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Chebyshev polynomials and their some interesting applications

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Abstract

The main purpose of this paper is by using the definitions and properties of Chebyshev polynomials to study the power sum problems involving Fibonacci polynomials and Lucas polynomials and to obtain some interesting divisible properties.

MSC: Primary 11B39

Keywords: Chebyshev polynomials; Fibonacci polynomials; Lucas polynomials; power sums; divisible property

1 Introduction

For any integer $n \geq 0$, the first kind Chebyshev polynomials $\{T_n(x)\}$ and the second kind Chebyshev polynomials $\{U_n(x)\}$ are defined by $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$, $U_1(x) = 2x$ and $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$ for all $n \geq 0$. If we write $\alpha = \alpha(x) = x + \sqrt{x^2 - 1}$ and $\beta = \beta(x) = x - \sqrt{x^2 - 1}$ for the sake of simplicity, then we have

$$T_n(x) = \frac{1}{2}\left(\alpha^n + \beta^n\right) = \frac{n}{2}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}, \quad |x| < 1 \tag{1}$$

and

$$U_n(x) = \frac{1}{\alpha - \beta}\left(\alpha^{n+1} - \beta^{n+1}\right) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k}, \quad |x| < 1. \tag{2}$$

Fibonacci polynomials $\{F_n(x)\}$ and Lucas polynomials $\{L_n(x)\}$ are defined by $F_0(x) = 0$, $F_1(x) = 1$, $L_0(x) = 2$, $L_1(x) = x$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$ for all $n \geq 0$. If we write $U(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $V(x) = \frac{x-\sqrt{x^2+4}}{2}$, then we have

$$F_n(x) = \frac{1}{U(x) - V(x)}\left(U^n(x) - V^n(x)\right) \quad \text{and} \quad L_n(x) = U^n(x) + V^n(x) \quad \text{for all } n \geq 0.$$

These polynomials occupy a very important position in the theory and application of mathematics, so many scholars have studied their various properties and obtained a series of interesting and important results. See references [1–13] for Chebyshev polynomials and

[14–16] for Fibonacci and Lucas polynomials. For example, Li Xiaoxue [1] proved some identities involving power sums of $T_n(x)$ and $U_n(x)$. As some applications of these results, she obtained some divisible properties involving Chebyshev polynomials. Precisely, she proved the congruence

$$U_0(x)U_2(x)U_4(x)\cdots U_{2n}(x)\cdot\sum_{m=1}^{h}T_{2m}^{2n+1}(x)\equiv 0 \bmod \bigl(U_{2h}(x)-1\bigr).$$

In this paper, we shall use the definition and properties of Chebyshev polynomials to study the power sum problem involving Fibonacci and Lucas polynomials and prove some new divisible properties involving these polynomials. That is, we shall prove the following two generalized conclusions.

Theorem 1 *Let n and h be non-negative integers with $h \geq 1$. Then, for any odd number $l \geq 1$, we have the congruence*

$$L_l(x)L_{3l}(x)\cdots L_{l(2n+1)}(x)\cdot\sum_{m=0}^{h}L_{2ml}^{2n+1}(x)\equiv 0 \bmod \bigl(L_{l(2h+1)}(x)+L_l(x)\bigr).$$

Theorem 2 *Let n and h be non-negative integers with $h \geq 1$. Then, for any even number $l \geq 1$, we have the congruence*

$$F_l(x)F_{3l}(x)\cdots F_{l(2n+1)}(x)\cdot\sum_{m=0}^{h}L_{2ml}^{2n+1}(x)\equiv 0 \bmod \bigl(F_{l(2h+1)}(x)+F_l(x)\bigr).$$

Especially for $l = 1$ and 2, from our theorems we may immediately deduce the following two corollaries.

