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On a system of fractional finite difference inclusions

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Abstract

By making a special product Banach space and using the famous result of Covitz and Nadler on fixed point of multifunctions we investigate the existence of a solution for a system of fractional finite difference inclusions via some boundary conditions. We provide an example to illustrate our main result.

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1 Introduction

There are many works about different applied models by using distinct types of fractional derivatives via or without singular kernel ([1–6]), discrete fractional boundary value problems within the Riesz space cases ([7, 8]), finite difference calculations ([9–13]), distinct types of fractional finite difference equations ([14–26]), and some equations including the nabla operator ([27–30]). In fact, working on discrete fractional boundary value problems is useful for modeling in distinct thermal or physical sciences, including steady heat flows, heat-transfer problems, description of anomalous diffusions, and so on. This leads us to working on discrete calculations, whereas there is also rich work on continuous fractional ones. It is well known that each differential equation is a particular case of a related differential inclusion. For this reason, we better investigate fractional inclusions. It seems that researchers of thermal sciences (and some other related fields) will investigate more systems of discrete fractional boundary value inclusions in the future. Recently, some results on fractional finite difference inclusions have been obtained ([18, 19]).

As is well known, the gamma function has some known properties such as $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. It is well known that the falling function is defined by $t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for all $t, \nu \in \mathbb{R}$ whenever the right-hand side is defined ([31]). If $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then we define $t^{\underline{\nu}} = 0$ ([31]). We can verify that $\nu^{\underline{\nu}} = \nu^{\underline{\nu-1}} = \Gamma(\nu+1)$ and $t^{\underline{\nu+1}} = (t-\nu)t^{\underline{\nu}}$. We use the notations $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ for all real numbers a and b whenever $b-a$ is a natural number ([31]). Let $\nu > 0$ be such that $m-1 < \nu \leq m$ for some natural number m. Then the νth fractional sum of f based at a is defined by $\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s)$ for all $t \in \mathbb{N}_{a+\nu}$ ([12]). We consider the trivial case $\Delta_a^{-0} f(t) = f(t)$ for $t \in \mathbb{N}_a$. Similarly, we define $\Delta_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{k=a}^{t+\nu} (t-\sigma(k))^{\underline{-\nu-1}} f(k)$ for all $t \in \mathbb{N}_{a+m-\nu}$ (see [28] and [32]).

To use the Covitz-Nadler theorem in our main result, we need to introduce some notion about multifunctions on metric spaces. Let (X,d) be a metric space. Denote by $P(X)$, 2^X, $P_c(X)$, and $P_{cp}(X)$ the class of all subsets, the class of all nonempty subsets, the class of all closed subsets, and the class of all compact subsets of X, respectively. A mapping $Q : X \to 2^X$ is called a multifunction on X, and $u \in X$ is called a fixed point of Q whenever $u \in Qu$. The (generalized) Pompeiu-Hausdorff metric H_d on $P_c(X)$ is defined as $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a,b)$ (see [33] and [34]). A multifunction $T : X \to 2^X$ is called a contraction if there exists $\lambda \in (0,1)$ such that $H_d(T(x),T(y)) \le \lambda d(x,y)$ for all $x,y \in X$. In 1970, Covitz and Nadler [35] proved that each contractive closed-valued multifunction on a complete metric space has a fixed point.

In 2011, Goodrich [36] investigated the general discrete fractional boundary problem

$$\begin{cases} -\Delta^\nu y(t) = f(t+\nu-1, y(t+\nu-1)), \\ \alpha y(\nu-2) - \beta\Delta y(\nu-2) = 0, \\ \gamma y(\nu+b) - \delta\Delta y(\nu+b) = 0, \end{cases}$$

where $t \in [0,b]_{\mathbb{N}_0}$, $\nu \in (1,2]$ and $\alpha\gamma + \alpha\delta + \beta\gamma \neq 0$ with $\alpha,\beta,\gamma,\delta \ge 0$. In 2015, by using idea of [36], Baleanu, Rezapour, and Salehi [18] investigated the existence of a solution for the fractional finite difference inclusion

$$\begin{cases} \Delta^\nu x(t) \in F(t, x(t), \Delta x(t), \Delta^2 x(t)), \\ \xi x(\nu-3) + \beta\Delta x(\nu-3) = 0, \\ x(\eta) = 0, \\ \gamma x(b+\nu) + \delta\Delta x(b+\nu) = 0, \end{cases}$$

where $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$, and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact-valued multifunction. Also, they investigated the fractional finite difference inclusion $\Delta_{\mu-2}^{\mu} x(t) \in F(t, x(t), \Delta x(t))$ via the boundary conditions $\Delta x(b+\mu) = A$ and $x(\mu-2) = B$, where $1 < \mu \le 2$, $A, B \in \mathbb{R}$, and $F : \mathbb{N}_{\mu-2}^{b+\mu+2} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact-valued multifunction in 2016 ([19]). By mixing ideas of the works, we investigate the existence of a solution for the k-dimensional system of fractional difference inclusions

$$\begin{cases} \Delta_{\nu_1-1}^{\nu_1} x_1(t) \in F_1(t, x_1(t), \dots, x_k(t), \Delta x_1(t), \dots, \Delta x_k(t), \Delta^2 x_1(t), \dots, \Delta^2 x_k(t), \\ \qquad \Delta_{\nu_1+1}^{\mu_{11}} x_1(t), \dots, \Delta_{\nu_1+1}^{\mu_{1k}} x_k(t), \Delta_{\nu_1+1}^{\gamma_{11}} x_1(t), \dots, \Delta_{\nu_1+1}^{\gamma_{1k}} x_k(t)), \\ \Delta_{\nu_2-1}^{\nu_2} x_2(t) \in F_2(t, x_1(t), \dots, x_k(t), \Delta x_1(t), \dots, \Delta x_k(t), \Delta^2 x_1(t), \dots, \Delta^2 x_k(t), \\ \qquad \Delta_{\nu_2+1}^{\mu_{21}} x_1(t), \dots, \Delta_{\nu_2+1}^{\mu_{2k}} x_k(t), \Delta_{\nu_2+1}^{\gamma_{21}} x_1(t), \dots, \Delta_{\nu_2+1}^{\gamma_{2k}} x_k(t)), \\ \vdots \\ \Delta_{\nu_k-1}^{\nu_k} x_k(t) \in F_k(t, x_1(t), \dots, x_k(t), \Delta x_1(t), \dots, \Delta x_k(t), \Delta^2 x_1(t), \dots, \Delta^2 x_k(t), \\ \qquad \Delta_{\nu_k+1}^{\mu_{k1}} x_1(t), \dots, \Delta_{\nu_k+1}^{\mu_{kk}} x_k(t), \Delta_{\nu_k+1}^{\gamma_{k1}} x_1(t), \dots, \Delta_{\nu_k+1}^{\gamma_{kk}} x_k(t)), \end{cases} \tag{1}$$

with boundary conditions $x_i(\nu_i - 1) = x_i(\nu_i) = x_i(b+\nu_i) = 0$, where $x_i : \mathbb{N}_{\nu_i-1}^{b+\nu_i} \to \mathbb{R}$, $b \in \mathbb{N}_0$, $0 < \mu_{ij} < 1$, $1 < \gamma_{ij} < 2$, $2 < \nu_i < 3$, and $F_i : \mathbb{N}_{\nu_i-1}^{b+\nu_i} \times \mathbb{R}^{5k} \to 2^{\mathbb{R}}$ are compact-valued multifunction for $i = 1,2,\dots,k$ and $j = 1,2,\dots,k$.

For prove of our main results, we need the following results.

