Stević *Advances in Difference Equations* (2017) 2017:283
DOI 10.1186/s13662-017-1350-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Bounded and periodic solutions to the linear first-order difference equation on the integer domain

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Abstract

The existence of bounded solutions to the linear first-order difference equation on the set of all integers is studied. Some sufficient conditions for the existence of solutions converging to zero when $n \to -\infty$, as well as when $n \to +\infty$, are also given. For the case when the coefficients of the equation are periodic, the long-term behavior of non-periodic solutions is studied.

MSC: Primary 39A06; secondary 39A22; 39A23; 39A45

Keywords: linear first-order difference equation; bounded solution; periodic solution; difference equation on integer domain

1 Introduction

Many classes of difference equations have been studied for a long time (see, for example, [1–23] and the references therein). The following difference equation:

$$x_{n+1} = q_n x_n + f_n, \tag{1}$$

where the coefficients $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ and the initial value x_0 are given numbers, is called the *linear first-order difference equation*, which is one of the most important and useful *solvable* equations. The general solution to equation (1) is given by

$$x_n = x_0 \prod_{j=0}^{n-1} q_j + \sum_{i=0}^{n-1} f_i \prod_{j=i+1}^{n-1} q_j, \quad n \in \mathbb{N}. \tag{2}$$

If, as usual, the conventions $\prod_{j=k}^{k-1} a_j = 1$ and $\sum_{j=k}^{k-1} a_j = 0$, $k \in \mathbb{Z}$ are used, then (2) also holds for $n = 0$, that is, the formula holds for every $n \in \mathbb{N}_0$. How formula (2) is obtained can be found, for example, in [1, 6, 14] (the case when sequences q_n and f_n are constant can be also found in [11] and many other books dealing completely or partly with difference equations).

Many nonlinear difference equations of interest are closely related to equation (1). For example, some of the nonlinear equations in [17, 19–21] and systems in [19, 22] have been solved by transforming them to some special equations of the form in (1), which shows its importance (some of the equations and systems, such as the one in [22], are transformed into special cases of the delayed version of equation (1), which is obviously also solvable; see also [19] and the comments on scaling indices of difference equations and systems). A deeper analysis can show that even the solvability of some product-type equations and systems is essentially influenced by the solvability of equation (1) (see, e.g., [23] and the references therein).

Usefulness of (1) has also been recently shown in [18], where, among others, a small but nice result on convergence of its solutions was proved by using formula (2), partially extending the result in the following problem in [7] (in the eight edition of the Russian version of the problem book from 1972 it is Problem 637.2).

Problem 1 *Let a sequence* $(y_n)_{n\in\mathbb{N}_0}$ *be defined by a sequence* $(x_n)_{n\in\mathbb{N}_0}$ *as follows*:

$$y_0 = x_0, \qquad y_n = x_n - \alpha x_{n-1}, \quad n \in \mathbb{N},$$

where $|\alpha| < 1$. *If* $\lim_{n\to\infty} y_n = 0$, *find* $\lim_{n\to\infty} x_n$.

For some other solvable equations, their applications, as well as invariants for some classes of equations, see, for example, [1–6, 9, 11–16].

Note that the domain of the above defined sequences x_n is $\mathbb{N}_0$, and it is difficult to find papers which consider equation (1) on $\mathbb{Z}$-the set of all integers, in the literature on difference equations. One of our aims is to fulfill the possible gap in the study of the equation. Motivated also by our recent paper [18], here, among others, we study bounded solutions to equation (1), but also on the whole $\mathbb{Z}$. We also present some sufficient conditions for the existence of solutions to (1) converging to zero when $n \to -\infty$ as well as when $n \to +\infty$. Some of the results presented in the next section could be folklore, but we could not locate them in the literature.

A solution x_n to (1) is said to be (eventually) periodic with period $T \in \mathbb{N}$ if there is $n_0 \in \mathbb{N}_0$ such that

$$x_n = x_{n+T} \quad \text{for } n \geq n_0.$$

If $T = 1$, then such a solution is called eventually constant [8].

Periodic solutions to (1) on $\mathbb{N}$ were studied in [2], which was another motivation for this paper. Our main result on periodicity is a nice complement to that in [2]. Namely, for the case when the coefficients of equation (1) are periodic, we describe the long-term behavior of its non-periodic solutions when $n \to -\infty$ as well as when $n \to +\infty$.

Assume that $S \subset \mathbb{Z}$ is an unbounded set and that $f := (f_n)_{n\in S}$ is a sequence defined on S. Then, if

$$\|f\|_{\infty,S} := \sup_{n\in S} |f_n| < +\infty, \tag{3}$$

we say that the sequence f is bounded on S. It is easy to see that the quantity defined in (3) is a norm on the space of all bounded sequences on S. From now on the set S will not be of special importance, so we will simply use the notation $\|f\|_\infty$ instead of $\|f\|_{\infty,S}$ for any unbounded set S which appears as a domain.

2 Bounded solutions to equation (1) on $\mathbb{Z}$

In this section we study the existence of bounded solutions to equation (1). The cases when the domains are $\mathbb{N}_0$ and $\mathbb{Z}\setminus\mathbb{N}$ are treated separately, while the results in the case when the domain is $\mathbb{Z}$ are obtained as some consequences of the considerations on the domains $\mathbb{N}_0$ and $\mathbb{Z}\setminus\mathbb{N}$.

Our first result is, among others, an extension of the result in Problem 1, so it could be folklore.

