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Stochastic Volterra integral equations with a parameter

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Abstract
In this paper, we study the properties of continuity and differentiability of solutions to stochastic Volterra integral equations and backward stochastic Volterra integral equations depending on a parameter.

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1 Introduction and preliminaries

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space on which a d-dimensional standard Brownian motion $W(\cdot)$ is defined such that $\mathbb{F}\equiv\{\mathcal{F}_t\}_{t\geq 0}$ is its natural filtration augmented by all the $\mathbb{P}$-null sets.

In this paper, for given $T>0$, we consider the continuity and differentiability of solutions to the following stochastic Volterra integral equations (SVIEs) with respect to a parameter α:

$$X_\alpha(t)=\varphi_\alpha(t)+\int_0^t b_\alpha\bigl(t,s,X_\alpha(s)\bigr)\,ds+\int_0^t \sigma_\alpha\bigl(t,s,X_\alpha(s)\bigr)\,dW(s),\quad t\in[0,T], \tag{1.1}$$

and backward stochastic Volterra integral equations (BSVIEs) with respect to α:

$$\begin{aligned} Y_\alpha(t)={}&\psi_\alpha(t)+\int_t^T f_\alpha\bigl(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)\bigr)\,ds\\ &+\int_t^T Z_\alpha(t,s)\,dW(s),\quad t\in[0,T]. \end{aligned} \tag{1.2}$$

Here $\alpha\in\mathbb{R}$ and, for any given α, $\varphi_\alpha:[0,T]\times\Omega\to\mathbb{R}^m$, $b_\alpha:[0,T]^2\times\mathbb{R}^m\to\mathbb{R}^m$, $\sigma_\alpha\equiv(\sigma_\alpha^1,\sigma_\alpha^2,\dots,\sigma_\alpha^d):[0,T]^2\times\mathbb{R}^m\to\mathbb{R}^{m\times d}$, $f_\alpha:[0,T]^2\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^{n\times d}\to\mathbb{R}^n$ are Borel measurable functions.

The theory of stochastic differential equations (SDEs), including SDEs with parameters, is an important topic in stochastic processes. There exists much research work on this topic (see, *e.g.*, [1–4]). SVIEs, as a natural and nontrivial extensions of SDEs, not only have their distinctive features (*e.g.*, containing memories), but they also have interesting applications such as in stochastic control (see, *e.g.*, [5, 6]). The theory for SVIEs is much richer than that of SDEs. SVIEs with regular kernels and driven by Brownian motions were first studied in the 1970s and early-1980s (see, *e.g.*, [7–9]). Later, Protter in [10] studied SVIEs driven by general semimartingales. BSVIEs also can be viewed generalizations of BSDEs, first studied by Lin ([11]); and then deeply investigated by Yong ([12]).

Building on previous work concerning SDEs and BSDEs with a parameter ([1–4, 13]), this article is devoted to studying properties of solutions to SVIE (1.1) and BSVIE (1.2) with respect to a parameter, mainly continuity and differentiability. For the continuity of solutions to SVIEs with respect to a parameter, the reader can refer to [7, 8]; and for the differentiability the reader can refer to [14]. In this work, we presents more general and complete results on this topic.

We now introduce some basic notations. Let $H = \mathbb{R}^n, \mathbb{R}^{n\times m}$, *etc.*

- $L^2_{\mathcal{F}_T}(\Omega;H)$ is the set of all $\mathcal{F}_T$-measurable random variables ξ valued in H such that

$$\|\xi\|_{L^2_{\mathcal{F}_T}(\Omega;H)} \equiv \left(\mathbb{E}|\xi|^2\right)^{\frac{1}{2}} < \infty.$$

- $L^2_{\mathbb{F}}(\Omega\times(0,T);H)$ is the set of all $\mathbb{F}$-progressively measurable processes $\varphi(\cdot)$ valued in H such that

$$\|\varphi(\cdot)\|_{L^2_{\mathbb{F}}(\Omega\times(0,T);H)} \equiv \left[\mathbb{E}\int_0^T \left|\varphi(t)\right|^2 dt\right]^{\frac{1}{2}} < \infty.$$

- $C_{\mathbb{F}}([0,T];L^2(\Omega;H))$ is the set of all $\mathbb{F}$-progressively measurable processes $\varphi(\cdot)$ valued in H such that, for almost all $\omega\in\Omega$, $t\mapsto\varphi(t,\omega)$ is continuous and

$$\|\varphi(\cdot)\|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H))} \equiv \left[\sup_{t\in[0,T]} \mathbb{E}\left(\left|X(t)\right|^2\right)\right]^{\frac{1}{2}} < \infty.$$

- $L^2(0,T;L^2_{\mathbb{F}}(\Omega\times(0,T);H))$ is the set of all processes $Z:[0,T]^2\times\Omega\to H$ such that, for almost $t\in[0,T]$, $Z(t,\cdot)\in L^2_{\mathbb{F}}(\Omega\times(0,T);H)$ and

$$\|Z(\cdot,\cdot)\|_{L^2(0,T;L^2_{\mathbb{F}}(\Omega\times(0,T);H))} \equiv \left[\mathbb{E}\int_0^T\int_0^T \left|Z(t,s)\right|^2 ds\,dt\right]^{\frac{1}{2}} < \infty.$$

- $\mathcal{H}^2[0,T] \equiv L^2_{\mathbb{F}}(\Omega\times(0,T);\mathbb{R}^n)\times L^2(0,T;L^2_{\mathbb{F}}(\Omega\times(0,T);\mathbb{R}^{n\times d}))$.