Corollary 1 *For any non-negative integers n and h with $h \geq 1$, we have*

$$L_1(x)L_3(x)\cdots L_{2n+1}(x)\cdot\sum_{m=0}^{h}L_{2m}^{2n+1}(x)\equiv 0 \bmod \bigl(L_{2h+1}(x)+x\bigr).$$

Corollary 2 *For any non-negative integers n and h with $h \geq 1$, we have*

$$F_2(x)F_6(x)\cdots F_{2(2n+1)}(x)\cdot\sum_{m=0}^{h}L_{4m}^{2n+1}(x)\equiv 0 \bmod \bigl(F_{2(2h+1)}(x)+x\bigr).$$

Some notes: In our theorems, the range of the summation for m is from 0 to h. If the range of the summation is $1 \leq m \leq h$, then it is very easy to prove the following corresponding results:

For any odd number $l \geq 1$, we have the polynomial congruence

$$L_l(x)L_{3l}(x)\cdots L_{l(2n+1)}(x)\cdot\sum_{m=1}^{h}L_{2ml}^{2n+1}(x)\equiv 0 \bmod \bigl(L_{l(2h+1)}(x)-L_l(x)\bigr).$$

For any even number $l \geq 1$, we have the polynomial congruence

$$F_l(x)F_{3l}(x)\cdots F_{l(2n+1)}(x)\cdot\sum_{m=1}^{h}L_{2ml}^{2n+1}(x)\equiv 0 \bmod \bigl(F_{l(2h+1)}(x)-F_l(x)\bigr).$$

Taking $l = 1$ and 2, and noting that $L_1(x) = F_2(x) = x$, then from these two congruences we can deduce the following:

$$L_1(x)L_3(x)\cdots L_{2n+1}(x)\cdot\sum_{m=1}^{h}L_{2m}^{2n+1}(x)\equiv 0 \bmod \bigl(L_{2h+1}(x)-x\bigr)$$

and

$$F_2(x)F_6(x)\cdots F_{2(2n+1)}(x)\cdot\sum_{m=1}^{h}L_{4m}^{2n+1}(x)\equiv 0 \bmod \bigl(F_{2(2h+1)}(x)-x\bigr).$$

Therefore, our theorems are actually an extension of references [1] and [16].

2 Several simple lemmas

To complete the proofs of our theorems, we need some new properties of Chebyshev polynomial, which we summarize as the following three lemmas.

Lemma 1 *For any integers $m, n \geq 0$, we have the identity*

$$T_n\left(\frac{1}{2}L_{2m}(x)\right)=\frac{1}{2}\cdot L_{2mn}(x).$$

Proof Let $\alpha = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta = \frac{x-\sqrt{x^2+4}}{2}$, then $L_{2m}(x) = \alpha^{2m} + \beta^{2m}$, $\alpha\cdot\beta = -1$ and $\alpha^{2m}\cdot\beta^{2m} = 1$. Replace x by $\frac{1}{2}L_{2m}(x)$ in $T_n(x)$ and note that

$$\begin{aligned}
&\frac{1}{2}L_{2m}(x)+\sqrt{\frac{1}{4}L_{2m}^2(x)-1}\\
&\quad=\frac{1}{2}\bigl(\alpha^{2m}+\beta^{2m}\bigr)+\frac{1}{2}\cdot\sqrt{\alpha^{4m}+\beta^{4m}+2-4}\\
&\quad=\frac{1}{2}\bigl(\alpha^{2m}+\beta^{2m}\bigr)+\frac{1}{2}\bigl(\alpha^{2m}-\beta^{2m}\bigr)\\
&\quad=\alpha^{2m},\\
&\frac{1}{2}L_{2m}(x)-\sqrt{\frac{1}{4}L_{2m}^2(x)-1}\\
&\quad=\frac{1}{2}\bigl(\alpha^{2m}+\beta^{2m}\bigr)-\frac{1}{2}\cdot\sqrt{\alpha^{4m}+\beta^{4m}+2-4}\\
&\quad=\frac{1}{2}\bigl(\alpha^{2m}+\beta^{2m}\bigr)-\frac{1}{2}\bigl(\alpha^{2m}-\beta^{2m}\bigr)\\
&\quad=\beta^{2m}.
\end{aligned}$$

