anal., Theory Methods Appl. **74**, 224-234 (2011)
31. Matignon, D: Stability results for fractional differential equations with applications to control processing. Comput. Eng. Syst. Appl. **2**, 963-968 (1997)
32. El-Saka, HA, Ahmed, E, Shehata, MI, El-Sayed, AMA: On stability, persistence, and Hopf bifurcation in fractional order dynamical systems. Nonlinear Dyn. **56**, 121-126 (2008)
33. Ahmed, E, El-Sayed, AMA, El-Saka, HAA: Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models. J. Math. Anal. Appl. **325**, 542-553 (2007)
34. Huang, S, Wang, B: Stability and stabilization of a class of fractional-order nonlinear systems for $0 < \alpha < 2$. Nonlinear Dyn., 1-12 (2016)
35. Delavari, H, Baleanu, D, Sadati, J: Stability analysis of Caputo fractional-order nonlinear system revisited. Nonlinear Dyn. **67**, 2433-2439 (2012)