In what follows, we will make use of the following elementary assumptions:

(A1) For any $\alpha\in\mathbb{R}$, $\varphi_\alpha(\cdot)$ is an $\mathbb{F}$-adapted continuous process, and there exists a positive constant L (independently on α), such that

$$\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}\left|\varphi_\alpha(t)\right|^2 \le L^2,\\
&\left|b_\alpha(t,s,x)-b_\alpha(t,s,y)\right| + \left|\sigma_\alpha(t,s,x)-\sigma_\alpha(t,s,y)\right| \le L|x-y|,\\
&\quad t,s\in[0,T], x,y\in\mathbb{R}^d,
\end{aligned}$$

and

$$\left|b_\alpha(\cdot,\cdot,0)\right| + \left|\sigma_\alpha(\cdot,\cdot,0)\right| \le L.$$

(A2) There exists a positive constant L, such that, for any $\alpha \in \mathbb{R}$,

$$\sup_{0\le t\le T} \mathbb{E}\left|\psi_\alpha(t)\right|^2 \le L^2,$$

$$\left|f_\alpha(t,s,y_1,z_1,\nu_1) - f_\alpha(t,s,y_2,z_2,\nu_2)\right| \le L\left(|y_1-y_2| + |z_1-z_2| + |\nu_1-\nu_2|\right),$$
$$t,s\in[0,T], y_1,y_2\in\mathbb{R}^m, z_1,z_2,\nu_1,\nu_2\in\mathbb{R}^{m\times d},$$

and

$$\left|f_\alpha(\cdot,\cdot,0,0,0)\right| \le L.$$

The rest of this paper is organized as follows. In Section 2, we review the well-posedness and continuity results for SVIEs depending on a parameter, and give the property of solutions' differentiability with respect to that parameter. Section 3 is devoted to studying the continuity and differentiability of solutions to BSVIEs with respect to a parameter.

2 SVIEs with a parameter

Firstly, we present the definition of solution to SVIE (1.1).

Definition 2.1 For any $\alpha \in \mathbb{R}$, a stochastic process $X_\alpha(\cdot) \in C_{\mathbb{F}}([0,T];L^2(\Omega;\mathbb{R}^m))$ is called an adapted solution to SVIE (1.1) on $[0,T]$ if (1.1) holds in the usual Itô's sense for all $t\in[0,T]$.

In the following, the well-posedness result comes from [7–9] and the property of continuity of solutions to (1.1) with respect to the parameter α comes from [7, 8].

Theorem 2.2 *Let assumption* (A1) *hold. Then, for any $\alpha\in\mathbb{R}$, SVIE* (1.1) *admits a unique solution $X_\alpha(\cdot)$. Furthermore, we have*

$$\sup_{0\le t\le T}\mathbb{E}\left|X_\alpha(t)\right|^2 \le C\sup_{0\le t\le T}\left\{\mathbb{E}\left|\varphi_\alpha(t)\right|^2 + \mathbb{E}\left(\int_0^t \left|b_\alpha(t,s,0)\right|ds\right)^2 + \mathbb{E}\left(\int_0^t \left|\sigma_\alpha(t,s,0)\right|^2 ds\right)\right\},$$

where C is a constant.

Theorem 2.3 *Assume that the coefficients $\varphi_\alpha(\cdot)$, $b_\alpha(\cdot,\cdot,\cdot)$, $\sigma_\alpha(\cdot,\cdot,\cdot)$ in* (1.1) *satisfy assumption* (A1), *and*

$$\lim_{\alpha'\to\alpha}\sup_{0\le t\le T}\mathbb{E}\left|\varphi_{\alpha'}(t)-\varphi_\alpha(t)\right|^2 = 0,$$

and that, for any $N>0$, $s,t\in[0,T]$, $s\le t$ and $\varepsilon>0$,

$$\lim_{\alpha'\to\alpha}P\left(\sup_{|x|<N}\left(\left|b_{\alpha'}(t,s,x)-b_\alpha(t,s,x)\right| + \left|\sigma_{\alpha'}(t,s,x)-\sigma_\alpha(t,s,x)\right|\right) > \varepsilon\right) = 0.$$

Then

$$\lim_{\alpha'\to\alpha}\sup_{0\le t\le T}\mathbb{E}\left|x_{\alpha'}(t)-x_\alpha(t)\right|^2 = 0.$$

Now, we are in the step to study the differentiability of solutions to SVIEs with respect to a parameter. Firstly, we give the definition of derivative of random variables.

Definition 2.4 A random variable η is called the derivative of a family of random variables $\{\xi_\alpha\}$ in L^2-norm if the following holds:

$$\lim_{\Delta\alpha\to 0} \mathbb{E}\left|\frac{\xi_{\alpha+\Delta\alpha}-\xi_\alpha}{\Delta\alpha}-\eta\right|^2 = 0.$$

For convenience, we denote η by $\partial_\alpha\xi_\alpha$. The following result states the solutions' differentiability with respect to a parameter.

Theorem 2.5 *Suppose assumption* (A1) *hold. Let $X_\alpha(\cdot)$ be solution to*

$$X_\alpha(t) = \varphi_\alpha(t) + \int_0^t b_\alpha\bigl(t,s,X_\alpha(s)\bigr)\,ds + \int_0^t \sigma_\alpha\bigl(t,s,X_\alpha(s)\bigr)\,dW(s), \quad t\in[0,T],$$

whereby the following conditions are satisfied:

(1) $\partial_\alpha\varphi_\alpha(t)$ *exists, and*

$$\lim_{\Delta\alpha\to 0}\sup_{0\le t\le T}\mathbb{E}\left|\partial_\alpha\varphi_\alpha(t)-\frac{1}{\Delta\alpha}\bigl(\varphi_{\alpha+\Delta\alpha}(t)-\varphi_\alpha(t)\bigr)\right|^2 \to 0;$$