From the definition of $T_n(x)$

$$T_n(x)=\frac{1}{2}\Bigl[\bigl(x+\sqrt{x^2-1}\bigr)^n+\bigl(x-\sqrt{x^2-1}\bigr)^n\Bigr],$$

we have the identity

$$\begin{aligned}
&T_n\left(\frac{1}{2}L_{2m}(x)\right)\\
&= \frac{1}{2}\left[\left(\frac{1}{2}L_{2m}(x)+\sqrt{\frac{1}{4}L_{2m}^2(x)-1}\right)^n+\left(\frac{1}{2}L_{2m}(x)-\sqrt{\frac{1}{4}L_{2m}^2(x)-1}\right)^n\right]\\
&= \frac{1}{2}\left[\alpha^{2mn}+\beta^{2mn}\right]\\
&= \frac{1}{2}\cdot L_{2mn}(x).
\end{aligned}$$

This proves Lemma 1. □

Lemma 2 *Let n and h be non-negative integers with $h \geq 1$. Then, for any odd number $l \geq 1$, we have the congruence*

$$L_{l(2h+1)(2n+1)}(x)+L_{l(2n+1)}(x)\equiv 0 \bmod \left(L_{l(2h+1)}(x)+L_l(x)\right).$$

Proof We prove this polynomial congruence by complete induction for $n \geq 0$. It is clear that Lemma 2 is true for $n = 0$. If $n = 1$, then note that $2 \nmid l$ and $L_{l(2h+1)}^3(x) = L_{3l(2h+1)}(x) - 3L_{l(2h+1)}(x)$, we have

$$\begin{aligned}
&L_{3l(2h+1)}(x)+L_{3l}(x)\\
&= L_{l(2h+1)}^3(x)+3L_{l(2h+1)}(x)+L_l^3(x)+3L_l(x)\\
&= \left(L_{l(2h+1)}(x)+L_l(x)\right)\left(L_{l(2h+1)}^2(x)+L_{l(2h+1)}(x)L_l(x)+L_l^2(x)+3\right)\\
&\equiv 0 \bmod \left(L_{l(2h+1)}(x)+L_l(x)\right).
\end{aligned}$$

That is to say, Lemma 2 is true for $n = 1$.

Suppose that Lemma 2 is true for all integers $n = 1, 2, \dots, k$. That is,

$$L_{l(2h+1)(2n+1)}(x)+L_{l(2n+1)}(x)\equiv 0 \bmod \left(L_{l(2h+1)}(x)+L_l(x)\right) \tag{3}$$

for all $0 \leq n \leq k$.

Then, for $n = k + 1 \geq 2$, note the identities

$$L_{2l(2h+1)}(x)L_{l(2h+1)(2n+1)}(x)=L_{l(2h+1)(2n+3)}(x)+L_{l(2h+1)(2n-1)}(x)$$

and

$$L_{2l(2h+1)}(x)=L_{l(2h+1)}^2(x)+2\equiv L_l^2(x)+2 \bmod \left(L_{l(2h+1)}(x)+L_l(x)\right),$$

applying inductive hypothesis (3), we have

$$\begin{aligned}
&L_{l(2h+1)(2n+1)}(x)+L_{l(2n+1)}(x)\\
&= L_{l(2h+1)(2k+3)}(x)+L_{l(2k+3)}(x)
\end{aligned}$$

$$
\begin{aligned}
&= L_{2l(2h+1)}(x)L_{l(2h+1)(2k+1)}(x) - L_{l(2h+1)(2k-1)}(x) \\
&\quad + L_{2l}(x)L_{l(2k+1)}(x) - L_{l(2k-1)}(x) \\
&\equiv \bigl(L_l^2(x) + 2\bigr)L_{l(2h+1)(2k+1)}(x) - L_{l(2h+1)(2k-1)}(x) \\
&\quad + \bigl(L_l^2(x) + 2\bigr)L_{l(2k+1)}(x) - L_{l(2k-1)}(x) \\
&\equiv \bigl(L_l^2(x) + 2\bigr)\bigl(L_{l(2h+1)(2k+1)}(x) + L_{l(2k+1)}(x)\bigr) \\
&\quad - \bigl(L_{l(2h+1)(2k-1)}(x) + L_{l(2k-1)}(x)\bigr) \\
&\equiv 0\bigl(L_{l(2h+1)}(x) + L_l(x)\bigr).
\end{aligned}
$$