Theorem 1 *Assume that*

$$\limsup_{n\to+\infty}|q_n| := q < 1 \tag{4}$$

and that $(f_n)_{n\in\mathbb{N}_0}$ *is a bounded sequence. Then the following statements are true.*

(a) *Every solution to equation* (1) *is bounded.*

(b) *If* $\lim_{n\to+\infty} f_n = 0$, *then every solution to equation* (1) *converges to zero.*

Proof (a) Since (4) holds, we have that there is $n_1\in\mathbb{N}$ such that

$$|q_n| \le \frac{1+q}{2} \quad \text{for } n \ge n_1. \tag{5}$$

Let

$$M_1 := \max\Big\{\sqrt{2}, \max_{j=\overline{0,n_1-1}}|q_j|\Big\}. \tag{6}$$

Using (5) and (6) in (2), as well as some standard estimates and sums, we have

$$\begin{aligned}
|x_n| &\le |x_0|\prod_{j=0}^{n-1}|q_j| + \sum_{i=0}^{n-1}|f_i|\prod_{j=i+1}^{n-1}|q_j| \qquad (7)\\
&\le |x_0|\prod_{j=0}^{n-1}|q_j| + \|f\|_\infty\left(\sum_{i=0}^{n_1-2}\prod_{j=i+1}^{n-1}|q_j| + \sum_{i=n_1-1}^{n-1}\prod_{j=i+1}^{n-1}|q_j|\right)\\
&\le |x_0|\left(\frac{1+q}{2}\right)^{n-n_1}M_1^{n_1} + \|f\|_\infty\left(\frac{1+q}{2}\right)^{n-n_1}\sum_{i=0}^{n_1-2}M_1^{n_1-i-1}\\
&\quad + \|f\|_\infty\sum_{i=n_1-1}^{n-1}\left(\frac{1+q}{2}\right)^{n-i-1}\\
&\le |x_0|\left(\frac{2M_1}{1+q}\right)^{n_1} + M_1\|f\|_\infty\left(\frac{2}{1+q}\right)^{n_1}\frac{M_1^{n_1-1}-1}{M_1-1} + \frac{2\|f\|_\infty}{1-q} < \infty \qquad (8)
\end{aligned}$$

for $n \ge n_1$, from which the boundedness of $(x_n)_{n\in\mathbb{N}_0}$ follows.

(b) Since f_n converges to zero, we have that for each $\varepsilon > 0$, there is $n_2\in\mathbb{N}$ such that

$$|f_n| < \varepsilon \quad \text{for } n \ge n_2. \tag{9}$$

Let $n_3 = \max\{n_1, n_2\}$, where n_1 is from (5), and

$$M_2 := \max\Big\{\sqrt{2}, \max_{j=\overline{0,n_3-1}} |q_j|\Big\}. \tag{10}$$

Using (5), (9) and (10) in (2), as well as some standard estimates and sums, we have

$$\begin{aligned} |x_n| &\le |x_0|\left(\frac{1+q}{2}\right)^{n-n_3} M_2^{n_3} + \|f\|_\infty \sum_{i=0}^{n_3-1}\prod_{j=i+1}^{n-1}|q_j| + \varepsilon \sum_{i=n_3}^{n-1}\prod_{j=i+1}^{n-1}|q_j| \\ &\le |x_0|\left(\frac{1+q}{2}\right)^{n-n_3} M_2^{n_3} + \|f\|_\infty \left(\frac{1+q}{2}\right)^{n-n_3}\sum_{i=0}^{n_3-1} M_2^{n_3-i-1} + \varepsilon \sum_{i=n_3}^{n-1}\left(\frac{1+q}{2}\right)^{n-i-1} \\ &\le |x_0|\left(\frac{2M_2}{1+q}\right)^{n_3}\left(\frac{1+q}{2}\right)^{n} + \|f\|_\infty \left(\frac{2}{1+q}\right)^{n_3}\frac{M_2^{n_3}-1}{M_2-1}\left(\frac{1+q}{2}\right)^{n} + \frac{2\varepsilon}{1-q} \end{aligned} \tag{11}$$

for $n \ge n_3$.

Letting $n \to +\infty$ in (11), we get

$$\limsup_{n\to+\infty} |x_n| \le \frac{2\varepsilon}{1-q}.$$

From this, and since ε is an arbitrary positive number, the result follows. □

Remark 1 Theorem 1 is optimal in the sense that condition (4) cannot be replaced by the following one: there is $n_4 \in \mathbb{N}_0$ such that

$$|q_n| < 1 \quad \text{for } n \ge n_4. \tag{12}$$

Indeed, assume that $q_n \in (0,1)$, $n \in \mathbb{N}_0$, is an increasing sequence such that the sequence

$$Q_n := \prod_{j=0}^{n-1} q_j, \quad n \in \mathbb{N}_0,$$

converges to $Q \in (0,1)$ as $n \to +\infty$, and that there are some numbers l and L such that

$$0 < l \le f_n \le L < \infty, \quad n \in \mathbb{N}_0. \tag{13}$$

Then, by using (2) and the fact $Q \le Q_n \le 1$, $n \in \mathbb{N}_0$, we have

$$|x_n| = Q_n\left|x_0 + \sum_{j=0}^{n-1}\frac{f_j}{Q_{j+1}}\right| \ge Q\big|-|x_0| + nl\big| \to +\infty \tag{14}$$

as $n \to +\infty$, from which it follows that not only there are unbounded solutions to equation (1) in this case, but that even none of the solutions to the equation in the case is bounded.

For example, let

$$q_n = 1 - \frac{1}{(n+2)^2}, \quad n \in \mathbb{N}_0$$

and

$$f_n = 2 + \sin n, \quad n \in \mathbb{N}_0.$$

Then

$$Q_n = \prod_{j=0}^{n-1}\left(1 - \frac{1}{(j+2)^2}\right) = \prod_{j=0}^{n-1}\frac{(j+1)(j+3)}{(j+2)^2} = \frac{n!(n+2)!}{2(n+1)!^2} = \frac{n+2}{2(n+1)}$$

for $n \in \mathbb{N}_0$, from which it follows that $\lim_{n\to+\infty} Q_n = 1/2$,

$$\frac{1}{2} \le Q_n \le 1, \quad n \in \mathbb{N}_0, \tag{15}$$

and along with (2) that

$$x_n = \frac{n+2}{2(n+1)}\left(x_0 + \sum_{j=0}^{n-1}(2+\sin j)\frac{2(j+2)}{j+3}\right). \tag{16}$$

Using (15), the following obvious estimate

$$1 \le f_n \le 3, \quad n \in \mathbb{N}_0, \tag{17}$$

and the triangle inequality in (16), we get

$$|x_n| \ge \frac{1}{2}\left(\frac{4}{3}n - |x_0|\right), \quad n \in \mathbb{N}_0,$$

from which it follows that each solution to equation (1) in this case is unbounded indeed.