(2) $\partial_\alpha b_\alpha(t,s,x)$ *and* $\partial_\alpha\sigma_\alpha(t,s,x)$ *exist, and*

$$\begin{aligned}\lim_{\Delta\alpha\to 0}\mathbb{E}\int_0^T\Biggl\{&\left|\frac{b_{\alpha+\Delta\alpha}(t,s,X_\alpha(s))-b_\alpha(t,s,X_\alpha(s))}{\Delta\alpha}-\partial_\alpha b_\alpha\bigl(t,s,X_\alpha(s)\bigr)\right|^2\\&+\left|\frac{\sigma_{\alpha+\Delta\alpha}(t,s,X_\alpha(s))-\sigma_\alpha(t,s,X_\alpha(s))}{\Delta\alpha}-\partial_\alpha\sigma_\alpha\bigl(t,s,X_\alpha(s)\bigr)\right|^2\Biggr\}\,ds = 0;\end{aligned}$$

(3) $\partial_x b_\alpha(t,s,x)$ *and* $\partial_x\sigma_\alpha(t,s,x)$ *exist and are continuous with respect to all arguments, and for some* K

$$P\bigl(\bigl|\partial_x b_\alpha(t,s,x)\bigr|\le K\bigr)=1, \qquad P\bigl(\bigl|\partial_x\sigma_\alpha^i(t,s,x)\bigr|\le K\bigr)=1, \quad i=1,2,\ldots,d.$$

Then $X_\alpha(\cdot)$ is differentiable with respect to α, and $\partial_\alpha X_\alpha(\cdot)$, the derivative of $X_\alpha(\cdot)$ with respect to α, solves the following SVIE:

$$\begin{aligned}\partial_\alpha X_\alpha(t) = {}&\partial_\alpha\varphi_\alpha(t) + \int_0^t\bigl[\partial_x b_\alpha\bigl(t,s,X_\alpha(s)\bigr)\partial_\alpha X_\alpha(s)+\partial_\alpha b_\alpha\bigl(t,s,X_\alpha(s)\bigr)\bigr]\,ds\\&+\int_0^t\bigl[\bigl\langle\partial_x\sigma_\alpha\bigl(t,s,X_\alpha(s)\bigr),\partial_\alpha X_\alpha(s)\bigr\rangle+\partial_\alpha\sigma_\alpha\bigl(t,s,X_\alpha(s)\bigr)\bigr]\,dW(s), \quad t\in[0,T], \quad (2.1)\end{aligned}$$

where $\langle\partial_x\sigma_\alpha,\partial_\alpha X_\alpha\rangle = (\partial_x\sigma_\alpha^i\partial_\alpha X_\alpha)_{1\le i\le d}$.

Proof For notational convenience, we can assume that $m = d = 1$. Denote $X_{\alpha,\Delta\alpha}(\cdot) \equiv \frac{1}{\Delta\alpha}(X_{\alpha+\Delta\alpha}(\cdot) - X_\alpha(\cdot))$. Then this stochastic process satisfies

$$\begin{aligned} X_{\alpha,\Delta\alpha}(t) = {} & \frac{\varphi_{\alpha+\Delta\alpha}(t) - \varphi_\alpha(t)}{\Delta\alpha} \\ & + \int_0^t \frac{b_{\alpha+\Delta\alpha}(t,s,X_{\alpha+\Delta\alpha}(s)) - b_\alpha(t,s,X_\alpha(s))}{\Delta\alpha}\,ds \\ & + \int_0^t \frac{\sigma_{\alpha+\Delta\alpha}(t,s,X_{\alpha+\Delta\alpha}(s)) - \sigma_\alpha(t,s,X_\alpha(s))}{\Delta\alpha}\,dW(s), \quad t \in [0,T]. \end{aligned} \tag{2.2}$$

Denote by $X_{\alpha,0}(\cdot)$ the solution to

$$\begin{aligned} X_{\alpha,0}(t) = {} & \partial_\alpha \varphi_\alpha(t) + \int_0^t \partial_x b_\alpha\big(t,s,X_\alpha(s)\big)X_{\alpha,0}(s)\,ds + \int_0^t \partial_x\sigma_\alpha\big(t,s,X_\alpha(s)\big)X_{\alpha,0}(s)\,dW(s) \\ & + \int_0^t \partial_\alpha b_\alpha\big(t,s,X_\alpha(s)\big)\,ds + \int_0^t \partial_\alpha\sigma_\alpha\big(t,s,X_\alpha(s)\big)\,dW(s), \quad t \in [0,T]. \end{aligned} \tag{2.3}$$

To simplify the above two equations, we introduce the following functions: for $\Delta\alpha \neq 0$,

$$\begin{aligned} \widetilde{\varphi}_{\Delta\alpha}(t) \equiv {} & \frac{\varphi_{\alpha+\Delta\alpha}(t) - \varphi_\alpha(t)}{\Delta\alpha} + \int_0^t \frac{b_{\alpha+\Delta\alpha}(t,s,X_\alpha(s)) - b_\alpha(t,s,X_\alpha(s))}{\Delta\alpha}\,ds \\ & + \int_0^t \frac{\sigma_{\alpha+\Delta\alpha}(t,s,X_\alpha(s)) - \sigma_\alpha(t,s,X_\alpha(s))}{\Delta\alpha}\,dW(s); \\ \widetilde{b}_{\Delta\alpha}(t,s,x) \equiv {} & \begin{cases} \frac{b_{\alpha+\Delta\alpha}(t,s,X_{\alpha+\Delta\alpha}(s)) - b_{\alpha+\Delta\alpha}(t,s,X_\alpha(s))}{X_{\alpha+\Delta\alpha}(s) - X_\alpha(s)}x, & X_{\alpha+\Delta\alpha}(s) \neq X_\alpha(s), \\ \partial_x b_{\alpha+\Delta\alpha}(t,s,X_\alpha(s))x, & X_{\alpha+\Delta\alpha}(s) = X_\alpha(s); \end{cases} \end{aligned}$$