Lemma 1.1 ([11]) *Let $a \in \mathbb{R}$, $\nu > 0$ with $m-1 < \nu \leq m$, and $\mu > 0$. Then, $\Delta(t-a)^{\underline{\mu}} = \mu(t-a)^{\underline{\mu-1}}$ for all t for which both sides are well defined. Also, $\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{-\nu}}(t-a)^{\underline{\mu+\nu}}$ for $t \in \mathbb{N}_{a+\mu+\nu}$ and $\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{\nu}}(t-a)^{\underline{\mu-\nu}}$ for $t \in \mathbb{N}_{a+\mu+m-\nu}$.*

Lemma 1.2 ([26]) *Let $\mu > 0$ with $m-1 < \mu \leq m$, $a \in \mathbb{R}$, and $f : \mathbb{N}_a \to \mathbb{R}$ be a map. Then, $\Delta_{a+m-\mu}^{-\mu}\Delta_a^{\mu} f(t) = f(t) + c_1(t-a-m+\mu)^{\underline{\mu-1}} + c_2(t-a-m+\mu)^{\underline{\mu-2}} + \cdots + c_m(t-a-m+\mu)^{\underline{\mu-m}}$, where $c_1, \ldots, c_m \in \mathbb{R}$ are some constants.*

2 Main results

Now, we are ready to provide our main results.

Lemma 2.1 *Let $2 < \mu \leq 3$, and $y : \mathbb{N}_2^b \to \mathbb{R}$ be a map. Then x_0 is a solution for the fractional finite difference equation*

$$\Delta_{\mu-1}^{\mu} x(t) = y(t) \tag{2}$$

via the boundary conditions $x(\mu-1) = x(\mu) = x(b+\mu) = 0$ if and only if x_0 is a solution for the fractional sum equation $x(t) = \sum_{s=2}^{b} G(t,s)y(s)$, where

$$G(t,s) = \frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} - \frac{(t-2)^{\underline{\mu-1}}(b+\mu-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)(b+\mu-2)^{\underline{\mu-1}}}$$

for $s \leq t-\mu$ and $G(t,s) = \frac{-(t-2)^{\underline{\mu-1}}(b+\mu-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)(b+\mu-2)^{\underline{\mu-1}}}$ for $t-\mu+1 \leq s$.

Proof Suppose that x_0 satisfies equation (2) with $x_0(\mu-1) = x_0(\mu) = x_0(b+\mu) = 0$. Then, $\Delta_{\mu-1}^{\mu} x_0(t) = y(t)$ for $t \in \mathbb{N}_2^b$ and $\Delta_2^{-\mu}\Delta_{\mu-1}^{\mu} x(t) = \Delta_2^{-\mu} y(t)$ for $t \in \mathbb{N}_{\mu-1}^{b+\mu}$. Now, using Lemma 1.2, we get $x_0(t) = c_1(t-2)^{\underline{\mu-1}} + c_2(t-2)^{\underline{\mu-2}} + c_3(t-2)^{\underline{\mu-3}} + \frac{1}{\Gamma(\mu)}\sum_{s=2}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}} y(s)$ for $t \in \mathbb{N}_{\mu-1}^{b+\mu}$. Using the boundary condition $x(\mu-1) = 0$, we obtain

$$0 = c_1(\mu-3)^{\underline{\mu-1}} + c_2(\mu-3)^{\underline{\mu-2}} + c_3(\mu-3)^{\underline{\mu-3}} + \frac{1}{\Gamma(\mu)}\sum_{s=2}^{-1}\bigl(\mu-1-\sigma(s)\bigr)^{\underline{\mu-1}} y(s).$$

Since $(\mu-3)^{\underline{\mu-1}} = (\mu-3)^{\underline{\mu-2}} = 0$ and $\frac{1}{\Gamma(\mu)}\sum_{s=2}^{-1}(\mu-1-\sigma(s))^{\underline{\mu-1}} y(s) = 0$, we get $c_3 = 0$. Since $x(\mu) = 0$, $0 = c_1(\mu-2)^{\underline{\mu-1}} + c_2(\mu-2)^{\underline{\mu-2}} + \frac{1}{\Gamma(\mu)}\sum_{s=2}^{0}(\mu-\sigma(s))^{\underline{\mu-1}} y(s)$. Since $(\mu-2)^{\underline{\mu-1}} = 0$ and $\frac{1}{\Gamma(\mu)}\sum_{s=2}^{0}(\mu-\sigma(s))^{\underline{\mu-1}} y(s) = 0$, we get $c_2 = 0$. Since $x(b+\mu) = 0$, we get

$$0 = c_1(b+\mu-2)^{\underline{\mu-1}} + \frac{1}{\Gamma(\mu)}\sum_{s=2}^{b}\bigl(b+\mu-\sigma(s)\bigr)^{\underline{\mu-1}} y(s),$$

and so $c_1 = -\frac{1}{\Gamma(\mu)(b+\mu-2)^{\underline{\mu-1}}}\sum_{s=2}^{b}(b+\mu-\sigma(s))^{\underline{\mu-1}} y(s)$. Hence,

$$\begin{aligned} x_0(t) &= \frac{-(t-2)^{\underline{\mu-1}}}{\Gamma(\mu)(b+\mu-2)^{\underline{\mu-1}}}\sum_{s=2}^{b}\bigl(b+\mu-\sigma(s)\bigr)^{\underline{\mu-1}} y(s) + \frac{1}{\Gamma(\mu)}\sum_{s=2}^{t-\mu}\bigl(t-\sigma(s)\bigr)^{\underline{\mu-1}} y(s) \\ &= \sum_{s=2}^{b} G(t,s)y(s). \end{aligned}$$

Now, let x_0 be a solution for the sum equation $x(t) = \sum_{s=2}^{b} G(t,s)y(s)$. Then,

$$x_0(t) = -\frac{(t-2)^{\underline{\mu-1}}}{\Gamma(\mu)(b+\mu-2)^{\underline{\mu-1}}}\sum_{s=2}^{b}\big(b+\mu-\sigma(s)\big)^{\underline{\mu-1}}y(s) + \frac{1}{\Gamma(\mu)}\sum_{s=2}^{t-\mu}\big(t-\sigma(s)\big)^{\underline{\mu-1}}y(s).$$

Since $\frac{1}{\Gamma(\mu)}\sum_{s=2}^{-1}(\mu-1-\sigma(s))^{\underline{\mu-1}}y(s) = \frac{1}{\Gamma(\mu)}\sum_{s=2}^{0}(\mu-\sigma(s))^{\underline{\mu-1}}y(s) = 0$ and $(\mu-3)^{\underline{\mu-1}} = (\mu-3)^{\underline{\mu-2}} = (\mu-2)^{\underline{\mu-1}} = 0$, a simple calculation shows that $x(\mu-1) = x(\mu) = x(b+\mu) = 0$. On the other hand, it is easy to check that

$$x_0(t) = c_1(t-2)^{\underline{\mu-1}} + c_2(t-2)^{\underline{\mu-2}} + c_3(t-2)^{\underline{\mu-3}} + \frac{1}{\Gamma(\mu)}\sum_{s=2}^{t-\mu}\big(t-\sigma(s)\big)^{\underline{\mu-1}}y(s)$$

is a solution for the equation $\Delta_{\mu-1}^{\mu}x(t) = y(t)$. □

Now, we are ready to provide our result on the existence of a solution for the k-dimensional system of fractional finite difference inclusions (1). Let $i \in \{1,\dots,k\}$ be given, and $\mathcal{X}_i$ be the set of all functions $x : \mathbb{N}_{\nu_i-1}^{b+\nu_i} \to \mathbb{R}$ endowed with the norm

$$\begin{aligned}\|x\|_i = {} & \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\big|x(t)\big| + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\big|\Delta x(t)\big| + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\big|\Delta^2 x(t)\big| \\ & + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\big|\Delta_{\nu_i+1}^{\mu_{ii}}x(t)\big| + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\big|\Delta_{\nu_i+1}^{\gamma_{ii}}x(t)\big|.\end{aligned}$$