If we assume that f_n is a positive sequence such that $\lim_{n\to+\infty} f_n = 0$ and $\sum_{j=0}^{\infty} f_j = +\infty$ (for example, $f_n = 1/(n+1)$, $n \in \mathbb{N}_0$), and that q_n is chosen as above, then from (2) and the fact $Q \le Q_n \le 1$, $n \in \mathbb{N}_0$, we have

$$|x_n| = Q_n\left|x_0 + \sum_{j=0}^{n-1}\frac{f_j}{Q_{j+1}}\right| \ge Q\left|-|x_0| + \sum_{j=0}^{n-1} f_j\right| \to +\infty$$

as $n \to +\infty$, from which it follows that none of the solutions to equation (1) in this case converges to zero. Specially, every solution to the difference equation

$$x_{n+1} = \frac{(n+1)(n+3)}{(n+2)^2}x_n + \frac{1}{n+1}, \quad n \in \mathbb{N}_0,$$

is unbounded.

Now we consider the case $\liminf_{n\to+\infty} |q_n| > 1$. In this case, we may assume that $q_n \ne 0$, $n \in \mathbb{N}_0$, otherwise we can consider (1) for sufficiently large n for which, due to the condition $\liminf_{n\to+\infty} |q_n| > 1$, will hold $q_n \ne 0$.

Theorem 2 *Assume that $(q_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}\setminus\{0\}$ is a sequence satisfying the following condition*:

$$\liminf_{n\to+\infty} |q_n| := \hat{q} > 1, \tag{18}$$

and that $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers. Then the following statements are true.

(a) *There is a unique bounded solution to equation* (1).

(b) *If $f_n \to 0$ as $n \to +\infty$, then the bounded solution also converges to zero as $n \to +\infty$.*

Proof (a) From (18), we have

$$\lim_{n\to+\infty} \left|\prod_{j=0}^{n-1} q_j\right| = +\infty. \tag{19}$$

Thus, from (2) and (19) we see that for a bounded solution to (1) it must be

$$x_0 = -\sum_{i=0}^{\infty} \frac{f_i}{\prod_{j=0}^{i} q_j}, \tag{20}$$

and that the sum on the right-hand side of (20) is finite (see [18]).

Using (20) in (2), it follows that

$$x_n = -\prod_{j=0}^{n-1} q_j \sum_{i=n}^{\infty} \frac{f_i}{\prod_{j=0}^{i} q_j} = -\sum_{i=n}^{\infty} \frac{f_i}{\prod_{j=n}^{i} q_j}, \quad n \in \mathbb{N}_0. \tag{21}$$

Condition (18) implies that there is $n_5 \in \mathbb{N}$ such that

$$|q_n| \ge \frac{1+\hat{q}}{2} > 1 \quad \text{for } n \ge n_5. \tag{22}$$

Let

$$M_3 := \min\Bigl\{1/\sqrt{2}, \min_{j=\overline{0,n_5-1}} |q_j|\Bigr\}. \tag{23}$$

Note that $M_3 > 0$ due to the assumption $q_n \neq 0$, $n \in \mathbb{N}_0$.

Hence, by using (22) and (23), we have

$$\begin{aligned}
\left|\sum_{i=0}^{\infty} \frac{f_i}{\prod_{j=0}^{i} q_j}\right| &\le \|f\|_\infty \sum_{i=0}^{\infty} \frac{1}{\prod_{j=0}^{i} |q_j|} \\
&\le \|f\|_\infty \left(\sum_{i=0}^{n_5-1} \frac{1}{\prod_{j=0}^{i} |q_j|} + \frac{1}{M_3^{n_5}} \sum_{i=n_5}^{\infty} \left(\frac{2}{1+\hat{q}}\right)^{i-n_5+1}\right) \\
&\le \|f\|_\infty \left(\frac{1}{M_3^{n_5}(1-M_3)} + \frac{2}{M_3^{n_5}(\hat{q}-1)}\right) < \infty,
\end{aligned} \tag{24}$$

from which it follows that x_0 defined in (20) is finite.

From (21) and (22), we have

$$|x_n| \le \sum_{i=n}^{\infty} \frac{|f_i|}{\prod_{j=n}^{i}|q_j|} \le \|f\|_\infty \sum_{i=n}^{\infty} \left(\frac{2}{1+\hat{q}}\right)^{i+1-n} = \frac{2\|f\|_\infty}{\hat{q}-1},$$

for $n \ge n_5$, from which the boundedness of the sequence in (21) follows. It is directly verified that the sequence satisfies equation (1), from which along with the unique choice of x_0 in (20) it follows that it is a unique bounded solution to the equation.

(b) Since f_n tends to zero as $n \to +\infty$, it follows that (9) holds for, say, $n \ge n_6$. Using (9) and (22) in (21), we have

$$|x_n| \le \sum_{i=n}^{\infty} \frac{|f_i|}{\prod_{j=n}^{i}|q_j|} < \varepsilon \sum_{i=n}^{\infty} \left(\frac{2}{1+\hat{q}}\right)^{i+1-n} = \frac{2\varepsilon}{\hat{q}-1} \tag{25}$$

for $n \ge \max\{n_5, n_6\}$.