for $\Delta\alpha = 0$,

$$\begin{aligned} \widetilde{\varphi}_0(t) \equiv {} & \partial_\alpha\varphi_\alpha(t) + \int_0^t \partial_\alpha b_\alpha\big(t,s,X_\alpha(s)\big)\,ds + \int_0^t \partial_\alpha\sigma_\alpha\big(t,s,X_\alpha(s)\big)\,dW(s); \\ \widetilde{b}_0(t,s,x) \equiv {} & \partial_x b_\alpha\big(t,s,X_\alpha(s)\big)x. \end{aligned}$$

Similarly, we can define $\widetilde{\sigma}_{\Delta\alpha}(t,s,x)$ and $\widetilde{\sigma}_0(t,s,x)$. Using these functions we can express (2.2) and (2.3) in the form

$$X_{\alpha,\Delta\alpha}(t) = \widetilde{\varphi}_{\Delta\alpha}(t) + \int_0^t \widetilde{b}_{\Delta\alpha}\big(t,s,X_{\alpha,\Delta\alpha}(s)\big)\,ds + \int_0^t \widetilde{\sigma}_{\Delta\alpha}\big(t,s,X_{\alpha,\Delta\alpha}(s)\big)\,dW(s)$$

and

$$X_{\alpha,0}(t) = \widetilde{\varphi}_0(t) + \int_0^t \widetilde{b}_0\big(t,s,X_{\alpha,0}(s)\big)\,ds + \int_0^t \widetilde{\sigma}_0\big(t,s,X_{\alpha,0}(s)\big)\,dW(s).$$

To prove the theorem, it is sufficient to show that

$$\lim_{\Delta\alpha\to 0} \mathbb{E}\big|X_{\alpha,\Delta\alpha}(t) - X_{\alpha,0}(t)\big|^2 = 0. \tag{2.4}$$

To obtain (2.4), due to Theorem 2.3, it is only necessary to verify that

$$\lim_{\Delta\alpha\to 0} \sup_{0\le t\le T} \mathbb{E}\big|\widetilde{\varphi}_{\Delta\alpha}(t) - \widetilde{\varphi}_0(t)\big|^2 = 0, \tag{2.5}$$

and that, for each $N > 0$, $\varepsilon > 0$,

$$\lim_{\Delta\alpha\to 0} P\Big(\sup_{|x|\le N}\big(\big|\widetilde{b}_{\Delta\alpha}(t,s,x)-\widetilde{b}_0(t,s,x)\big|^2+\big|\widetilde{\sigma}_{\Delta\alpha}(t,s,x)-\widetilde{\sigma}_0(t,s,x)\big|^2\big)>\varepsilon\Big)=0. \tag{2.6}$$

Equation (2.6) can be obtained by Condition (3), and (2.5) can be obtained by Condition (1) and Condition (2). That completes the proof. □

The next result is an immediate consequence of Theorem 2.5.

Corollary 2.6 *Suppose that $X(\cdot)$ solves SVIE*:

$$X(t)=x+\int_0^t b\big(t,s,X(s)\big)\,ds+\int_0^t \sigma\big(t,s,X(s)\big)\,dW(s),\quad t\in[0,T], \tag{2.7}$$

where $b(\cdot,\cdot,\cdot)$, $\sigma(\cdot,\cdot,\cdot)$ satisfy assumption (A1), *and $\partial_x b(t,s,x)$ and $\partial_x\sigma(t,s,x)$ exist and are continuous with respect to all arguments, and for some K*

$$P\big(\big|\partial_x b(t,s,x)\big|\le K\big)=1,\qquad P\big(\big|\partial_x\sigma^i(t,s,x)\big|\le K\big)=1,\quad i=1,2,\ldots,d.$$

Then $\nabla X(\cdot)$ ($\equiv(\partial_{x_i}X^j)_{1\le i,j\le m}$) solves the following SVIE:

$$\begin{aligned}\nabla X(t)=I_m&+\int_0^t \partial_x b\big(t,s,X(s)\big)\nabla X(s)\,ds\\&+\int_0^t \big\langle\partial_x\sigma\big(t,s,X(s)\big),\nabla X(s)\big\rangle\,dW(s),\quad t\in[0,T],\end{aligned}$$

where $I_m\in\mathbb{R}^{m\times m}$ is the unit matrix.

3 BSVIEs with a parameter

In this section, we consider the continuity and differentiability of solutions to BSVIEs of the form (1.2) with respect to α. Firstly, we give the definition of solution to (1.2).

Definition 3.1 Let $S\in[0,T)$. For any $\alpha\in\mathbb{R}$, stochastic processes pair $(Y_\alpha(\cdot),Z_\alpha(\cdot,\cdot))\in\mathcal{H}^2[S,T]$ is called an adapted M-solution to BSVIE (1.2) on $[S,T]$ if (1.2) holds in the usual Itô's sense for almost all $t\in[S,T]$ and, in addition, the following holds:

$$Y_\alpha(t)=\mathbb{E}\big(Y_\alpha(t)|\mathcal{F}_S\big)+\int_S^t Z_\alpha(t,s)\,dW(s),\quad \text{a.e. } t\in[S,T].$$

The following result comes from [12, Theorem 3.7].

