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# Local uniqueness of positive solutions for a coupled system of fractional differential equations with integral boundary conditions

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**Abstract**
In this paper, we study a coupled system of fractional boundary value problems subject to integral boundary conditions. By applying a recent fixed point theorem in ordered Banach spaces, we investigate the local existence and uniqueness of positive solutions for the coupled system. We show that the unique positive solution can be found in a product set, and that it can be approximated by constructing iterative sequences for any given initial point of the product set. As an application, an interesting example is presented to illustrate our main result.

**MSC:** 34A08; 26A33; 34B18; 34B27

**Keywords:** local existence and uniqueness; positive solutions; coupled system of fractional boundary value problems; integral boundary conditions

## 1 Introduction

In this paper we discuss the local existence and uniqueness of positive solutions for the following coupled system of fractional boundary value problem subject to integral boundary conditions:

$$\begin{cases} D^{\alpha}u(t)+f(t,v(t))=0, \qquad D^{\beta}v(t)+g(t,u(t))=0, \quad 0<t<1, \\ u(0)=0, \qquad u(1)=\int_0^1\phi(t)u(t)\,dt, \qquad v(0)=0, \qquad v(1)=\int_0^1\psi(t)v(t)\,dt, \end{cases} \tag{1.1}$$

where $1<\alpha,\ \beta\leq 2$, $\phi,\psi\in L^1[0,1]$ are nonnegative and $f,g\in C([0,1]\times[0,+\infty),[0,+\infty))$ and $D$ is the standard Riemann-Liouville fractional derivative. By a positive solution of the problem (1.1), we mean a pair of functions $(u,v)\in C([0,1])\times C([0,1])$ satisfying (1.1) with $u(t)\geq 0$, $v(t)\geq 0$, $t\in[0,1]$ and $(u,v)\neq(0,0)$. The functions $\phi(t)$, $\psi(t)$ satisfy the following conditions:

$$(Q)\quad \phi,\psi:[0,1]\to[0,+\infty) \quad \text{with } \phi,\psi\in L^1[0,1] \quad \text{and}$$

$$\sigma_1:=\int_0^1\phi(t)t^{\alpha-1}\,dt, \qquad \sigma_2:=\int_0^1\psi(t)t^{\beta-1}\,dt\in(0,1);$$

$$\sigma_3:=\int_0^1 t^{\alpha-1}(1-t)\phi(t)\,dt, \qquad \sigma_4:=\int_0^1 t^{\beta-1}(1-t)\psi(t)\,dt>0.$$

Recently, coupled systems of fractional differential equations with a variety of boundary value conditions have been studied by many people; see [1–28] and the references therein. As is well known, coupled systems with boundary conditions appear in the investigations of many problems such as reaction-diffusion equations and Sturm-Liouville problems (see [29, 30]), and they have many applications in different fields of sciences and engineering (see heat equations [31–33], steady-state heat flow and beam deformation [34, 35] for example), mathematical biology (see [36, 37]) and so on. So the subject of coupled systems is gaining much attention and importance. From the literature, we can see that there are a large number of articles dealing with the existence or multiplicity of solutions or positive solutions for some nonlinear coupled systems with boundary conditions; see [2, 3, 5, 8, 10–13, 16, 17, 19, 22–24, 26–28] for details.

In [19], Su considered the following two-point boundary value problem for a coupled system of fractional differential equations:

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), \qquad D^{\beta}v(t) = g(t, u(t), D^{\nu}u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases} \tag{1.2}$$

where $1 < \alpha, \beta < 2, \mu, \nu > 0, \alpha - \nu \geq 1, \beta - \mu \geq 1, f, g : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are given functions, and $D$ is also the standard Riemann-Liouville fractional derivative. By applying Schauder fixed point theorem, the author established sufficient conditions for the existence of solutions for the problem (1.2). By using the same method, Ahmad and Nieto [2] extended the results of [19] to a three-point boundary value problem for a coupled system of fractional differential equations. By using Banach fixed point theorem and nonlinear alternative of Leray-Schauder type, Wang *et al.* [21] gave the existence and uniqueness of positive solutions to the following boundary values problem for a coupled system of nonlinear fractional differential equations:

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t)), \qquad D^{\beta}v(t) = g(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = au(\xi), \qquad v(0) = 0, \qquad v(1) = bv(\xi), \end{cases}$$

where $1 < \alpha, \beta < 2, 0 \leq a, b \leq 1, 0 < \xi < 1, f, g : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are given functions, and $D$ is also the standard Riemann-Liouville fractional derivative.