Letting $n \to +\infty$ in (25) the following is obtained:

$$\limsup_{n\to+\infty} |x_n| \le \frac{2\varepsilon}{\hat{q}-1}.$$

From this and by using the fact that ε is an arbitrary positive number, we obtain that $x_n \to 0$ as $n \to +\infty$, as desired. □

Remark 2 Theorem 2 is optimal in the sense that condition (18) cannot be replaced by the following one: there is $n_7 \in \mathbb{N}_0$ such that

$$|q_n| > 1 \quad \text{for } n \ge n_7. \tag{26}$$

Indeed, assume that $q_n > 1$, $n \in \mathbb{N}_0$, is an decreasing sequence such that the sequence

$$Q_n := \prod_{j=0}^{n-1} q_j, \quad n \in \mathbb{N}_0,$$

converges to $Q > 1$ as $n \to +\infty$, and that there are some numbers l and L such that (13) holds. Then, by using (2) and the fact $Q \ge Q_n \ge 1$, $n \in \mathbb{N}_0$, we have

$$|x_n| = Q_n \left| x_0 + \sum_{j=0}^{n-1} \frac{f_j}{Q_{j+1}} \right| \ge \big| -|x_0| + nl/Q \big| \to +\infty \tag{27}$$

as $n \to +\infty$, from which it follows that all the solutions to equation (1) are unbounded in the case, so, it does not have bounded solutions.

For example, let

$$q_n = 1 + \frac{1}{n^2+4n+3}, \quad n \in \mathbb{N}_0,$$

and

$$f_n = 5 + 2\cos n, \quad n \in \mathbb{N}_0.$$

Then

$$Q_n = \prod_{j=0}^{n-1}\left(1+\frac{1}{j^2+4j+3}\right) = \prod_{j=0}^{n-1}\frac{(j+2)^2}{(j+1)(j+3)} = \frac{2(n+1)}{n+2}$$

for $n \in \mathbb{N}_0$, from which it follows that $\lim_{n\to+\infty} Q_n = 2$,

$$1 \le Q_n \le 2, \quad n \in \mathbb{N}_0, \tag{28}$$

and along with (2) that

$$x_n = \frac{2(n+1)}{n+2}\left(x_0 + \sum_{j=0}^{n-1}(5+2\cos j)\frac{j+3}{2(j+2)}\right). \tag{29}$$

Using (28), the following obvious estimate

$$3 \le f_n \le 7, \quad n \in \mathbb{N}_0, \tag{30}$$

and the triangle inequality in (29), we get

$$|x_n| \ge \frac{3}{2}n - |x_0|, \quad n \in \mathbb{N}_0,$$

from which it follows that each solution to equation (1) in this case is unbounded.

If we assume that f_n is a positive sequence such that $\lim_{n\to+\infty} f_n = 0$ and $\sum_{j=0}^{\infty} f_j = +\infty$ (for example, $f_n = 1/\ln(n+2)$, $n \in \mathbb{N}_0$), and that q_n is chosen as above, then from (2) and the fact $1 \le Q_n \le Q$, $n \in \mathbb{N}_0$, we have

$$|x_n| = Q_n\left|x_0 + \sum_{j=0}^{n-1}\frac{f_j}{Q_{j+1}}\right| \ge \left|-|x_0| + \frac{1}{Q}\sum_{j=0}^{n-1} f_j\right| \to +\infty,$$

as $n \to +\infty$, from which it follows that none of the solutions to equation (1) in this case converges to zero. Specially, every solution to the difference equation

$$x_{n+1} = \frac{(n+2)^2}{(n+1)(n+3)}x_n + \frac{1}{\ln(n+2)}, \quad n \in \mathbb{N}_0,$$

is unbounded.

Now we consider the case when $n \in \mathbb{Z} \setminus \mathbb{N}$. If in (1) is $q_n \neq 0$ for every $n \in \mathbb{Z} \setminus \mathbb{N}_0$, then the sequence x_n is not only well-defined on the set $\mathbb{N}_0$, but also for every $n \in \mathbb{Z}$. Indeed, if $n \le 0$, then from (1) we have

$$x_{-n} = \frac{x_{-(n-1)}}{q_{-n}} - \frac{f_{-n}}{q_{-n}}, \quad n \in \mathbb{N}. \tag{31}$$

Using one of the methods for solving equation (1), from (31) the following is obtained:

$$x_{-n} = \frac{x_0 - \sum_{j=1}^{n} f_{-j}\prod_{l=1}^{j-1} q_{-l}}{\prod_{j=1}^{n} q_{-j}} \tag{32}$$

for $n \in \mathbb{N}$.

Closed form formulas (2) and (32) together present the general solution to equation (1) on $\mathbb{Z}$, when $q_n \neq 0$, for $n \in \mathbb{Z} \setminus \mathbb{N}_0$.

Now we formulate and prove the corresponding results to Theorems 1 and 2 concerning bounded solutions to equation (31). The results are dual to Theorems 1 and 2 and are essentially obtained from them by using the change of variables $y_n = x_{-n}$. However, there are some different details which are used later in the text. Because of this and for the completeness, we will sketch their proofs.

First, we consider equation (31) for the case

$$\liminf_{n\to\infty} |q_{-n}| =: \tilde{q} > 1. \tag{33}$$

Theorem 3 *Assume that* $(q_{-n})_{n\in\mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ *is a sequence satisfying condition* (33), *and that* $(f_{-n})_{n\in\mathbb{N}}$ *is a bounded sequence of complex numbers. Then the following statements are true.*

(a) *Every solution to* (31) *is bounded.*

(b) *If* $\lim_{n\to+\infty} f_{-n} = 0$, *then every solution to* (31) *converges to zero as* $n \to +\infty$.

Proof (a) From (33) it follows that there is $n_8 \in \mathbb{N}$ such that

$$\frac{1}{|q_{-n}|} \le \frac{2}{1+\tilde{q}} \quad \text{for } n \ge n_8. \tag{34}$$

Let

$$M_5 := \min\Big\{1/\sqrt{2}, \min_{j=\overline{1,n_8-1}} |q_{-j}|\Big\}. \tag{35}$$

Using (34) and (35) in (32), as well as some standard estimates and sums, we have

$$\begin{aligned} |x_{-n}| &\le \frac{|x_0|}{\prod_{j=1}^n |q_{-j}|} + \sum_{j=1}^n \frac{|f_{-j}|}{\prod_{l=j}^n |q_{-l}|} \\ &\le \frac{|x_0|}{M_5^{n_8-1}} + \frac{\|f\|_\infty}{(1-M_5)M_5^{n_8-1}} + \frac{2\|f\|_\infty}{\tilde{q}-1} < \infty \end{aligned} \tag{36}$$

for $n \ge n_8$, from which the boundedness of $(x_n)_{n\in\mathbb{N}_0}$ follows.