Theorem 3.2 *For any α, let* (A2) *hold. Then BSVIE* (1.2) *admits a unique adapted M-solution $(Y_\alpha(\cdot),Z_\alpha(\cdot,\cdot))\in\mathcal{H}^2[0,T]$. Moreover, the following estimate holds*:

$$\begin{aligned}\big\|\big(Y_\alpha(\cdot),Z_\alpha(\cdot,\cdot)\big)\big\|^2_{\mathcal{H}^2[S,T]}&\equiv\mathbb{E}\left\{\int_S^T\big|Y_\alpha(t)\big|^2\,dt+\int_S^T\int_S^T\big|Z_\alpha(t,s)\big|^2\,ds\,dt\right\}\\&\le C\left(1+\mathbb{E}\int_S^T\big|\psi_\alpha(t)\big|^2\,dt\right),\quad\forall S\in[0,T].\end{aligned}$$

Let $\bar{f}_\alpha$ *and* $\bar{\psi}_\alpha$ *also satisfy* (A2), *and* $(\bar{Y}_\alpha(\cdot), \bar{Z}_\alpha(\cdot,\cdot)) \in \mathcal{H}^2[0,T]$ *be the adapted M-solution to* (1.2) *with* f_α *and* ψ_α *replaced by* $\bar{f}_\alpha$ *and* $\bar{\psi}_\alpha$, *respectively. Then there exists a constant C such that, for any* $S \in [0,T]$,

$$\begin{aligned}
&\big\| \big(Y_\alpha(\cdot) - \bar{Y}_\alpha(\cdot), Z_\alpha(\cdot,\cdot) - \bar{Z}_\alpha(\cdot,\cdot)\big) \big\|^2_{\mathcal{H}^2[S,T]} \\
&\leq C\mathbb{E}\biggl\{ \int_S^T \big|\psi_\alpha(t) - \bar{\psi}_\alpha(t)\big|^2\,dt \\
&\quad + \int_S^T \int_t^T \big| f_\alpha\big(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)\big) - \bar{f}_\alpha\big(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\big|^2\,ds\,dt \biggr\}.
\end{aligned}$$

In the sequel, we study the properties of continuity and differentiability of solutions to BSVIEs depending on a parameter. For notational convenience, for fixed α_0, we write $(Y_0(\cdot), Z_0(\cdot,\cdot))$ for $(Y_{\alpha_0}(\cdot), Z_{\alpha_0}(\cdot,\cdot))$. Let us make the following assumptions:

(A3) The function $\alpha \to (f_\alpha, \psi_\alpha)$ is continuous; *i.e.*, for any α_0, $f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)) - f_{\alpha_0}(t,s,Y_0(s),Z_0(t,s),Z_0(s,t))$ converges to 0 in $L^2(0,T;L^2_{\mathbb{F}}(\Omega\times(0,T);\mathbb{R}^n))$ and $\psi_\alpha - \psi_{\alpha_0}$ converges to 0 in $C_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$ as $\alpha \to \alpha_0$.

(A4) For any $\alpha \in \mathbb{R}$, f_α is differentiable with respect to (y,z,v) with uniformly bounded derivatives denoted by $\partial_y f_\alpha$, $\partial_z f_\alpha$ and $\partial_v f_\alpha$ which are uniformly continuous; *i.e.*, for any $\varepsilon > 0$, there exists δ such that, for any $t,s \in [0,T]$, $y \in \mathbb{R}^n$, $z, v \in \mathbb{R}^{n\times d}$,

$$\big|\partial_y f_\alpha(t,s,y+h,z,v) - \partial_y f_\alpha(t,s,y,z,v)\big| < \varepsilon, \quad |h| \leq \delta,$$

and the same holds for z, v parts and for $\partial_z f_\alpha$, $\partial_v f_\alpha$.

(A5) The function $\alpha \mapsto (f_\alpha, \psi_\alpha)$ is differentiable; *i.e.*, for any α_0, the functions $\alpha \mapsto \psi_\alpha(t)$, $\mathbb{R} \to C_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$ and $\alpha \mapsto f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_0(s,t))$, $\mathbb{R} \to L^2(0,T;L^2_{\mathbb{F}}(\Omega\times(0,T);\mathbb{R}^n))$ are differentiable at α_0 with derivatives $\partial_\alpha f_{\alpha_0}(t,s,Y_0(s),Z_0(t,s),Z_0(s,t))$ and $\partial_\alpha \psi_{\alpha_0}(t)$.

Theorem 3.3 *Let the coefficients* f_α, ψ_α $(\alpha \in \mathbb{R})$ *of BSVIEs* (1.2) *satisfy assumption* (A2), *and* $(Y_\alpha(\cdot), Z_\alpha(\cdot,\cdot))$ *be solutions.*

1. *Suppose that* f_α, ψ_α *satisfy assumption* (A3). *Then the function* $\alpha \mapsto (Y_\alpha(\cdot), Z_\alpha(\cdot,\cdot))$, $\mathbb{R} \to \mathcal{H}^2[0,T]$ *is continuous.*
2. *Suppose that* f_α, ψ_α *satisfy assumptions* (A4)-(A5). *Then the function* $\alpha \mapsto (Y_\alpha(\cdot), Z_\alpha(\cdot,\cdot))$, $\mathbb{R} \to \mathcal{H}^2[0,T]$ *is differentiable with derivatives given by* $(\partial_\alpha Y_\alpha(\cdot), \partial_\alpha Z_\alpha(\cdot,\cdot))$, *which is the solution to the following BSVIE*:

$$\begin{aligned}
\partial_\alpha Y_\alpha(t) = \partial_\alpha \psi_\alpha(t) + \int_t^T &\big[\partial_y f_\alpha\big(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\partial_\alpha Y_\alpha(s) \\
&+ \big\langle \partial_z f_\alpha\big(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)\big), \partial_\alpha Z_\alpha(t,s)\big\rangle \\
&+ \big\langle \partial_v f_\alpha\big(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)\big), \partial_\alpha Z_\alpha(s,t)\big\rangle \\
&+ \partial_\alpha f_\alpha\big(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\big]\,ds \\
&+ \int_t^T \partial_\alpha Z_\alpha(t,s)\,dW(s), \quad t \in [0,T],
\end{aligned} \tag{3.1}$$

where $\langle \partial_z f_\alpha, \partial_\alpha Z_\alpha\rangle \equiv (\partial_{z_i} f_\alpha \partial_\alpha Z^i_\alpha)_{1\leq i\leq d}$ *and* $\langle \partial_v f_\alpha, \partial_\alpha Z_\alpha\rangle \equiv (\partial_{v_i} f_\alpha \partial_\alpha Z^i_\alpha)_{1\leq i\leq d}$.

Proof Property 1 is an immediate consequence of Theorem 3.2. Now, let us prove property 2.

For simplicity, we can assume that $\alpha_0 = 0$ and $m = d = 1$. For any $\alpha \neq 0$, put

$$\Delta_\alpha Y(\cdot) \equiv \frac{Y_\alpha(\cdot) - Y_0(\cdot)}{\alpha}, \qquad \Delta_\alpha Z(\cdot,\cdot) \equiv \frac{Z_\alpha(\cdot,\cdot) - Z_0(\cdot,\cdot)}{\alpha}.$$

Then

$$\begin{aligned}\Delta_\alpha Y(t) = {}& \frac{\psi_\alpha(t) - \psi_0(t)}{\alpha} \\ & + \int_t^T \frac{f_\alpha(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)) - f_0(t,s,Y_0(s),Z_0(t,s),Z_0(s,t))}{\alpha}\,ds \\ & + \int_t^T \Delta_\alpha Z(t,s)\,dW(s), \quad t \in [0,T]. \end{aligned} \tag{3.2}$$

For any $\alpha \neq 0$, define

$$A_\alpha(t,s) \equiv \begin{cases} \frac{f_\alpha(t,s,Y_\alpha(s),Z_\alpha(t,s),Z_\alpha(s,t)) - f_\alpha(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t))}{Y_\alpha(s)-Y_0(s)}, & Y_\alpha(s) \neq Y_0(s), \\ \partial_y f_\alpha(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)), & Y_\alpha(s) = Y_0(s), \end{cases}$$

$$B_\alpha(t,s) \equiv \begin{cases} \frac{f_\alpha(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)) - f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_\alpha(s,t))}{Z_\alpha(t,s)-Z_0(t,s)}, & Z_\alpha(t,s) \neq Z_0(t,s), \\ \partial_z f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_\alpha(s,t)), & Z_\alpha(t,s) = Z_0(t,s), \end{cases}$$

$$C_\alpha(t,s) \equiv \begin{cases} \frac{f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_\alpha(s,t)) - f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_0(s,t))}{Z_\alpha(s,t)-Z_0(s,t)}, & Z_\alpha(s,t) \neq Z_0(s,t), \\ \partial_v f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)), & Z_\alpha(s,t) = Z_0(s,t), \end{cases}$$

and

$$h_\alpha(t,s) \equiv \frac{f_\alpha(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)) - f_0(t,s,Y_0(s),Z_0(t,s),Z_0(s,t))}{\alpha}.$$

By virtue of $A_\alpha(\cdot,\cdot)$, $B_\alpha(\cdot,\cdot)$, $B_\alpha(\cdot,\cdot)$, $h_\alpha(\cdot,\cdot)$, we can define

$$F_\alpha(t,s,y,z,v) \equiv A_\alpha(t,s)y + B_\alpha(t,s)z + C_\alpha(t,s)v + h_\alpha(t,s).$$

Hence, BSVIE (3.2) can be written as

$$\begin{aligned}\Delta_\alpha Y(t) = {}& \frac{\psi_\alpha(t) - \psi_0(t)}{\alpha} + \int_t^T F_\alpha\big(t,s,\Delta_\alpha Y(s),\Delta_\alpha Z(t,s),\Delta_\alpha Z(s,t)\big)\,ds \\ & + \int_t^T \Delta_\alpha Z(t,s)\,dW(s), \quad t \in [0,T]. \end{aligned} \tag{3.3}$$

On the other hand, denote

$$\begin{aligned} F_0(t,s,y,z,v) \equiv {}& \partial_y f_0\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)y + \partial_z f_0\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)z \\ & + \partial_v f_0\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)v + \partial_\alpha f_0\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big), \end{aligned}$$

and introduce the following BSVIE:

$$\partial_\alpha Y_0(t) = \partial_\alpha \psi_0(t) + \int_t^T F_0\big(t,s,\partial_\alpha Y_0(s),\partial_\alpha Z_0(t,s),\partial_\alpha Z_0(s,t)\big)\,ds + \int_t^T \partial_\alpha Z_0(t,s)\,dW(s), \quad t\in[0,T]. \tag{3.4}$$