Consider the space $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_k$ with the norm $\|(x_1,x_2,\dots,x_k)\|_{\mathcal{X}} = \sum_{i=1}^{k}\|x_i\|_i$. We show that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space. Let $\{x_n\}$ be a Cauchy sequence in $\mathcal{X}$, and let $\epsilon > 0$ be arbitrary. Choose a natural number N such that $\|x_n - x_m\| < \epsilon$ for all $m,n > N$. Then we get $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|x_n(t)-x_m(t)| < \epsilon$, $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta x_n(t)-\Delta x_m(t)| < \epsilon$, $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta^2 x_n(t)-\Delta^2 x_m(t)| < \epsilon$, $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta^\mu x_n(t)-\Delta^\mu x_m(t)| < \epsilon$, and $\max_{t\in\mathbb{N}_\nu^{b+\nu+2}}|\Delta^\gamma x_n(t)-\Delta^\gamma x_m(t)| < \epsilon$. Since $\mathbb{R}$ is complete, there are real numbers $x(t)$, $z(t)$, $w(t)$, $p(t)$, and $q(t)$ such that $x_n(t) \to x(t)$, $\Delta x_n(t) \to z(t)$, $\Delta^2 x_n(t) \to w(t)$, $\Delta^\mu x_n(t) \to p(t)$, and $\Delta^\gamma x_n(t) \to q(t)$ for all $t \in \mathbb{N}_\nu^{b+\nu+2}$. Note that $\Delta x_n(t) = x_n(t+1) - x_n(t)$, and so $\Delta x(t) = x(t+1) - x(t) = z(t)$. Similarly, we get $\Delta^2 x(t) = w(t)$. Also, we have $\Delta^\mu x_n(t) = \frac{1}{\Gamma(-\mu)}\sum_{s=0}^{t+\mu}(t-\sigma(s))^{\underline{-\mu-1}}x_n(s)$. Since $x_n(s) \to x(s)$, we get $\Delta^\mu x(t) = p(t)$. Similarly, we have $\Delta^\gamma x(t) = q(t)$. This implies that there exists a natural number M such that $|x_n(t)-x(t)| < \frac{\epsilon}{5}$, $|\Delta x_n(t) - \Delta x(t)| < \frac{\epsilon}{5}$, $|\Delta^2 x_n(t) - \Delta^2 x(t)| < \frac{\epsilon}{5}$, $|\Delta^\mu x_n(t) - \Delta^\mu x(t)| < \frac{\epsilon}{5}$, and $|\Delta^\gamma x_n(t) - \Delta^\gamma x(t)| < \frac{\epsilon}{5}$ for all $t \in \mathbb{N}_\nu^{b+\nu+2}$ and $n > M$. Thus,

$$\begin{aligned}\|x_n - x\| = {} & \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}\big|x_n(t)-x(t)\big| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}\big|\Delta x_n(t)-\Delta x(t)\big| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}\big|\Delta^2 x_n(t)-\Delta^2 x(t)\big| \\ & + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}\big|\Delta^\mu x_n(t)-\Delta^\mu x(t)\big| + \max_{t\in\mathbb{N}_\nu^{b+\nu+2}}\big|\Delta^\gamma x_n(t)-\Delta^\gamma x(t)\big| < \epsilon\end{aligned}$$

for all $n > M$. This shows that $\mathcal{X}$ is a Banach space. Define the set of selections of F_i at $(x_1,\dots,x_k) \in \mathcal{X}$ by

$$\begin{aligned}S_{F_i,(x_1,x_2,\dots,x_k)} = {} & \big\{y : \mathbb{N}_2^b \to \mathbb{R} : y(t) \in F_i\big(t, x_1(t),\dots,x_k(t), \Delta x_1(t),\dots,\Delta x_k(t), \\ & \Delta^2 x_1(t),\dots,\Delta^2 x_k(t), \Delta_{\nu_i+1}^{\mu_{i1}}x_1(t),\dots,\Delta_{\nu_i+1}^{\mu_{ik}}x_k(t), \\ & \Delta_{\nu_i+1}^{\gamma_{i1}}x_1(t),\dots,\Delta_{\nu_i+1}^{\gamma_{ik}}x_k(t)\big) \text{ for all } t \in \mathbb{N}_2^b\big\}\end{aligned}$$

for $x = (x_1,\dots,x_k) \in \mathcal{X}$ and $i = 1,\dots,k$. We say that a function $(x_1,x_2,\dots,x_k) \in \mathcal{X}$ is a solution for the k-dimensional system of inclusions if there exist real-valued functions $y_1,\dots,y_k$ on $\mathbb{N}_2^b$ such that

$$\begin{aligned} y_i(t) \in F_i\bigl(&t,x_1(t),\dots,x_k(t),\Delta x_1(t),\dots,\Delta x_k(t),\Delta^2 x_1(t),\dots,\Delta^2 x_k(t),\\ &\Delta^{\mu_{i1}}_{\nu_i+1}x_1(t),\dots,\Delta^{\mu_{ik}}_{\nu_i+1}x_k(t),\Delta^{\gamma_{i1}}_{\nu_i+1}x_1(t),\dots,\Delta^{\gamma_{ik}}_{\nu_i+1}x_k(t)\bigr) \end{aligned}$$

for all $t \in \mathbb{N}_2^b$, $x_i(t) = \frac{-(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}(b+\nu_i-\sigma(s))^{\underline{\nu_i-1}}y_i(s) + \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}(t-\sigma(s))^{\underline{\nu_i-1}}y_i(s)$ and $x_i(\nu_i-1) = x_i(\nu_i) = x_i(b+\nu_i) = 0$ for $i = 1,\dots,k$. Since $F_i(t,x_1(t),\dots,x_k(t)) \neq \varnothing$ for $i = 1,\dots,k$, the selection principle implies that $S_{F_i,(x_1,x_2,\dots,x_k)}$ is nonempty.

Theorem 2.2 *Suppose that $\psi_1,\dots,\psi_k : \mathbb{N}_{\nu_i-1}^{b+\nu_i} \to \mathbb{R}$ be some maps with*

$$0 < L = \sum_{i=1}^{k}\Bigl(\max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\bigl|\psi_i(t)\bigr|\Bigr)\bigl(\Lambda_1^i + \cdots + \Lambda_5^i\bigr) < 1,$$

where

$$\begin{aligned} \Lambda_j^i = \max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}}\Biggl|&\frac{1}{(b+\nu_i-2)^{\underline{\nu_i-1}}\Gamma(\nu_i-j+1)}\sum_{s=2}^{b}\bigl((b+\nu_i-2)^{\underline{\nu_i-1}}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-j}}\\ &-(t-2)^{\underline{\nu_i-j}}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}\bigr)\Biggr| \end{aligned}$$

for $j = 1,2,3$,

$$\begin{aligned} \Lambda_4^i = \max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}}\Biggl|&\frac{1}{\Gamma(\nu_i-\mu_{ii})(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl((b+\nu_i-2)^{\underline{\nu_i-1}}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-\mu_{ii}}}\\ &-(t-2)^{\underline{\nu_i-\mu_{ii}}}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}\bigr)\Biggr|, \end{aligned}$$

and

$$\begin{aligned} \Lambda_5^i = \max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}}\Biggl|&\frac{1}{\Gamma(\nu_i-\gamma_{ii})(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl((b+\nu_i-2)^{\underline{\nu_i-1}}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-\gamma_{ii}}}\\ &-(t-2)^{\underline{\nu_i-\gamma_{ii}}}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}\bigr)\Biggr|. \end{aligned}$$