(b) Since f_{-n} converges to zero as $n \to +\infty$, we have that for every $\varepsilon > 0$, there is $n_9 \in \mathbb{N}$ such that

$$|f_{-n}| < \varepsilon \quad \text{for } n \ge n_9. \tag{37}$$

Let $n_{10} = \max\{n_8, n_9\}$, where n_8 is from (34), and

$$M_6 := \min\Big\{1/\sqrt{2}, \min_{j=\overline{1,n_{10}-1}} |q_{-j}|\Big\}. \tag{38}$$

Using (34), (37) and (38) in (36), as well as some standard estimates and sums, we have

$$|x_n| \le |x_0|\left(\frac{1+\tilde{q}}{2M_6}\right)^{n_{10}-1}\left(\frac{2}{1+\tilde{q}}\right)^n + \|f\|_\infty\left(\frac{1+\tilde{q}}{2M_6}\right)^{n_{10}-1}\left(\frac{2}{1+\tilde{q}}\right)^n \frac{1}{1-M_6} + \frac{2\varepsilon}{\tilde{q}-1} \tag{39}$$

for $n \ge n_{10}$.

Letting $n \to +\infty$ in (39) and using the fact that ε is an arbitrary positive number, the result follows. □

Now we consider the case $\limsup_{n\to+\infty} |q_{-n}| < 1$. If so, then a bounded solution $(x_{-n})_{n\in\mathbb{N}}$ to (31) is obtained only if

$$x_0 = \sum_{j=1}^{\infty} f_{-j} \prod_{l=1}^{j-1} q_{-l}, \tag{40}$$

and for such chosen x_0, it is obtained

$$x_{-n} = \frac{\sum_{j=n+1}^{\infty} f_{-j} \prod_{l=1}^{j-1} q_{-l}}{\prod_{l=1}^{n} q_{-l}} = \sum_{j=n+1}^{\infty} f_{-j} \prod_{l=n+1}^{j-1} q_{-l} \tag{41}$$

for $n \in \mathbb{N}$.

Theorem 4 *Assume that* $(q_{-n})_{n\in\mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ *is a sequence such that*

$$\limsup_{n\to\infty} |q_{-n}| := q < 1, \tag{42}$$

and that $(f_{-n})_{n\in\mathbb{N}}$ *is a bounded sequence of complex numbers. Then the following statements are true.*

(a) *There is a unique bounded solution to* (31).

(b) *If* $\lim_{n\to+\infty} f_{-n} = 0$, *then the bounded solution* x_{-n} *also converges to zero as* $n \to +\infty$.

Proof (a) From (42) we have that there is $n_{11} \in \mathbb{N}$ such that

$$|q_{-n}| < \frac{1+q}{2} \quad \text{for } n \geq n_{11}. \tag{43}$$

By using (43) and some simple estimates in (41), we have

$$|x_{-n}| \leq \sum_{j=n+1}^{\infty} |f_{-j}| \prod_{l=n+1}^{j-1} |q_{-l}| < \|f\|_\infty \sum_{j=n+1}^{\infty} \left(\frac{1+q}{2}\right)^{j-n-1} = \frac{2\|f\|_\infty}{1-q} \tag{44}$$

for $n \geq n_{11}$, from which the boundedness of sequence (41) easily follows. A simple calculation shows that the sequence satisfies equation (31). Since x_0 is uniquely determined by the convergent series in (40), it follows that the sequence is a unique bounded solution to (31).

(b) Since $\lim_{n\to+\infty} f_{-n} = 0$, we have that for every $\varepsilon > 0$, there is $n_{12} \in \mathbb{N}$ such that (37) holds for $n \geq n_{12}$.

From this, (41) and (43), we have that

$$|x_{-n}| \leq \sum_{j=n+1}^{\infty} |f_{-j}| \prod_{l=n+1}^{j-1} |q_{-l}| < \varepsilon \sum_{j=n+1}^{\infty} \left(\frac{1+q}{2}\right)^{j-n-1} = \frac{2\varepsilon}{1-q} \tag{45}$$

for $n \geq \max\{n_{11}, n_{12}\}$. Letting $n \to +\infty$ in (45) and since ε is an arbitrary positive number, we obtain $\lim_{n\to+\infty} x_{-n} = 0$, as desired. □

From Theorems 1-4 the following four interesting corollaries are obtained.

From Theorems 1 and 3 we obtain the following corollary.

Corollary 1 *Consider equation* (1) *for* $n \in \mathbb{Z}$. *Assume that* $(q_n)_{n\in\mathbb{Z}}$ *and* $(f_n)_{n\in\mathbb{Z}}$ *are sequences of complex numbers such that* $q_{-n} \neq 0$, $n \in \mathbb{N}$, $\limsup_{n\to+\infty} |q_n| < 1$ *and* $\liminf_{n\to+\infty} |q_{-n}| > 1$, *and*

$$\sup_{n\in\mathbb{Z}} |f_n| < \infty. \tag{46}$$

Then the following statements are true.

(a) *Every solution to* (1) *is bounded on* $\mathbb{Z}$.

(b) *If*

$$\lim_{n\to\pm\infty} f_n = 0, \tag{47}$$

then, for every solution $(x_n)_{n\in\mathbb{Z}}$ *to* (1), *we have* $\lim_{n\to\pm\infty} x_n = 0$.

From Theorems 1 and 4 we obtain the following corollary.

Corollary 2 *Consider equation* (1) *for* $n \in \mathbb{Z}$. *Assume that* $(q_n)_{n\in\mathbb{Z}}$ *and* $(f_n)_{n\in\mathbb{Z}}$ *are sequences of complex numbers such that* $q_{-n} \neq 0$, $n \in \mathbb{N}$, $\limsup_{n\to+\infty} |q_n| < 1$ *and* $\limsup_{n\to+\infty} |q_{-n}| < 1$, *and that* (46) *holds. Then the following statements are true.*

(a) *There is a unique bounded solution to* (1) *on* $\mathbb{Z}$.