Now Lemma 2 follows from complete induction. □

Lemma 3 *Let n and h be non-negative integers with $h \geq 1$. Then, for any even number $l \geq 1$, we have the congruence*

$$F_{l(2h+1)(2n+1)}(x) + F_{l(2n+1)}(x) \equiv 0 \bmod \bigl(F_{l(2h+1)}(x) + F_l(x)\bigr).$$

Proof We can also prove Lemma 3 by complete induction. If $n = 0$, then it is clear that Lemma 3 is true. If $n = 1$, then note that $2 \mid l$ and $F_{3l(2h+1)}(x) = (x^2+4)F_{l(2h+1)}^3(x) + 3F_{l(2h+1)}(x)$, we have

$$
\begin{aligned}
&F_{3l(2h+1)}(x) + F_{3l}(x) \\
&\quad = \bigl(x^2 + 4\bigr)F_{l(2h+1)}^3(x) + 3F_{l(2h+1)}(x) + \bigl(x^2 + 4\bigr)F_l^3(x) + 3F_l(x) \\
&\quad = \bigl(x^2 + 4\bigr)\bigl(F_{l(2h+1)}(x) + F_l(x)\bigr)\bigl(F_{l(2h+1)}^2(x) + F_{l(2h+1)}(x)F_l(x) + F_l^2(x)\bigr) \\
&\qquad + 3\bigl(F_{l(2h+1)}(x) + F_l(x)\bigr) \\
&\quad \equiv 0 \bmod \bigl(F_{l(2h+1)}(x) + F_l(x)\bigr).
\end{aligned}
$$

So Lemma 3 is true for $n = 1$. Suppose that Lemma 3 is true for all integers $n = 1, 2, \ldots, k$. That is,

$$F_{l(2h+1)(2n+1)}(x) + F_{l(2n+1)}(x) \equiv 0 \bmod \bigl(F_{l(2h+1)}(x) + F_l(x)\bigr) \tag{4}$$

for all $0 \leq n \leq k$.

Then, for $n = k + 1$, note the identities

$$L_{2l(2h+1)}(x)F_{l(2h+1)(2n+1)}(x) = F_{l(2h+1)(2n+3)}(x) + F_{l(2h+1)(2n-1)}(x)$$

and

$$
\begin{aligned}
L_{2l(2h+1)}(x) &= \bigl(x^2 + 4\bigr)F_{l(2h+1)}^2(x) + 2 \\
&\equiv \bigl(x^2 + 4\bigr)F_l^2(x) + 2 \bmod \bigl(F_{l(2h+1)}(x) + F_l(x)\bigr),
\end{aligned}
$$

applying inductive hypothesis (4), we have

$$F_{l(2h+1)(2n+1)}(x) + F_{l(2n+1)}(x) = F_{l(2h+1)(2k+3)}(x) + F_{l(2k+3)}(x)$$

$$
\begin{aligned}
&= L_{2l(2h+1)}(x)F_{l(2h+1)(2k+1)}(x) - F_{l(2h+1)(2k-1)}(x) \\
&\quad + L_{2l}(x)F_{l(2k+1)}(x) - F_{l(2k-1)}(x) \\
&\equiv \big[(x^2+4)F_l^2(x)+2\big]F_{l(2h+1)(2k+1)}(x) - F_{l(2h+1)(2k-1)}(x) \\
&\quad + \big[(x^2+4)F_l^2(x)+2\big]F_{l(2k+1)}(x) - F_{l(2k-1)}(x) \\
&\equiv \big[(x^2+4)F_l^2(x)+2\big]\big(F_{l(2h+1)(2k+1)}(x) + F_{l(2k+1)}(x)\big) \\
&\quad - \big(F_{l(2h+1)(2k-1)}(x) + F_{l(2k-1)}(x)\big) \\
&\equiv 0\big(F_{l(2h+1)}(x) + F_l(x)\big).
\end{aligned}
$$