Assume that $F_i : \mathbb{N}_{\nu_i-1}^{b+\nu_i} \times \mathbb{R}^{5k} \to P_{cp}(\mathbb{R})$ is a multifunction such that

$$H_d\bigl(F_i(t,x_1,\dots,x_{5k}),F_i(t,z_1,\dots,z_{5k})\bigr) \le \psi_i(t)\left(\sum_{i=1}^{5k}|x_i - z_i|\right)$$

for $t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}$, $i = 1,\dots,k$, and $x_1,\dots,x_{5k},z_1,\dots,z_{5k} \in \mathbb{R}$. Then the system of fractional difference inclusions (1) *has a solution.*

Proof Choose $y_i \in S_{F_i,(x_1,x_2,\ldots,x_k)}$ for $i = 1,\ldots,k$. Define

$$h_i(t) = -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s) + \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s)$$

for all $t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}$. Then $h_i \in \mathcal{X}_i$, and so the set

$$\Biggl\{h_i \in \mathcal{X}_i : \text{there exists } y \in S_{F_i,(x_1,x_2,\ldots,x_k)} \text{ such that}$$
$$h_i(t) = -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s)$$
$$+ \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s) \text{ for all } t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}\Biggr\}$$

is nonempty. Define the operator $T : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$T(x_1,x_2,\ldots,x_k)(t_1,t_2,\ldots,t_k) = \begin{pmatrix} T_1(x_1,x_2,\ldots,x_k)(t_1) \\ T_2(x_1,x_2,\ldots,x_k)(t_2) \\ \vdots \\ T_k(x_1,x_2,\ldots,x_k)(t_k) \end{pmatrix},$$

where

$$T_i(x_1,x_2,\ldots,x_k) = \Biggl\{f \in \mathcal{X}_i : \text{there exists } y_i \in S_{F_{i,(x_1,\ldots,x_k)}} \text{ such that}$$
$$f(t) = -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s)$$
$$+ \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s), \text{for all } t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}\Biggr\}.$$

We show that the multifunction T has a fixed point. First, we prove that $T(x_1,x_2,\ldots,x_k)$ is a closed subset of $\mathcal{X}$ for all $(x_1,x_2,\ldots,x_k) \in \mathcal{X}$. Let $(x_1,x_2,\ldots,x_k) \in \mathcal{X}$, and let $\{(x_1^n,\ldots,x_k^n)\}_{n\geq 1}$ be a sequence in $T(x_1,x_2,\ldots,x_k)$ with $(x_1^n,\ldots,x_k^n) \to (x_1^0,\ldots,x_k^0)$. For each n, choose $(y_1^n,\ldots,y_k^n) \in S_{F_1,(x_1,\ldots,x_k)} \times S_{F_2,(x_1,\ldots,x_k)} \times \cdots \times S_{F_k,(x_1,\ldots,x_k)}$ such that

$$x_i^n = -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i^n(s) + \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i^n(s)$$

for $t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}$, $n \geq 1$ and $i = 1,\ldots,k$. Since the multifunctions $F_1,\ldots,F_k$ are compact-valued, $\{y_i^n\}_{n\geq 1}$ has a subsequence that converges to some $y_i^0 : \mathbb{N}_2^b \to \mathbb{R}$. We denote this subsequence again by $\{y_i^n\}_{n\geq 1}$. It is easy to check that $y_i^0 \in S_{F_i,(x_1,\ldots,x_k)}$ and

$$x_i^n(t) \to x_i^0(t) = -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i^0(s)$$
$$+ \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i^0(s)$$

for all $t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}$ and $i = 1,\dots,k$. This implies that $x_i^0 \in T_i(x_1,\dots,x_s)$ for all $i = 1,\dots,k$. Hence, $(x_1^0,\dots,x_k^0) \in T(x_1,\dots,x_k)$, and so the multifunction T has closed values. Since T is a compact-valued multifunction, it is easy to see that $T(x_1,\dots,x_k)$ is a bounded set in $\mathcal{X}$ for all $(x_1,\dots,x_k) \in \mathcal{X}$. Let $(u_1,\dots,u_k),(v_1,\dots,v_k) \in \mathcal{X}$, $(h_1,\dots,h_k) \in T(u_1,\dots,u_k)$ and $(h'_1,\dots,h'_k) \in T(v_1,\dots,v_k)$. Choose $(y_1,\dots,y_k) \in S_{F_1,(u_1,\dots,u_k)} \times S_{F_2,(u_1,\dots,u_k)} \times \cdots \times S_{F_k,(u_1,\dots,u_k)}$ and $(y'_1,\dots,y'_k) \in S_{F_1,(v_1,\dots,v_k)} \times S_{F_2,(v_1,\dots,v_k)} \times \cdots \times S_{F_k,(v_1,\dots,v_k)}$ such that

$$h_i(t) = -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s) + \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}y_i(s)$$

and

$$h'_i(t) = -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}y'_i(s) + \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}y'_i(s)$$

for all $t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}$ and $i = 1,\dots,k$. Since

$$\begin{aligned}
&H_d\bigl(F_i\bigl(t,u_1(t),\dots,u_k(t),\Delta u_1(t),\dots,\Delta u_k(t),\Delta^2u_1(t),\dots,\Delta^2u_k(t),\\
&\quad \Delta_{\nu_i+1}^{\mu_{i1}}u_1(t),\dots,\Delta_{\nu_i+1}^{\mu_{ik}}u_k(t),\Delta_{\nu_i+1}^{\gamma_{i1}}u_1(t),\dots,\Delta_{\nu_i+1}^{\gamma_{ik}}u_k(t)\bigr),\\
&\quad F_i\bigl(t,v_1(t),\dots,v_k(t),\Delta v_1(t),\dots,\Delta v_k(t),\Delta^2v_1(t),\dots,\Delta^2v_k(t),\\
&\quad \Delta_{\nu_i+1}^{\mu_{i1}}v_1(t),\dots,\Delta_{\nu_i+1}^{\mu_{ik}}v_k(t),\Delta_{\nu_i+1}^{\gamma_{i1}}v_1(t),\dots,\Delta_{\nu_i+1}^{\gamma_{ik}}v_k(t)\bigr)\bigr)\\
&\leq \psi_i(t)\Biggl(\sum_{j=1}^{k}\bigl(\bigl|u_j(t)-v_j(t)\bigr| + \bigl|\Delta u_j(t)-\Delta v_j(t)\bigr| + \bigl|\Delta^2u_j(t)-\Delta^2v_j(t)\bigr|\\
&\quad + \bigl|\Delta_{\nu_i+1}^{\mu_{ij}}u_j(t)-\Delta_{\nu_i+1}^{\mu_{ij}}v_j(t)\bigr| + \bigl|\Delta_{\nu_i+1}^{\gamma_{ij}}u_j(t)-\Delta_{\nu_i+1}^{\gamma_{ij}}v_j(t)\bigr|\bigr)\Biggr)
\end{aligned}$$

for all $(u_1,\dots,u_k),(v_1,\dots,v_k) \in \mathcal{X}$ and $t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}$, we get

$$\begin{aligned}
\bigl|y_i(t)-y'_i(t)\bigr| &\leq \psi_i(t)\Biggl(\sum_{j=1}^{k}\bigl(\bigl|u_j(t)-v_j(t)\bigr| + \bigl|\Delta u_j(t)-\Delta v_j(t)\bigr| + \bigl|\Delta^2u_j(t)-\Delta^2v_j(t)\bigr|\\
&\quad + \bigl|\Delta_{\nu_i+1}^{\mu_{ij}}u_j(t)-\Delta_{\nu_i+1}^{\mu_{ij}}v_j(t)\bigr| + \bigl|\Delta_{\nu_i+1}^{\gamma_{ij}}u_j(t)-\Delta_{\nu_i+1}^{\gamma_{ij}}v_j(t)\bigr|\bigr)\Biggr)
\end{aligned}$$