(b) *If* (47) *holds, then, for the bounded solution* $(x_n)_{n\in\mathbb{Z}}$, *we have* $\lim_{n\to\pm\infty} x_n = 0$.

From Theorems 2 and 3 we obtain the following corollary.

Corollary 3 *Consider equation* (1) *for* $n \in \mathbb{Z}$. *Assume that* $(q_n)_{n\in\mathbb{Z}}$ *and* $(f_n)_{n\in\mathbb{Z}}$ *are sequences of complex numbers such that* $q_{-n} \neq 0$, $n \in \mathbb{N}$, $\liminf_{n\to+\infty} |q_n| > 1$ *and* $\liminf_{n\to+\infty} |q_{-n}| > 1$, *and that* (46) *holds. Then the following statements are true.*

(a) *There is a unique bounded solution to* (1) *on* $\mathbb{Z}$.

(b) *If* (47) *holds, then, for the bounded solution* $(x_n)_{n\in\mathbb{Z}}$, *we have* $\lim_{n\to\pm\infty} x_n = 0$.

From Theorems 2 and 4 we obtain the following corollary.

Corollary 4 *Consider equation* (1) *for* $n \in \mathbb{Z}$. *Assume that* $(q_n)_{n\in\mathbb{Z}}$ *and* $(f_n)_{n\in\mathbb{Z}}$ *are sequences of complex numbers such that* $q_{-n} \neq 0$, $n \in \mathbb{N}$, $\liminf_{n\to+\infty} |q_n| > 1$ *and* $\limsup_{n\to+\infty} |q_{-n}| < 1$, *and that* (46) *holds. Then the following statements are true.*

(a) *There is a unique bounded solution to* (1) *on* $\mathbb{Z}$ *if and only if*

$$x_0 = \sum_{j=1}^{\infty} f_{-j} \prod_{l=1}^{j-1} q_{-l} = -\sum_{i=0}^{\infty} \frac{f_i}{\prod_{j=0}^{i} q_j}.$$

(b) *If* (47) *holds, then, for the bounded solution* $(x_n)_{n\in\mathbb{Z}}$, *we have* $\lim_{n\to\pm\infty} x_n = 0$.

3 Periodic solutions to equation (1)

In this section we study equation (1) in the case when the sequences q_n and f_n are periodic with the same period T. Note that if sequence q_n is periodic with period T_1 and sequence f_n is periodic with period T_2, where $T_1 \neq T_2$, then $T := \operatorname{lcm}(T_1, T_2)$ (the least common multiple of integers T_1 and T_2) is a common period of these two sequences, since $T = k_1T_1 = k_2T_2$ for some integers k_1 and k_2. Hence, the case leads to the investigation of equation (1) whose coefficients are periodic with the same period.

Periodic solutions to equation (1) have been studied in [2]. Among others, the authors of [2] quote, in an equivalent form, the following basic result, which is certainly folklore.

Theorem 5 *Assume that $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are two periodic sequences with period T. Then the following statements are true.*

(a) *If*

$$\lambda := \prod_{j=0}^{T-1} q_j \neq 1, \tag{48}$$

then (1) *has a unique T-periodic solution given by the initial condition*

$$x_0 = \frac{\sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j}{1-\lambda}. \tag{49}$$

(b) *If*

$$\lambda = 1 \quad \text{and} \quad \sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j = 0, \tag{50}$$

then (1) *has a one-parameter family of T-periodic solutions.*

(c) *If*

$$\lambda = 1 \quad \text{and} \quad \sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j \neq 0, \tag{51}$$

then (1) *has no T-periodic solutions.*

The case in (a) is interesting for guaranteeing the uniqueness of a T-periodic solution to equation (1). Note that, due to the T-periodicity of sequences q_n and f_n, from (2) we see that a solution to equation (1) will be periodic with period T if and only if

$$x_0 = x_T = x_0 \prod_{j=0}^{T-1} q_j + \sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j,$$

from which, if (48) holds, it follows that x_0 must be given by (49). However, the authors of [2] did not consider there the relationship between the periodic and other solutions to equation (1). We prove a nice result which gives an answer to the natural problem. Before this, we formulate and prove a closely related result to Theorem 5.

Theorem 6 *Assume that* $(q_{-n})_{n\in\mathbb{N}}$ *and* $(f_{-n})_{n\in\mathbb{N}}$ *are two periodic sequences with period* T *such that* $q_{-n}\neq 0$, $n=\overline{1,T}$. *Then the following statements are true.*

(a) *If*

$$\hat{\lambda}:=\prod_{j=1}^{T}q_{-j}\neq 1, \tag{52}$$

then (31) *has a unique* T*-periodic solution given by the initial condition*

$$\hat{x}_0=\frac{\sum_{j=1}^{T}f_{-j}\prod_{l=1}^{j-1}q_{-l}}{1-\hat{\lambda}}. \tag{53}$$

(b) *If*

$$\hat{\lambda}=1 \quad and \quad \sum_{j=1}^{T}f_{-j}\prod_{l=1}^{j-1}q_{-l}=0, \tag{54}$$

then all the solutions to (31) *are* T*-periodic.*

(c) *If*

$$\hat{\lambda}=1 \quad and \quad \sum_{j=1}^{T}f_{-j}\prod_{l=1}^{j-1}q_{-l}\neq 0, \tag{55}$$

then (31) *has no* T*-periodic solutions.*

Proof (a)-(c) Since q_{-n} and f_{-n} are periodic, by using (32) we see that a solution to (31) will be T-periodic if and only if

$$\hat{x}_0=\hat{x}_{-T}=\frac{\hat{x}_0-\sum_{j=1}^{T}f_{-j}\prod_{l=1}^{j-1}q_{-l}}{\prod_{j=1}^{T}q_{-j}}, \tag{56}$$

from which, if $\hat{\lambda}\neq 1$, (53) follows. If $\hat{\lambda}=1$, then from the second equality in (56), we see that if (54) holds we have $\hat{x}_{-T}=\hat{x}_0$, so all the solutions are T-periodic, while if (55) holds, such $\hat{x}_0$ does not exist, so there are no T-periodic solutions in this case, which completes the proof. □

Now we formulate and prove the main result in this section. The result deals with the asymptotic behavior of solutions to equation (1) in the case when the quantity in (48) is different from 0 and 1.