By property 1 of this theorem, we know that $(Y_\alpha(\cdot),Z_\alpha(\cdot,\cdot))$ converges to $(Y_0(\cdot),Z_0(\cdot,\cdot))$ in $\mathcal{H}^2[0,T]$. We now have to prove that $(\Delta_\alpha Y(\cdot),\Delta_\alpha X(\cdot,\cdot))$, the solution to BSVIE (3.3), converges to $(\partial_\alpha Y(\cdot),\partial_\alpha Z(\cdot,\cdot))$, the solution to BSVIE (3.4), in $\mathcal{H}^2[0,T]$, as α tends to 0. Also by property 1, it is sufficient to check

$$F_\alpha\big(t,s,\partial_\alpha Y_0(s),\partial_\alpha Z_0(t,s),\partial_\alpha Z_0(s,t)\big) \longrightarrow F_0\big(t,s,\partial_\alpha Y_0(s),\partial_\alpha Z_0(t,s),\partial_\alpha Z_0(s,t)\big) \quad \text{in } L^2\big(0,T;L^2_{\mathbb{F}}\big(\Omega\times(0,T)\big)\big) \tag{3.5}$$

and

$$\frac{\psi_\alpha(\cdot)-\psi_0(\cdot)}{\alpha} \longrightarrow \partial_\alpha\psi_0(\cdot) \quad \text{in } L^2_{\mathbb{F}}\big(\Omega\times(0,T)\big). \tag{3.6}$$

Equation (3.6) can be gotten by assumption (A5). Now, we check (3.5).

Notice that

$$A_\alpha(t,s) = \int_0^1 \partial_y f_\alpha\big(t,s,Y_0(s)+\lambda\big(Y_\alpha(s)-Y_0(s)\big),Z_\alpha(t,s),Z_\alpha(s,t)\big)\,d\lambda.$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T \big|A_\alpha(t,s)-\partial_y f_\alpha\big(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\big|^2\big|\partial_\alpha Y_0(s)\big|^2\,ds\,dt \\
&\quad = \mathbb{E}\int_0^T\int_0^T \bigg|\int_0^1 \partial_y f_\alpha\big(t,s,Y_0(s)+\lambda\big(Y_\alpha(s)-Y_0(s)\big),Z_\alpha(t,s),Z_\alpha(s,t)\big)\,d\lambda \\
&\qquad - \partial_y f_\alpha\big(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\bigg|^2\big|\partial_\alpha Y_0(s)\big|^2\,ds\,dt \\
&\quad \le \mathbb{E}\int_0^T\int_0^T\int_0^1 \big|\partial_y f_\alpha\big(t,s,Y_0(s)+\lambda\big(Y_\alpha(s)-Y_0(s)\big),Z_\alpha(t,s),Z_\alpha(s,t)\big) \\
&\qquad - \partial_y f_\alpha\big(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\big|^2\big|\partial_\alpha Y_0(s)\big|^2\,d\lambda\,ds\,dt.
\end{aligned} \tag{3.7}$$

Set $\Omega_1=\{|Y_\alpha(s)-Y_0(s)|\le\delta\}$. Hence, by assumption (A4),

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T\int_0^1 \big|\partial_y f_\alpha\big(t,s,Y_0(s)+\lambda\big(Y_\alpha(s)-Y_0(s)\big),Z_\alpha(t,s),Z_\alpha(s,t)\big) \\
&\qquad - \partial_y f_\alpha\big(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\big|^2\big|\partial_\alpha Y_0(s)\big|^2\,d\lambda\,ds\,dt \\
&\quad \le \mathbb{E}\int_0^T\int_0^T \big[\varepsilon^2\big|\partial_\alpha Y_0(s)\big|^2 + 2L^2\chi_{\Omega_1^c}\big|\partial_\alpha Y_0(s)\big|^2\big]\,ds\,dt \\
&\quad \le \varepsilon^2 T\mathbb{E}\int_0^T \big|\partial_\alpha Y_0(s)\big|^2\,ds + 2L^2T\mathbb{E}\int_0^T \chi_{\Omega_1^c}\big|\partial_\alpha Y_0(s)\big|^2\,ds.
\end{aligned} \tag{3.8}$$

Now, we estimate the second term on the right side of the above inequality. For given $M > 0$, define $\Omega_2 = \{|\partial_\alpha Y_0(s)| \le M\}$. Then

$$\begin{aligned}
&\mathbb{E}\int_0^T \chi_{\Omega_1^c}\big|\partial_\alpha Y_0(s)\big|^2\,ds \\
&\quad= \mathbb{E}\int_0^T \chi_{\Omega_1^c}\chi_{\Omega_2}\big|\partial_\alpha Y_0(s)\big|^2\,ds + \mathbb{E}\int_0^T \chi_{\Omega_1^c}\chi_{\Omega_2^c}\big|\partial_\alpha Y_0(s)\big|^2\,ds \\
&\quad\le \mathbb{E}\int_0^T \chi_{\Omega_1^c}\chi_{\Omega_2}M^2\frac{|Y_\alpha(s)-Y_0(s)|^2}{\delta^2}\,ds + \mathbb{E}\int_0^T \chi_{\Omega_2^c}\big|\partial_\alpha Y_0(s)\big|^2\,ds \\
&\quad\le \frac{M^2}{\delta^2}\mathbb{E}\int_0^T \big|Y_\alpha(s)-Y_0(s)\big|^2\,ds + \mathbb{E}\int_0^T \chi_{\{|\partial_\alpha Y_0(s)|>M\}}\big|\partial_\alpha Y_0(s)\big|^2\,ds.
\end{aligned} \tag{3.9}$$

By the Lebesgue dominated convergence theorem, since $\partial_\alpha Y_0(\cdot) \in L^2_{\mathbb{F}}(\Omega \times (0,T))$, one deduces that

$$\mathbb{E}\int_0^T \chi_{\{|\partial_\alpha Y_0(s)|>M\}}\big|\partial_\alpha Y_0(s)\big|^2\,ds \to 0. \tag{3.10}$$

By property 1, $Y_\alpha(\cdot) \to Y_0(\cdot)$ in $L^2_{\mathbb{F}}(\Omega \times (0,T))$, one gets

$$\frac{M^2}{\delta^2}\mathbb{E}\int_0^T \big|Y_\alpha(s)-Y_0(s)\big|^2\,ds \to 0. \tag{3.11}$$