for all $t \in \mathbb{N}_2^b$. Since

$$\begin{aligned}
\bigl|h_i(t)-h'_i(t)\bigr| &\leq -\frac{(t-2)^{\underline{\nu_i-1}}}{\Gamma(\nu_i)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\bigl(b+\nu_i-\sigma(s)\bigr)^{\underline{\nu_i-1}}\bigl|y_i(t)-y'_i(t)\bigr|\\
&\quad + \frac{1}{\Gamma(\nu_i)}\sum_{s=2}^{t-\nu_i}\bigl(t-\sigma(s)\bigr)^{\underline{\nu_i-1}}\bigl|y_i(t)-y'_i(t)\bigr|
\end{aligned}$$

and $\sum_{s=t-\nu_i+1}^{b} \frac{(t-\sigma(s))^{\underline{\nu_i-1}}}{\Gamma(\nu)} = 0$, we obtain

$$\begin{aligned} \left|h_i(t) - h_i'(t)\right| \le \sum_{s=2}^{b} & \frac{(b+\nu_i-2)^{\underline{\nu_i-1}}(t-\sigma(s))^{\underline{\nu_i-1}} - (t-2)^{\underline{\nu_i-1}}(b+\nu_i-\sigma(s))^{\underline{\nu_i-1}}}{(b+\nu_i-2)^{\underline{\nu_i-1}}\Gamma(\nu_i)} \\ & \times \left|y_i(s) - y_i'(s)\right|. \end{aligned}$$

Hence,

$$\begin{aligned} \left|h_i(t) - h_i'(t)\right| \le{}& \max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}} \left| \frac{1}{(b+\nu_i-2)^{\underline{\nu_i-1}}\Gamma(\nu_i)} \sum_{s=2}^{b} \left((b+\nu_i-2)^{\underline{\nu_i-1}}\left(t-\sigma(s)\right)^{\underline{\nu_i-1}} \right.\right. \\ & \left.\left. - (t-2)^{\underline{\nu_i-1}}\left(b+\nu_i-\sigma(s)\right)^{\underline{\nu_i-1}}\right)\right| \\ & \times \max_{t\in\mathbb{N}_2^b}\left|y_i(t) - y_i'(t)\right| \\ \le{}& \Lambda_1^i \max_{t\in\mathbb{N}_2^b}\left|y_i(t) - y_i'(t)\right| \\ \le{}& \Lambda_1^i \psi_i(t)\left(\sum_{j=1}^{k}\left(\left|u_j(t) - v_j(t)\right| + \left|\Delta u_j(t) - \Delta v_j(t)\right| + \left|\Delta^2 u_j(t) - \Delta^2 v_j(t)\right| \right.\right. \\ & \left.\left. + \left|\Delta_{\nu_i+1}^{\mu_{ij}} u_j(t) - \Delta_{\nu_i+1}^{\mu_{ij}} v_j(t)\right| + \left|\Delta_{\nu_i+1}^{\gamma_{ij}} u_j(t) - \Delta_{\nu_i+1}^{\gamma_{ij}} v_j(t)\right|\right)\right) \\ \le{}& \Lambda_1^i\left(\max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|\psi_i(t)\right|\right)\left(\sum_{j=1}^{k}\left(\max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|u_j(t) - v_j(t)\right| + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|\Delta u_j(t) - \Delta v_j(t)\right| \right.\right. \\ & + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|\Delta^2 u_j(t) - \Delta^2 v_j(t)\right| + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|\Delta_{\nu_i+1}^{\mu_{ij}} u_j(t) - \Delta_{\nu_i+1}^{\mu_{ij}} v_j(t)\right| \\ & \left.\left. + \max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|\Delta_{\nu_i+1}^{\gamma_{ij}} u_j(t) - \Delta_{\nu_i+1}^{\gamma_{ij}} v_j(t)\right|\right)\right) \\ \le{}& \Lambda_1^i\left(\max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|\psi_i(t)\right|\right)\left(\sum_{i=1}^{k}\|u_i - v_i\|_i\right) \\ \le{}& \Lambda_1^i\left(\max_{t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}}\left|\psi_i(t)\right|\right)\left\|(u_1 - v_1, \ldots, u_k - v_k)\right\|_{\mathcal{X}} \end{aligned}$$

for all $t \in \mathbb{N}_{\nu_i-1}^{b+\nu_i}$. Since

$$\begin{aligned} \Delta h_i(t) ={}& -\frac{(t-2)^{\underline{\nu_i-2}}}{\Gamma(\nu_i-1)(b+\nu_i-2)^{\underline{\nu_i-1}}} \sum_{s=2}^{b}\left(b+\nu_i-\sigma(s)\right)^{\underline{\nu_i-1}} y_i(s) \\ & + \frac{1}{\Gamma(\nu_i-1)} \sum_{s=2}^{t-\nu_i+1}\left(t-\sigma(s)\right)^{\underline{\nu_i-2}} y_i(s), \end{aligned}$$

we get

$$\begin{aligned}\left|\Delta h_i(t)-\Delta h_i'(t)\right| &\le \max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}}\left|\frac{1}{(b+\nu_i-2)^{\underline{\nu_i-1}}\Gamma(\nu_i-1)}\sum_{s=2}^{b}\Big((b+\nu_i-2)^{\underline{\nu_i-1}}\big(t-\sigma(s)\big)^{\underline{\nu_i-2}}\right.\\ &\quad\left.-(t-2)^{\underline{\nu_i-2}}\big(b+\nu_i-\sigma(s)\big)^{\underline{\nu_i-1}}\Big)\right|\\ &\quad\times\max_{t\in\mathbb{N}_0^b}\left|y_i(t)-y_i'(t)\right|\\ &\le \Lambda_2^i\max_{t\in\mathbb{N}_0^b}\left|y_i(t)-y_i'(t)\right|\\ &\;\;\vdots\\ &\le \Lambda_2^i\Big(\max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}}\left|\psi_i(t)\right|\Big)\left\|(u_1-v_1,\dots,u_k-v_k)\right\|_{\mathcal{X}}\end{aligned}$$

for all $t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}$. Since

$$\begin{aligned}\Delta^2 h_i(t) &= -\frac{(t-2)^{\underline{\nu_i-3}}}{\Gamma(\nu_i-2)(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\big(b+\nu_i-\sigma(s)\big)^{\underline{\nu_i-1}}y_i(s)\\ &\quad+\frac{1}{\Gamma(\nu_i-2)}\sum_{s=2}^{t-\nu_i+2}\big(t-\sigma(s)\big)^{\underline{\nu_i-3}}y_i(s),\end{aligned}$$

we get

$$\begin{aligned}\left|\Delta^2 h_i(t)-\Delta^2 h_i'(t)\right| &\le \max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}}\left|\frac{1}{(b+\nu_i-2)^{\underline{\nu_i-1}}\Gamma(\nu_i-2)}\sum_{s=2}^{b}\Big((b+\nu_i-2)^{\underline{\nu_i-1}}\big(t-\sigma(s)\big)^{\underline{\nu_i-3}}\right.\\ &\quad\left.-(t-2)^{\underline{\nu_i-3}}\big(b+\nu_i-\sigma(s)\big)^{\underline{\nu_i-1}}\Big)\right|\\ &\quad\times\max_{t\in\mathbb{N}_0^b}\left|y_i(t)-y_i'(t)\right|\\ &\le \Lambda_3^i\max_{t\in\mathbb{N}_0^b}\left|y_i(t)-y_i'(t)\right|\\ &\;\;\vdots\\ &\le \Lambda_3^i\Big(\max_{t\in\mathbb{N}_{\nu_i-1}^{\nu_i+b}}\left|\psi_i(t)\right|\Big)\left\|(u_1-v_1,\dots,u_k-v_k)\right\|_{\mathcal{X}}\end{aligned}$$