This completes the proof of Lemma 3. □

3 Proofs of the theorems

In this section, we shall prove our theorems by mathematical induction. Replace x by $\frac{1}{2}\cdot L_{2ml}(x)$ in (1), from Lemma 1 we have

$$
\begin{aligned}
&\sum_{m=0}^{h} T_{2n+1}\left(\frac{1}{2}L_{2ml}(x)\right) \\
&\quad = \frac{1}{2}\cdot\sum_{m=0}^{h} L_{2ml(2n+1)}(x) \\
&\quad = \frac{2n+1}{2}\cdot\sum_{k=0}^{n}(-1)^k\frac{(2n-k)!}{k!(2n+1-2k)!}\sum_{m=0}^{h} L_{2ml}^{2n+1-2k}(x)
\end{aligned}
$$

or

$$
\begin{aligned}
&\sum_{m=0}^{h}\big[L_{2ml(2n+1)}(x) - (-1)^n(2n+1)L_{2ml}(x)\big] \\
&\quad = (2n+1)\cdot\sum_{k=0}^{n-1}(-1)^k\frac{(2n-k)!}{k!(2n+1-2k)!}\sum_{m=0}^{h} L_{2ml}^{2n+1-2k}(x). \qquad (5)
\end{aligned}
$$

Note the identities

$$
\begin{aligned}
&\sum_{m=0}^{h} L_{2ml(2n+1)}(x) \\
&\quad = \sum_{m=0}^{h}\big(\alpha^{2ml(2n+1)} + \beta^{2ml(2n+1)}\big) \\
&\quad = \frac{\alpha^{2(h+1)l(2n+1)}-1}{\alpha^{2l(2n+1)}-1} + \frac{\beta^{2(h+1)l(2n+1)}-1}{\beta^{2l(2n+1)}-1} \\
&\quad = \frac{\alpha^{l(2h+1)(2n+1)} - (-1)^l\beta^{l(2n+1)}}{\alpha^{l(2n+1)} - (-1)^l\beta^{l(2n+1)}} + \frac{\beta^{l(2h+1)(2n+1)} - (-1)^l\alpha^{l(2n+1)}}{\beta^{l(2n+1)} - (-1)^l\alpha^{l(2n+1)}} \\
&\quad = \frac{L_{l(2h+1)(2n+1)}(x) + L_{l(2n+1)}(x)}{L_{l(2n+1)}(x)} \quad \text{if } 2\nmid l; \qquad (6)
\end{aligned}
$$

and

$$\sum_{m=0}^{h} L_{2ml(2n+1)}(x) = \frac{F_{l(2h+1)(2n+1)}(x) + F_{l(2n+1)}(x)}{F_{l(2n+1)}(x)} \quad \text{if } 2 \mid l. \tag{7}$$

If l is an odd number, then from (5) and (6) we have

$$\begin{aligned} &\frac{L_{l(2h+1)(2n+1)}(x) + L_{l(2n+1)}(x)}{L_{l(2n+1)}(x)} - (-1)^n(2n+1)\frac{L_{l(2h+1)}(x) + L_l(x)}{L_l(x)} \\ &\quad = (2n+1) \cdot \sum_{k=0}^{n-1}(-1)^k \frac{(2n-k)!}{k!(2n+1-2k)!} \sum_{m=0}^{h} L_{2ml}^{2n+1-2k}(x). \end{aligned} \tag{8}$$

Now we prove Theorem 1 by mathematical induction. If $n = 1$, then from (8), Lemma 2 and note that $2 \nmid l$ we have

$$\begin{aligned} &L_l(x)L_{3l}(x) \sum_{m=0}^{h} L_{2ml}^3(x) \\ &\quad = L_l(x)L_{3l}(x)\left[\frac{L_{3l(2h+1)}(x) + L_{3l}(x)}{L_{3l}(x)} + \frac{3L_{l(2h+1)}(x) + 3L_l(x)}{L_l(x)}\right] \\ &\quad \equiv 0 \bmod \bigl(L_{l(2h+1)}(x) + L_l(x)\bigr). \end{aligned} \tag{9}$$

That is, Theorem 1 is true for $n = 1$.