Theorem 7 *Assume that* $(q_n)_{n\in\mathbb{Z}}$ *and* $(f_n)_{n\in\mathbb{Z}}$ *are two periodic sequences with period* T *and that the quantity in* (48) *is different from* 0 *and* 1. *Then equation* (1) *has a unique* T*-periodic solution, say,* $(\hat{x}_n)_{n\in\mathbb{Z}}$, *and the following statements are true.*

(a) *If* $|\lambda|<1$, *then all the solutions to* (1) *converge geometrically to the periodic one as* $n\to+\infty$, *while they are getting away geometrically from the periodic one as* $n\to-\infty$.

(b) *If* $|\lambda| > 1$, *then all the solutions to* (1) *converge geometrically to the periodic one as* $n \to -\infty$, *while they are getting away geometrically from the periodic one as* $n \to +\infty$.

Proof Unique existence of a T-periodic solution $(\hat{x}_n)_{n\in\mathbb{Z}}$ to (1) follows from Theorem 6(a). Assume that $(x_n)_{n\in\mathbb{Z}}$ is an arbitrary solution to (1). Then, due to the T-periodicity of q_n, we have

$$\lambda = \prod_{j=n}^{n+T-1} q_j \tag{57}$$

for every $n \in \mathbb{Z}$ (note that due to an assumption we have $\lambda \neq 0$), from which along with (2), we obtain

$$\begin{aligned} x_{n+T} &= x_0 \prod_{j=0}^{n+T-1} q_j + \sum_{i=0}^{n+T-1} f_i \prod_{j=i+1}^{n+T-1} q_j \\ &= \lambda\left(x_0 \prod_{j=0}^{n-1} q_j + \sum_{i=0}^{n-1} f_i \prod_{j=i+1}^{n-1} q_j\right) + \sum_{i=n}^{n+T-1} f_i \prod_{j=i+1}^{n+T-1} q_j = \lambda x_n + d_n, \end{aligned} \tag{58}$$

where

$$d_n := \sum_{i=n}^{n+T-1} f_i \prod_{j=i+1}^{n+T-1} q_j, \quad n \in \mathbb{N}_0.$$

Using again the periodicity of p_n and q_n, we have

$$\begin{aligned} d_{n+T} &= \sum_{i=n+T}^{n+2T-1} f_i \prod_{j=i+1}^{n+2T-1} q_j = \sum_{l=n}^{n+T-1} f_{l+T} \prod_{j=l+T+1}^{n+2T-1} q_j \\ &= \sum_{l=n}^{n+T-1} f_l \prod_{k=l+1}^{n+T-1} q_{k+T} = \sum_{l=n}^{n+T-1} f_l \prod_{k=l+1}^{n+T-1} q_k = d_n, \end{aligned} \tag{59}$$

which means that the sequence d_n is T-periodic.

From (58) we see that if $\hat{x}_n$ is the periodic solution to (1), it must be

$$\hat{x}_n = \frac{d_n}{1-\lambda}, \quad n \in \mathbb{N}_0. \tag{60}$$

Also, we have

$$x_{mT+l} = \lambda^m x_l + d_l \sum_{j=0}^{m-1} \lambda^j, \quad m, l \in \mathbb{N}. \tag{61}$$

Hence, since $\lambda \neq 1$, we have

$$|x_{mT+l} - \hat{x}_{mT+l}| = \left|\lambda^m x_l + d_l \sum_{j=0}^{m-1} \lambda^j - \frac{d_l}{1-\lambda}\right| = |\lambda|^m \left|x_l - \frac{d_l}{1-\lambda}\right| \tag{62}$$

for every $m, l \in \mathbb{N}_0$.

Let

$$C_x := \max_{l=\overline{0,T-1}} \frac{1}{(\sqrt[T]{|\lambda|})^l}\left|x_l - \frac{d_l}{1-\lambda}\right|.$$

Note that since $\lambda \notin \{0,1\}$, the quantity is well defined.

Then from (62) it follows that

$$|x_n - \hat{x}_n| \le C_x\left(\sqrt[T]{|\lambda|}\right)^n \quad \text{for } n \in \mathbb{N}_0. \tag{63}$$

If $|\lambda| < 1$, then from (63) it follows that all the solutions to equation (1) converge geometrically to the periodic solution to the equation as $n \to +\infty$. Now note that if $x_{n_0} = \hat{x}_{n_0}$ for some $n_0 \in \mathbb{N}$, then it will be $x_n = \hat{x}_n$ for $n \ge n_0$. Moreover, since due to the condition $\lambda \ne 0$, we have $q_n \ne 0$, $n \in \mathbb{N}_0$, it follows that $x_n = \hat{x}_n$ for every $n \in \mathbb{N}_0$. So, for an arbitrary solution x_n different from $\hat{x}_n$, it must be $x_n \ne \hat{x}_n$ for every $n \in \mathbb{N}_0$. Hence, if $|\lambda| > 1$, then from (62) it follows that

$$|x_n - \hat{x}_n| \ge \left(\sqrt[T]{|\lambda|}\right)^n \min_{l=\overline{0,T-1}} \frac{1}{(\sqrt[T]{|\lambda|})^l}\left|x_l - \frac{d_l}{1-\lambda}\right| > 0, \quad n \in \mathbb{N}_0,$$

that is, every solution x_n different from $\hat{x}_n$ gets away geometrically from the periodic solution as $n \to +\infty$.

Now we are going to consider the case $n \le 0$. From Theorem 6(a) we see that equation (31) has a unique T-periodic solution if (52) holds with $\hat{x}_0$ given in (53).