for all $t\in\mathbb{N}_{\nu_i-1}^{b+\nu_i}$. Using Lemma 1.1, we obtain

$$\begin{aligned}\Delta_{\nu_i+1}^{\mu_{ii}}h_1(t) &= \frac{-(t-2)^{\underline{\nu_i-\mu_{ii}-1}}}{\Gamma(\nu_i-\mu_{ii})(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=0}^{b}\big(\nu_i+b-\sigma(s)\big)^{\underline{\nu_i-1}}y_i(s)\\ &\quad+\sum_{s=2}^{t-\nu_i+\mu_{ii}}\frac{(t-\sigma(s))^{\underline{\nu_i-1-\mu_{ii}}}}{\Gamma(\nu_i-\mu_{ii})}y_i(s),\end{aligned}$$

and so

$$\begin{aligned}
&\left|\Delta^{\mu_{ii}}_{\nu_i+1}h_i(t)-\Delta^{\mu_{ii}}_{\nu_i+1}h'_i(t)\right| \\
&\quad\leq \max_{t\in\mathbb{N}^{\nu_i+b}_{\nu_i-1}}\left|\frac{1}{\Gamma(\nu_i-\mu_{ii})(b+\nu_i-2)^{\underline{\nu_i-1}}}\sum_{s=2}^{b}\Big((b+\nu_i-2)^{\underline{\nu_i-1}}\big(t-\sigma(s)\big)^{\underline{\nu_i-\mu_{ii}}}\right. \\
&\qquad\left.-(t-2)^{\underline{\nu_i-\mu_{ii}}}\big(b+\nu_i-\sigma(s)\big)^{\underline{\nu_i-1}}\Big)\right| \\
&\qquad\times\max_{t\in\mathbb{N}^b_0}\left|y_i(t)-y'_i(t)\right| \\
&\quad\leq\Lambda^i_4\max_{t\in\mathbb{N}^b_0}\left|y_i(t)-y'_i(t)\right| \\
&\quad\vdots \\
&\quad\leq\Lambda^i_4\Big(\max_{t\in\mathbb{N}^{\nu_i+b}_{\nu_i-1}}\left|\psi_i(t)\right|\Big)\left\|(u_1-v_1,\dots,u_k-v_k)\right\|_{\mathcal{X}}
\end{aligned}$$

for $t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}$, and similarly

$$\left|\Delta^{\gamma_{ii}}_{\nu_i+1}h_i(t)-\Delta^{\gamma_{ii}}_{\nu_i+1}h'_i(t)\right|\leq\Lambda^i_5\Big(\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\psi_i(t)\right|\Big)\left\|(u_1-v_1,\dots,u_k-v_k)\right\|_{\mathcal{X}}$$

for $t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}$. Thus,

$$\begin{aligned}
\left\|h_i-h'_i\right\|_i&=\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|h_i(t)-h'_i(t)\right|+\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\Delta h_i(t)-\Delta h'_i(t)\right|+\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\Delta^2h_i(t)-\Delta^2h'_i(t)\right| \\
&\quad+\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\Delta^{\mu_{ii}}_{\nu_i+1}h_i(t)-\Delta^{\mu_{ii}}_{\nu_i+1}h'_i(t)\right|+\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\Delta^{\gamma_{ii}}_{\nu_i+1}h_i(t)-\Delta^{\gamma_{ii}}_{\nu_i+1}h'_i(t)\right| \\
&\leq\big(\Lambda^i_1+\cdots+\Lambda^i_5\big)\Big(\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\psi_i(t)\right|\Big)\left\|(u_1-v_1,\dots,u_k-v_k)\right\|_{\mathcal{X}}
\end{aligned}$$

for $i=1,\dots,k$ and all $(u_1,\dots,u_k),(v_1,\dots,v_k)\in\mathcal{X}$, $h_i\in T(u_1,\dots,u_k)$, and $h'_i\in T(v_1,\dots,v_k)$. This implies that

$$\begin{aligned}
&\left\|(h_1,\dots,h_k)-\big(h'_1,\dots,h'_k\big)\right\|_{\mathcal{X}} \\
&\quad=\sum_{i=1}^{k}\left\|h_i-h'_i\right\|_i \\
&\quad\leq\sum_{i=1}^{k}\Big(\big(\Lambda^i_1+\cdots+\Lambda^i_5\big)\Big(\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\psi_i(t)\right|\Big)\left\|(u_1-v_1,\dots,u_k-v_k)\right\|_{\mathcal{X}}\Big) \\
&\quad\leq\left(\sum_{i=1}^{k}\Big(\max_{t\in\mathbb{N}^{b+\nu_i}_{\nu_i-1}}\left|\psi_i(t)\right|\Big)\big(\Lambda^i_1+\cdots+\Lambda^i_5\big)\right)\left\|(u_1-v_1,\dots,u_k-v_k)\right\| \\
&\quad=L\left\|(u_1-v_1,\dots,u_k-v_k)\right\|.
\end{aligned}$$

By the result of Covitz and Nadler there exists $x^* \in \mathcal{X}$ such that $x^* \in T(x^*)$. We can check that x^* is a solution for the system of fractional difference inclusions (1). □

The following example illustrates our main result.

Example 1 Consider the two-dimensional system of fractional difference inclusions

$$\begin{cases} \Delta_{1.7}^{2.7} x_1(t) \in [0, 2\pi + e^{4t} + \frac{\sin x_1(t)}{e^{t+4}} + \frac{\sin x_2(t)}{3e^{t+3}} + \frac{|\Delta x_1(t)|+|\Delta x_2(t)|}{e^{10}} \\ \qquad + \frac{\cos\sin(\Delta^2 x_1(t))}{3e^{t+3}} + \frac{\cos\sin(\Delta^2 x_2(t))}{e^{t+4}} + \frac{|\Delta_{3.7}^{0.6} x_1(t)|}{\cosh(t+10)} + \frac{|\Delta_{3.7}^{0.5} x_2(t)|}{te^{t+2}} + \frac{|\Delta_{3.7}^{1.6} x_1(t)|}{\pi e^{t+3}} \\ \qquad + \frac{|\Delta_{3.7}^{1.2} x_2(t)|}{e^{t+4}}], \\ \Delta_{1.3}^{2.3} x_2(t) \in [0, 4 + e^{t^2} + \frac{|x_1(t)|}{e^{t^2+4}} + \frac{|x_2(t)|}{e^{t^2+3}} + \frac{|\Delta x_1(t)|}{e^{\pi t^2}} + \frac{|\Delta x_2(t)|}{\cosh(t^5+8)} \\ \qquad + \frac{\sin(\Delta^2 x_1(t))}{3e^{t^2+3}} + \frac{\sin(\Delta^2 x_1(t))}{e^{t^4+4}} + \frac{|\Delta_{3.3}^{0.8} x_1(t)|+|\Delta_{3.3}^{0.4} x_2(t)|}{e^{t^4+6}} + \frac{|\Delta_{3.3}^{1.9} x_1(t)|}{4e^{t^2+1}} + \frac{|\Delta_{3.3}^{1.3} x_2(t)|}{t^2 e^{t^4}}], \end{cases} \tag{3}$$

with the boundary conditions $x_1(1.7) = x_1(2.7) = x_1(7.7) = 0$ and $x_1(1.3) = x_1(2.3) = x_1(7.3) = 0$. Put $b = 5$, $\nu_1 = 2.7$, $\mu_{11} = 0.6$, $\mu_{12} = 0.5$, $\gamma_{11} = 1.6$, $\gamma_{12} = 1.2$, $\nu_2 = 2.3$, $\mu_{21} = 0.8$, $\mu_{22} = 0.4$, $\gamma_{21} = 1.9$, $\gamma_{22} = 1.3$,