In [23], Yang studied the boundary values problem for a coupled system of nonlinear fractional differential equations as follows:

$$\begin{cases} D^{\alpha}u(t) + a(t)f(t, v(t)) = 0, \qquad D^{\beta}v(t) + b(t)g(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \int_0^1 \phi(t)u(t)\,dt, \qquad v(0) = 0, \qquad v(1) = \int_0^1 \psi(t)v(t)\,dt, \end{cases} \tag{1.3}$$

where $1 < \alpha, \beta \leq 2, a, b \in C((0,1), [0,+\infty)), \phi, \psi \in L^1[0,1]$ are nonnegative and $f, g \in C([0,1] \times [0,+\infty), [0,+\infty))$, and $D$ is also the standard Riemann-Liouville fractional derivative. The existence and nonexistence of positive solutions were shown by applying Banach fixed point theorem, nonlinear alternative of Leray-Schauder type and the fixed point theorems of cone expansion and compression of norm type. Very recently, in [38], the authors studied a boundary value problem of coupled systems of nonlinear Riemann-Liouvillle fractional integro-differential equations supplemented with nonlocal Riemann-

Liouvillle fractional integro-differential boundary conditions. And in [39], the authors introduced a new concept of coupled non-separated boundary conditions and solved a coupled system of fractional differential equations supplemented with these conditions. By using Banach's contraction principle and Leray-Schauder's alternative, the authors gave some new existence and uniqueness results in [38, 39]. To the best of our knowledge, there are still very few papers (such as [20, 21]) considered the uniqueness of positive solutions of boundary value problems with fractional coupled systems.

Motivated greatly by the above mentioned work and [23, 40], we consider the local existence and uniqueness of positive solutions for the coupled system (1.1). To prove our main results, we present some definitions, notations and lemmas in Section 2. And we give some new properties of the corresponding Green's function for the system (1.1). In Section 3, we give sufficient conditions for the local existence and uniqueness of positive solutions for the system (1.1) by using a recent fixed point theorem in ordered Banach spaces. To demonstrate our result, we give an interesting example in Section 4.

## 2 Preliminaries

**Definition 2.1** (See [41, 42]) The fractional integral of order $q$ with the lower limit $a$ for a function $f$ is given as

$$I_{a^+}^{q} f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s)\,ds, \quad t > a, q > 0, \tag{2.1}$$

provided the right-hand side is pointwise defined on $[a,\infty)$, here $f \in C[a,b]$ and $\Gamma$ is the gamma function. For $a = 0$, the fractional integral (2.1) can be written by $I_{0^+}^{\alpha} h(t) = h(t) * \varphi_\alpha(t)$, here $\varphi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \le 0$.

**Definition 2.2** (See [41, 42]) Riemann-Liouville derivative of order $q$ with the lower limit $a$ for a function $f : [a,\infty) \to \mathbf{R}$ is defined as

$$D_{a^+}^{q} f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} f(s)\,ds, \quad t > a, n-1 < q < n.$$

**Lemma 2.1** (See [10]) *If* $\int_0^1 \phi(t)t^{\alpha-1}\,dt \neq 1$, *then, for any* $\sigma \in C[0,1]$, *the unique solution of the following boundary value problem*:

$$\begin{cases} D^\alpha u(t) + \sigma(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \int_0^1 \phi(t)u(t)\,dt, \end{cases}$$

*is given by*

$$u(t) = \int_0^1 G_{1\alpha}(t,s)\sigma(s)\,ds,$$

*where*

$$G_{1\alpha}(t,s) = G_{2\alpha}(t,s) + G_{3\alpha}(t,s),$$

$$
G_{2\alpha}(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}
$$

$$
G_{3\alpha}(t,s) = \frac{t^{\alpha-1}}{1-\int_0^1 \phi(t)t^{\alpha-1}\,dt}\int_0^1 \phi(t)G_{2\alpha}(t,s)\,dt.
$$

*Then* $G(t,s) = (G_{1\alpha}(t,s), G_{1\beta}(t,s))$ *is called the Green's function of the system* (1.1).