From (57) we see that $\hat{\lambda} = \lambda$. Using this fact along with the periodicity of q_{-n} and f_{-n}, we obtain

$$\begin{aligned}
\hat{x}_0 &= \frac{\sum_{j=1}^{T} f_{-j}\prod_{l=1}^{j-1} q_{-l}}{1-\hat{\lambda}} = \frac{\sum_{j=1}^{T} f_{T-j}\prod_{l=1}^{j-1} q_{-l}}{1-\lambda} \\
&= \frac{\sum_{i=0}^{T-1} f_i\prod_{l=1}^{T-i-1} q_{-l}}{1-\lambda} = \frac{\sum_{i=0}^{T-1} f_i\prod_{l=1}^{T-i-1} q_{T-l}}{1-\lambda} \\
&= \frac{\sum_{i=0}^{T-1} f_i\prod_{s=i+1}^{T-1} q_s}{1-\lambda}.
\end{aligned} \tag{64}$$

From (49) and (64) we see that $\hat{x}_0 = x_0$, so, there is only one 'initial value' for which a T-periodic solution to (1) on $\mathbb{Z}$ is obtained.

Using (32), we obtain

$$\begin{aligned}
x_{-(n+T)} &= \frac{x_0 - \sum_{j=1}^{n+T} f_{-j}\prod_{l=1}^{j-1} q_{-l}}{\prod_{l=1}^{n+T} q_{-l}} \\
&= \frac{1}{\lambda}\left(\frac{x_0 - \sum_{j=1}^{n} f_{-j}\prod_{l=1}^{j-1} q_{-l}}{\prod_{l=1}^{n} q_{-l}}\right) - \sum_{j=n+1}^{n+T} \frac{f_{-j}}{\prod_{l=j}^{n+T} q_{-l}} \\
&= \frac{1}{\lambda}x_{-n} - c_n
\end{aligned} \tag{65}$$

for $n \in \mathbb{N}_0$, where

$$c_n = \sum_{j=n+1}^{n+T} \frac{f_{-j}}{\prod_{l=j}^{n+T} q_{-l}}. \tag{66}$$

Since

$$\begin{aligned} c_{n+T} &= \sum_{j=n+T+1}^{n+2T} \frac{f_{-j}}{\prod_{l=j}^{n+2T} q_{-l}} = \sum_{j=n+T+1}^{n+2T} \frac{f_{T-j}}{\prod_{l=j}^{n+2T} q_{-l}} \\ &= \sum_{k=n+1}^{n+T} \frac{f_{-k}}{\prod_{l=k+T}^{n+2T} q_{-l}} = \sum_{k=n+1}^{n+T} \frac{f_{-k}}{\prod_{l=k+T}^{n+2T} q_{T-l}} \\ &= \sum_{k=n+1}^{n+T} \frac{f_{-k}}{\prod_{s=k}^{n+T} q_{-s}} = c_n, \end{aligned}$$

we see that c_n is a T-periodic sequence for $n \in \mathbb{N}$.

From (65) we see that if $\tilde{x}_{-n}$ is the periodic solution to equation (31), then it must be

$$\tilde{x}_{-n} = \frac{c_n}{\lambda^{-1} - 1}, \quad n \in \mathbb{N}_0. \tag{67}$$

Also, we have

$$x_{-(mT+l)} = \lambda^{-m} x_{-l} - c_l \sum_{j=0}^{m-1} \lambda^{-j} \tag{68}$$

for $m \in \mathbb{N}$.

Hence, since $\lambda \neq 1$, we have

$$\begin{aligned} |x_{-(mT+l)} - \tilde{x}_{-(mT+l)}| &= \left| \lambda^{-m} x_{-l} - c_l \sum_{j=0}^{m-1} \lambda^{-j} - \frac{c_l}{\lambda^{-1} - 1} \right| \\ &= |\lambda|^{-m} \left| x_{-l} - \frac{c_l}{\lambda^{-1} - 1} \right| \end{aligned} \tag{69}$$

for every $m, l \in \mathbb{N}_0$.

Let

$$\widehat{C}_x := \max_{l=\overline{0,T-1}} \left(\sqrt[T]{|\lambda|} \right)^l \left| x_{-l} - \frac{c_l}{\lambda^{-1} - 1} \right|.$$

Then from (69) it follows that

$$|x_{-n} - \tilde{x}_{-n}| \le \frac{\widehat{C}_x}{(\sqrt[T]{|\lambda|})^n}. \tag{70}$$

If $|\lambda| > 1$, then from (70) it follows that all the solutions to equation (31) converge geometrically to the periodic solution to the equation as $n \to +\infty$. Now note that if $x_{-n_0} = \tilde{x}_{-n_0}$ for some $n_0 \in \mathbb{N}$, then it will be $x_{-n} = \tilde{x}_{-n}$ for $n \ge n_0$. Moreover, since $q_{-n} \neq 0$, $n \in \mathbb{N}_0$, then we have $x_{-n} = \tilde{x}_{-n}$ for every $n \in \mathbb{N}_0$. So, for an arbitrary solution x_{-n} different from $\tilde{x}_{-n}$, it must be $x_{-n} \neq \tilde{x}_{-n}$ for every $n \in \mathbb{N}_0$. Hence, if $0 < |\lambda| < 1$, then from (69) it follows that

$$|x_{-n} - \tilde{x}_{-n}| \geq \left(\sqrt[T]{|\lambda|}\right)^{-n} \min_{l=\overline{0,T-1}} \left(\sqrt[T]{|\lambda|}\right)^{l} \left|x_{-l} - \frac{c_l}{\lambda^{-1} - 1}\right| > 0, \quad n \in \mathbb{N}_0,$$

and consequently, every solution x_{-n} different from $\tilde{x}_{-n}$ gets away geometrically from the periodic solution as $n \to +\infty$, which finishes the proof of the theorem. □

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author has contributed solely to the writing of this paper. He read and approved the manuscript.

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Received: 14 July 2017 Accepted: 4 September 2017 Published online: 12 September 2017

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