$$F_1(t, y_1, \ldots, y_{10}) = \Bigg[0, 2\pi + e^{4t} + \frac{\sin y_1}{e^{t+4}} + \frac{\sin y_2}{3e^{t+3}} + \frac{|y_3| + |y_4|}{e^{10}} \\ + \frac{\cos\sin y_5}{3e^{t+3}} + \frac{\cos\sin y_6}{e^{t+4}} + \frac{|y_7|}{\cosh(t+10)} + \frac{|y_8|}{te^{t+2}} + \frac{|y_9|}{\pi e^{t+3}} + \frac{|y_{10}|}{e^{t+4}}\Bigg],$$

and

$$F_2(t, y_1, \ldots, y_{10}) = \Bigg[0, 4 + e^{t^2} + \frac{|y_1|}{e^{t^2+4}} + \frac{|y_2|}{e^{t^2+3}} + \frac{|y_3|}{e^{\pi t^2}} + \frac{|y_4|}{\cosh(t^5+8)} \\ + \frac{\sin y_5}{3e^{t^2+3}} + \frac{\sin y_6}{e^{t^4+4}} + \frac{|y_7| + |y_8|}{e^{t^4+6}} + \frac{|y_9|}{4e^{t^2+1}} + \frac{|y_{10}|}{t^2 e^{t^4}}\Bigg].$$

Note that $2\pi + e^{4t} + \frac{\sin y_1}{e^{t+4}} + \frac{\sin y_2}{3e^{t+3}} + \frac{|y_3|+|y_4|}{e^{10}} + \frac{\cos\sin y_5}{3e^{t+3}} + \frac{\cos\sin y_6}{e^{t+4}} + \frac{|y_7|}{\cosh(t+10)} + \frac{|y_8|}{te^{t+2}} + \frac{|y_9|}{\pi e^{t+3}} + \frac{|y_{10}|}{e^{t+4}} > 0$ for $t \in \mathbb{N}_{1.7}^{7.7}$ and $y_1, \ldots, y_{10} \in \mathbb{R}$, and so $F_1 : \mathbb{N}_{1.7}^{7.7} \times \mathbb{R}^{10} \to 2^{\mathbb{R}}$ is a nonempty-valued multifunction. If $\psi_1(t) = \frac{1}{e^{(t+2)}}$, then $\max_{t\in\mathbb{N}_{1.7}^{7.7}} |\psi_1(t)| = \max_{t\in\mathbb{N}_{1.7}^{7.7}} \frac{1}{e^{t+2}} = \frac{1}{e^{3.7}} \cong \frac{1}{40.4473}$. Similarly, we have $4 + e^{t^2} + \frac{|y_1|}{e^{t^2+4}} + \frac{|y_2|}{e^{t^2+3}} + \frac{|y_3|}{e^{\pi t^2}} + \frac{|y_4|}{\cosh(t^5+8)} + \frac{\sin y_5}{3e^{t^2+3}} + \frac{\sin y_6}{e^{t^4+4}} + \frac{|y_7|+|y_8|}{e^{t^4+6}} + \frac{|y_9|}{4e^{t^2+1}} + \frac{|y_{10}|}{t^2 e^{t^4}} > 0$ for $t \in \mathbb{N}_{1.3}^{7.3}$ and $y_1, \ldots, y_{10} \in \mathbb{R}$, and so $F_2 : \mathbb{N}_{1.3}^{7.3} \times \mathbb{R}^{10} \to 2^{\mathbb{R}}$ is a nonempty-valued multifunction. If $\psi_2(t) = \frac{1}{e^{(t^2+2)}}$, then $\max_{t\in\mathbb{N}_{1.3}^{7.3}} |\psi_2(t)| = \max_{t\in\mathbb{N}_{1.3}^{7.3}} \frac{1}{e^{(t^2+2)}} = \frac{1}{e^{((1.3)^2+2)}} \cong \frac{1}{40.0448}$. Note that

$$\Lambda_1^1 = \max_{t\in\mathbb{N}_{1.7}^{7.7}} \left\{ \left| \frac{1}{(5.7)^{\underline{1.7}}\Gamma(2.7)} \sum_{s=2}^{5} \left((5.7)^{\underline{1.7}}\big(t - \sigma(s)\big)^{\underline{1.7}} - (t-2)^{\underline{1.7}}\big(7.7 - \sigma(s)\big)^{\underline{1.7}}\right) \right| \right\} \\ \cong \max\{0.0, 0.0, 1.48, 3, 3.7, 2.89, 0\} = 3.7,$$

$$\Lambda_2^1 = \max_{t\in\mathbb{N}_{1.7}^{7.7}} \left\{ \left| \frac{1}{(5.7)^{\underline{1.7}}\Gamma(1.7)} \sum_{s=2}^{5} \left((5.7)^{\underline{1.7}}\big(t - \sigma(s)\big)^{\underline{0.7}} - (t-2)^{\underline{0.7}}\big(7.7 - \sigma(s)\big)^{\underline{1.7}}\right) \right| \right\} \\ \cong \max\{0, 1.48, 1.51, 0.7, 0.8, 2.89, 4.53\} = 4.53,$$

$$\Lambda_3^1 = \max_{t\in\mathbb{N}_{1.7}^{7.7}} \left\{ \left| \frac{1}{(5.7)^{\underline{1.7}}\Gamma(0.7)} \sum_{s=2}^{5} \left((5.7)^{\underline{1.7}} \left(t-\sigma(s)\right)^{\underline{-0.3}} - (t-2)^{\underline{-0.3}} \left(7.7-\sigma(s)\right)^{\underline{1.7}} \right) \right| \right\}$$
$$\cong \max\{1.48, 0.03, 0.81, 1.5, 2.09, 1.63, 1.43\} = 2.09,$$

$$\Lambda_4^1 = \max_{t\in\mathbb{N}_{1.7}^{7.7}} \left\{ \left| \frac{1}{\Gamma(2.1)(5.7)^{\underline{1.7}}} \sum_{s=2}^{5} \left((5.7)^{\underline{1.7}} \left(t-\sigma(s)\right)^{\underline{2.1}} - (t-2)^{\underline{2.1}} \left(7.7-\sigma(s)\right)^{\underline{1.7}} \right) \right| \right\}$$
$$\cong \max\{1.87, 1.11, 0.88, 0.77, 0.59, 0.25, 0.16\} = 1.87,$$

$$\Lambda_5^1 = \max_{t\in\mathbb{N}_{1.7}^{7.7}} \left\{ \left| \frac{1}{\Gamma(1.1)(5.7)^{\underline{1.7}}} \sum_{s=2}^{5} \left((5.7)^{\underline{1.7}} \left(t-\sigma(s)\right)^{\underline{1.1}} - (t-2)^{\underline{1.1}} \left(7.7-\sigma(s)\right)^{\underline{1.7}} \right) \right| \right\}$$
$$\cong \max\{1.35, 0.39, 0.2, 2.42, 0.59, 0.17, 0.08\} = 2.42,$$

$$\Lambda_1^2 = \max_{t\in\mathbb{N}_{1.3}^{7.3}} \left| \frac{1}{(5.3)^{\underline{1.3}}\Gamma(2.3)} \sum_{s=2}^{5} \left((5.3)^{\underline{1.3}} \left(t-\sigma(s)\right)^{\underline{1.3}} - (t-2)^{\underline{1.3}} \left(7.3-\sigma(s)\right)^{\underline{1.3}} \right) \right| \}$$
$$\cong \max\{0, 0, 1.73, 3, 3.3, 2.36, 0\} = 3.3,$$