**Lemma 2.2** (See [10]) *If* $\int_0^1 \phi(t)t^{\alpha-1}\,dt \in [0,1)$, *then* $G_{1\alpha}(t,s)$ *given by* (2.2) *satisfies* $G_{1\alpha}(t,s) \ge 0$ *is continuous for all* $t,s \in [0,1]$, $G_{1\alpha}(t,s) > 0$ *for all* $t,s \in (0,1)$.

**Lemma 2.3** *The function* $G_{2\alpha}(t,s)$ *has the following properties*:

$$
G_{2\alpha}(t,s) \ge \frac{\alpha-1}{\Gamma(\alpha)}t^{\alpha-1}(1-t)(1-s)^{\alpha-1}s, \quad t,s \in [0,1] \ (\textit{Theorem } 1.1 \textit{ of } [43]);
$$

$$
G_{2\alpha}(t,s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad t,s \in [0,1].
$$

*Proof* The second inequality is obvious. □

**Lemma 2.4** *Let* $\alpha, \beta \in (1,2]$. *Assume that* $(Q)$ *holds. Then the functions* $G_{1\alpha}(t,s)$, $G_{1\beta}(t,s)$ *have the following properties*:

$$
\frac{(\alpha-1)\sigma_3 s(1-s)^{\alpha-1}t^{\alpha-1}}{(1-\sigma_1)\Gamma(\alpha)} \le G_{1\alpha} \le \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)(1-\sigma_1)}, \quad t,s \in [0,1];
$$

$$
\frac{(\beta-1)\sigma_4 s(1-s)^{\beta-1}t^{\beta-1}}{(1-\sigma_2)\Gamma(\beta)} \le G_{1\beta} \le \frac{(1-s)^{\beta-1}t^{\beta-1}}{\Gamma(\beta)(1-\sigma_2)}, \quad t,s \in [0,1].
$$

*Proof* We only prove the first inequality. From Lemma 2.1 and Lemma 2.3,

$$
\begin{aligned}
G_{1\alpha}(t,s) &= G_{2\alpha}(t,s) + G_{3\alpha}(t,s) \ge G_{3\alpha}(t,s) = \frac{t^{\alpha-1}}{1-\sigma_1}\int_0^1 \phi(t)G_{2\alpha}(t,s)\,dt \\
&\ge \frac{t^{\alpha-1}}{1-\sigma_1}\int_0^1 \phi(t)\cdot\frac{\alpha-1}{\Gamma(\alpha)}t^{\alpha-1}(1-t)(1-s)^{\alpha-1}s\,dt \\
&= \frac{(\alpha-1)t^{\alpha-1}s(1-s)^{\alpha-1}}{(1-\sigma_1)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)\phi(t)\,dt = \frac{(\alpha-1)\sigma_3 s(1-s)^{\alpha-1}t^{\alpha-1}}{(1-\sigma_1)\Gamma(\alpha)}.
\end{aligned}
$$

Also, from Lemma 2.3,

$$
\begin{aligned}
G_{1\alpha}(t,s) &= G_{2\alpha}(t,s) + G_{3\alpha}(t,s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{1-\sigma_1}\int_0^1 \phi(t)\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,dt \\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-1} + \frac{(1-s)^{\alpha-1}}{1-\sigma_1}\int_0^1 t^{\alpha-1}\phi(t)\,dt\right] \\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-1} + \frac{(1-s)^{\alpha-1}\sigma_1}{1-\sigma_1}\right] = \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)(1-\sigma_1)}.
\end{aligned}
$$
□

In the sequel, we list some definitions, notations in ordered Banach spaces and preliminary facts which will be used later. For details, see [40, 44–46].