Suppose that Theorem 1 is true for all integers $1 \le n \le s$. Then, for $n = s + 1$, from (8) we have

$$\begin{aligned} &\frac{L_{l(2h+1)(2s+3)}(x) + L_{l(2s+3)}(x)}{L_{l(2s+3)}(x)} + (-1)^s(2s+3)\frac{L_{l(2h+1)}(x) + L_l(x)}{L_l(x)} \\ &\quad = (2s+3) \cdot \sum_{k=0}^{s}(-1)^k \frac{(2s+2-k)!}{k!(2s+3-2k)!} \sum_{m=0}^{h} L_{2ml}^{2s+3-2k}(x) \\ &\quad = \sum_{m=0}^{h} L_{2ml}^{2s+3}(x) + (2s+3) \cdot \sum_{k=1}^{s}(-1)^k \frac{(2s+2-k)!}{k!(2s+3-2k)!} \sum_{m=0}^{h} L_{2ml}^{2s+3-2k}(x). \end{aligned} \tag{10}$$

From Lemma 2 we have

$$\begin{aligned} &L_l(x)L_{3l}(x) \cdots L_{l(2s+3)}(x) \cdot \frac{L_{l(2h+1)(2s+3)}(x) + L_{l(2s+3)}(x)}{L_{l(2s+3)}(x)} \\ &\quad \equiv 0 \bmod \bigl(L_{l(2h+1)}(x) + L_l(x)\bigr). \end{aligned} \tag{11}$$

Applying inductive assumption, we have

$$\begin{aligned} &L_l(x)L_{3l}(x) \cdots L_{l(2s+1)}(x) \sum_{k=1}^{s}(-1)^k \frac{(2s+2-k)!}{k!(2s+3-2k)!} \sum_{m=0}^{h} L_{2ml}^{2s+3-2k}(x) \\ &\quad \equiv 0 \bmod \bigl(L_{l(2h+1)}(x) + L_l(x)\bigr). \end{aligned} \tag{12}$$

Combining (9)-(12) and Lemma 2, we can deduce the congruence

$$L_l(x)L_{3l}(x)\cdots L_{l(2s+3)}(x)\cdot\sum_{m=0}^{h}L_{2ml}^{2s+3}(x)\equiv 0 \bmod \bigl(L_{l(2h+1)}(x)+L_l(x)\bigr).$$

This proves Theorem 1 by mathematical induction.

Now we prove Theorem 2. If $2 \mid l$, then from (5) and (7) we have

$$\begin{aligned}&\frac{F_{l(2h+1)(2n+1)}(x)+F_{l(2n+1)}(x)}{F_{l(2n+1)}(x)}-(-1)^n(2n+1)\frac{F_{l(2h+1)}(x)+F_l(x)}{F_l(x)}\\&\quad=(2n+1)\cdot\sum_{k=0}^{n-1}(-1)^k\frac{(2n-k)!}{k!(2n+1-2k)!}\sum_{m=0}^{h}L_{2ml}^{2n+1-2k}(x).\end{aligned} \tag{13}$$

Applying (13), Lemma 3 and the method of proving Theorem 1, we may immediately deduce the congruence

$$F_l(x)F_{3l}(x)\cdots F_{l(2n+1)}(x)\cdot\sum_{m=0}^{h}L_{2ml}^{2n+1}(x)\equiv 0 \bmod \bigl(F_{l(2h+1)}(x)+F_l(x)\bigr).$$

This completes the proof of Theorem 2.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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