$$\Lambda_2^2 = \max_{t\in\mathbb{N}_{1.3}^{7.3}} \left\{ \left| \frac{1}{(5.3)^{\underline{1.3}}\Gamma(1.3)} \sum_{s=2}^{5} \left((5.3)^{\underline{1.3}} \left(t-\sigma(s)\right)^{\underline{0.3}} - (t-2)^{\underline{0.3}} \left(7.3-\sigma(s)\right)^{\underline{1.3}} \right) \right| \right\}$$
$$\cong \max\{0, 1.73, 1.26, 0.3, 0.93, 2.36, 2.94\} = 2.94,$$

$$\Lambda_3^2 = \max_{t\in\mathbb{N}_{1.3}^{7.3}} \left\{ \left| \frac{1}{(5.3)^{\underline{1.3}}\Gamma(0.3)} \sum_{s=2}^{5} \left((5.3)^{\underline{1.3}} \left(t-\sigma(s)\right)^{\underline{-0.7}} - (t-2)^{\underline{-0.7}} \left(7.3-\sigma(s)\right)^{\underline{1.3}} \right) \right| \right\}$$
$$\cong \max\{1.73, 0.47, 0.96, 1.23, 1.43, 0.58, 0.41\} = 1.73,$$

$$\Lambda_4^2 = \max_{t\in\mathbb{N}_{1.3}^{7.3}} \left\{ \left| \frac{1}{\Gamma(1.9)(5.3)^{\underline{1.3}}} \sum_{s=2}^{5} \left((5.3)^{\underline{1.3}} \left(t-\sigma(s)\right)^{\underline{1.9}} - (t-2)^{\underline{1.9}} \left(7.3-\sigma(s)\right)^{\underline{1.3}} \right) \right| \right\}$$
$$\cong \max\{3.76, 0.82, 0.46, 0.33, 1.43, 0.29, 0.14\} = 3.76,$$

and

$$\Lambda_5^2 = \max_{t\in\mathbb{N}_{1.3}^{7.3}} \left\{ \left| \frac{1}{\Gamma(1)(5.3)^{\underline{1.3}}} \sum_{s=2}^{5} \left((5.3)^{\underline{1.3}} \left(t-\sigma(s)\right)^{\underline{1}} - (t-2)^{\underline{1}} \left(7.3-\sigma(s)\right)^{\underline{1.3}} \right) \right| \right\}$$
$$\cong \max\{4.68, 0.56, 0.21, 2.81, 1.82, 0.23, 0.088\} = 4.68.$$

Thus, we obtain the table:

	Λ_j^i				
	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	3.7	4.53	2.09	1.87	2.42
$i=2$	3.3	2.94	1.73	3.76	4.68

Note that

$$0 < L = \sum_{i=1}^{2} \left(\max_{t\in\mathbb{N}_{\nu_i-1}^{5+\nu_i}} \left|\psi_i(t)\right| \right) \left(\Lambda_1^i + \cdots + \Lambda_5^i \right)$$

$$\begin{aligned}
&\cong \frac{1}{40.44}(3.7 + 4.53 + 2.09 + 1.87 + 2.42) + \frac{1}{40.04}(3.3 + 2.94 + 1.73 + 3.76 + 4.68) \\
&= \frac{1}{40.44}(14.63) + \frac{1}{40.04}(16.44) \cong 0.77 < 1.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&H_d\big(F_1(t,y_1,\ldots,y_{10}),F_1(t,z_1,\ldots,z_{10})\big) \\
&\quad\le \left|\frac{\sin y_1}{e^{t+4}} + \frac{\sin y_2}{3e^{t+3}} + \frac{|y_3|+|y_4|}{e^{10}} + \frac{\cos\sin y_5}{3e^{t+3}}\right. \\
&\qquad + \frac{\cos\sin y_6}{e^{t+4}} + \frac{|y_7|}{\cosh(t+10)} + \frac{|y_8|}{te^{t+2}} + \frac{|y_9|}{\pi e^{t+3}} + \frac{|y_{10}|}{e^{t+4}} \\
&\qquad - \frac{\sin z_1}{e^{t+4}} - \frac{\sin z_2}{3e^{t+3}} - \frac{|z_3|+|z_4|}{e^{10}} - \frac{\cos\sin z_5}{3e^{t+3}} \\
&\qquad \left. - \frac{\cos\sin z_6}{e^{t+4}} - \frac{|z_7|}{\cosh(t+10)} - \frac{|z_8|}{te^{t+2}} - \frac{|z_9|}{\pi e^{t+3}} - \frac{|z_{10}|}{e^{t+4}}\right| \\
&\quad\le \frac{1}{e^{(t+2)}}\big(\sin y_1 - \sin z_1 + \sin y_2 - \sin z_2 + |y_3 - z_3| + |y_4 - z_4| + \cos\sin y_5 - \cos\sin z_5 \\
&\qquad + \cos\sin y_6 - \cos\sin z_6 + |y_7 - z_7| + |y_8 - z_8| + |y_9 - z_9| + |y_{10} - z_{10}|\big) \\
&\quad\le \frac{1}{e^{(t+2)}}\left(\sum_{i=1}^{10}|y_i - z_i|\right) \le \psi_1(t)\sum_{k=1}^{10}|y_k - z_k|
\end{aligned}$$

for all $t \in \mathbb{N}_{1.7}^{7.7}$, $y_1,\ldots,y_{10},z_1,\ldots,z_{10} \in \mathbb{R}$. Similarly, we have

$$\begin{aligned}
&H_d\big(F_2(t,y_1,\ldots,y_{10}),F_2(t,z_1,\ldots,z_{10})\big) \\
&\quad\le \left|\frac{|y_1|}{e^{t^2+4}} + \frac{|y_2|}{e^{t^2+3}} + \frac{|y_3|}{e^{\pi t^2}} + \frac{|y_4|}{\cosh(t^5+8)}\right. \\
&\qquad + \frac{\sin y_5}{3e^{t^2+3}} + \frac{\sin y_6}{e^{t^4+4}} + \frac{|y_7|+|y_8|}{e^{t^4+6}} + \frac{|y_9|}{4e^{t^2+1}} + \frac{|y_{10}|}{t^2e^{t^4}} \\
&\qquad - \frac{|z_1|}{e^{t^2+4}} - \frac{|z_2|}{e^{t^2+3}} - \frac{|z_3|}{e^{\pi t^2}} - \frac{|z_4|}{\cosh(t^5+8)} \\
&\qquad \left. - \frac{\sin z_5}{3e^{t^2+3}} - \frac{\sin z_6}{e^{t^4+4}} - \frac{|z_7|+|z_8|}{e^{t^4+6}} - \frac{|z_9|}{4e^{t^2+1}} - \frac{|z_{10}|}{t^2e^{t^4}}\right| \\
&\quad\le \frac{1}{e^{(t^2+2)}}\big(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3| + |y_4 - z_4| + \sin y_5 - \sin z_5 \\
&\qquad + \sin y_6 - \sin z_6 + |y_7 - z_7| + |y_8 - z_8| + |y_9 - z_9| + |y_{10} - z_{10}|\big) \\
&\quad\le \frac{1}{e^{(t^2+2)}}\left(\sum_{i=1}^{10}|y_i - z_i|\right) \le \psi_2(t)\sum_{k=1}^{10}|y_k - z_k|
\end{aligned}$$

for all $t \in \mathbb{N}_{1.7}^{7.7}$, $y_1,\ldots,y_{10},z_1,\ldots,z_{10} \in \mathbb{R}$. Now, using Theorem 2.2, we conclude that problem (3) has a solution.

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Competing interests
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Authors' contributions
Both authors have equal contributions. The whole work was carried out, read, and approved by the authors.

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