Equation (3.7), together with (3.8)-(3.11), yields

$$\mathbb{E}\int_0^T\int_0^T \big|A_\alpha(t,s) - \partial_y f_\alpha\big(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)\big)\big|^2\big|\partial_\alpha Y_0(s)\big|^2\,ds\,dt \to 0. \tag{3.12}$$

Denote $\Omega_3 = \{|Z_\alpha(t,s) - Z_0(t,s)| \le \delta\}$. Then

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T \big|\partial_y f_\alpha\big(t,s,Y_0(s),Z_\alpha(t,s),Z_\alpha(s,t)\big) \\
&\qquad - \partial_y f_\alpha\big(t,s,Y_0(s),Z_0(t,s),Z_\alpha(s,t)\big)\big|^2\big|\partial_\alpha Y_0(s)\big|^2\,ds\,dt \\
&\quad\le \varepsilon^2 T\mathbb{E}\int_0^T \big|\partial_\alpha Y_0(s)\big|^2\,ds + L^2\mathbb{E}\int_0^T\int_0^T \chi_{\Omega_3^c}\big|\partial_\alpha Y_0(s)\big|^2\,ds\,dt \\
&\quad\le \varepsilon^2 T\mathbb{E}\int_0^T \big|\partial_\alpha Y_0(s)\big|^2\,ds + \frac{L^2M^2}{\delta^2}\mathbb{E}\int_0^T\int_0^T \big|Z_\alpha(t,s)-Z_0(t,s)\big|^2\,ds\,dt \\
&\qquad + L^2 T\mathbb{E}\int_0^T \chi_{\{|\partial_\alpha Y_0(s)|>M\}}\big|\partial_\alpha Y_0(s)\big|^2\,ds \\
&\quad\to 0.
\end{aligned} \tag{3.13}$$

Similarly, we can check that

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T \big|\partial_y f_\alpha\big(t,s,Y_0(s),Z_0(t,s),Z_\alpha(s,t)\big) \\
&\quad - \partial_y f_\alpha\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)\big|^2\big|\partial_\alpha Y_0(s)\big|^2\,ds\,dt \to 0.
\end{aligned} \tag{3.14}$$

Combining (3.12) with (3.13), (3.14), we can obtain

$$\mathbb{E}\int_0^T\int_0^T \big|A_\alpha(t,s)-\partial_y f_\alpha\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)\big|^2\big|\partial_\alpha Y_0(s)\big|^2\,ds\,dt\to 0. \tag{3.15}$$

With a similar procedure, we can get

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_0^T \big|B_\alpha(t,s)-\partial_z f_\alpha\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)\big|^2\big|\partial_\alpha Z_0(t,s)\big|^2\,ds\,dt\to 0,\\
&\mathbb{E}\int_0^T\int_0^T \big|C_\alpha(t,s)-\partial_v f_\alpha\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)\big|^2\big|\partial_\alpha Z_0(s,t)\big|^2\,ds\,dt\to 0,
\end{aligned} \tag{3.16}$$

and

$$\mathbb{E}\int_0^T\int_0^T \big|h_\alpha(t,s)-\partial_\alpha f_0\big(t,s,Y_0(s),Z_0(t,s),Z_0(s,t)\big)\big|^2\,ds\,dt\to 0. \tag{3.17}$$

By (3.15)-(3.17), we prove (3.5). That completes the proof. □

An immediate consequence of Theorem 3.3 is now the following.

Corollary 3.4 *Suppose that* $(Y(\cdot),Z(\cdot,\cdot))$ *is the solution to BSVIE*:

$$\begin{aligned}
Y(t)={}&\psi\big(X(t)\big)+\int_t^T f\big(t,s,Y(s),Z(t,s),Z(s,t)\big)\,ds\\
&+\int_t^T Z(t,s)\,dW(s),\quad t\in[0,T],
\end{aligned} \tag{3.18}$$

where $\psi(\cdot)$ *is differentiable and* $f(\cdot,\cdot,\cdot,\cdot,\cdot)$ *is differentiable with respect to* (y,z,v) *with uniformly bounded derivatives. Here* $X(\cdot)$ *solves SVIE* (2.7) *and the coefficients* $b(\cdot,\cdot,\cdot)$, $\sigma(\cdot,\cdot,\cdot)$ *satisfy the assumptions in Corollary* 2.6. *Then* $(\nabla Y(\cdot),\nabla Z(\cdot,\cdot))$ *solves the following BSVIE*:

$$\begin{aligned}
\nabla Y(t)={}&\partial_x\psi\big(X(t)\big)\nabla X(t)+\int_t^T \partial_y f\big(t,s,Y(s),Z(t,s),Z(s,t)\big)\nabla Y(s)\\
&+\big\langle\partial_z f\big(t,s,Y(s),Z(t,s),Z(s,t)\big),\nabla Z(t,s)\big\rangle\\
&+\big\langle\partial_v f\big(t,s,Y(s),Z(t,s),Z(s,t)\big),\nabla Z(s,t)\big\rangle\,ds\\
&+\int_t^T \nabla Z(t,s)\,dW(s),\quad t\in[0,T].
\end{aligned} \tag{3.19}$$

4 Conclusions

In this paper, we consider the properties of continuity and differentiability of solutions to two kinds of stochastic Volterra integral equations: forward equations (Theorem 2.5) and backward equations (Theorem 3.3). As an application, we obtain the deeper well-posedness results on the structure of the solution to forward/backward stochastic Volterra integral equations (Corollaries 2.6 and 3.4).

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The author declares that he has no competing interests.

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