Let $(E, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. $\theta$ is the zero element of $E$. If there is a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$, then $P$ is called normal, in this case $N$ is the infimum of such constants, it is called the normality constant of $P$. We say that an operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (*i.e.*, $h \geq \theta$ and $h \neq \theta$), we define the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Let $\mathbf{\Phi}$ denote the class of those functions $\varphi : (0,1) \to (0,1)$ which satisfies the condition $\varphi(t) > t$ for $t \in (0,1)$.

**Lemma 2.5** (Theorem 2.1 of [40]) *Let $P$ be a normal cone in a real Banach space $E$, $h > \theta$. $T : P \to P$ is an increasing operator which satisfies*:

(i) *there is $h_0 \in P_h$ such that $Th_0 \in P_h$;*

(ii) *for any $x \in P$ and $t \in (0,1)$, there exists $\varphi \in \mathbf{\Phi}$ such that $T(tx) \geq \varphi(t)Tx$.*

*Then*:

(1) *the operator $T$ has a unique fixed point $x^*$ in $P_h$;*

(2) *for any initial value $x_0 \in P_h$, constructing successively the sequence $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, we have $x_n \to x^*$ as $n \to \infty$.*

## 3 Local existence and uniqueness of positive solutions

Throughout this section, we work in the product space $X \times X$, where $X = \{u(t) | u(t) \in C[0,1]\}$ endowed with the norm $\|u\|_X = \max_{t\in[0,1]} |u(t)|$. For $(u,v) \in X \times X$, let $\|(u,v)\|_{X\times X} = \max\{\|u\|_X, \|v\|_X\}$. Evidently, $(X \times X, \|(u,v)\|_{X\times X})$ is a Banach space. Define $K = \{(u,v) \in X \times X | u(t) \geq 0, v(t) \geq 0\}$, $P = \{u \in X | u(t) \geq 0, t \in [0,1]\}$, then the cone $K \subset X \times X$ and $K = P \times P$ is normal, and the space $X \times X$ can be equipped with a partial order:

$$(u_1, v_1) \leq (u_2, v_2) \Leftrightarrow u_1(t) \leq u_2(t), \quad v_1(t) \leq v_2(t), t \in [0,1].$$

From Lemma 2.1 and the discussion of [23], we can obtain the following fact.

**Lemma 3.1** *Assume that* (Q) *holds and $f(t,x), g(t,x)$ are continuous, then $(u,v) \in X \times X$ is a solution of the system* (1.1) *if and only if $(u,v) \in X \times X$ is a solution of the integral equations*

$$\begin{cases} u(t) = \int_0^1 G_{1\alpha}(t,s) f(s, v(s))\, ds, \\ v(t) = \int_0^1 G_{1\beta}(t,s) g(s, u(s))\, ds. \end{cases}$$

*For $(u,v) \in X \times X$, define operators $T_1$, $T_2$ and $T$ by*

$$T_1 u(t) = \int_0^1 G_{1\alpha}(t,s) f\bigl(s, v(s)\bigr)\, ds, \qquad T_2 v(t) = \int_0^1 G_{1\beta}(t,s) g\bigl(s, u(s)\bigr)\, ds,$$

$$T(u,v)(t) = \left( \int_0^1 G_{1\alpha}(t,s) f\bigl(s, v(s)\bigr)\, ds, \int_0^1 G_{1\beta}(t,s) g\bigl(s, u(s)\bigr)\, ds \right).$$

*Then $T_1, T_2 : X \to X$ and $T : X \times X \to X \times X$. Moreover,*

$$T(u,v)(t) = \bigl(T_1 u(t), T_2 v(t)\bigr). \tag{3.1}$$

*It follows from Lemma* 3.1 *that the fixed point of operator T coincides with the solution of the system* (1.1).

**Theorem 3.1** *Assume that* (*Q*) *and the following conditions hold*:

($H_1$) $f, g \in C([0,1] \times [0,+\infty), [0,+\infty))$ *and* $f(t,0), g(t,0) \not\equiv 0$;
($H_2$) $f(t,u_1) \le f(t,u_2)$, $g(t,u_1) \le g(t,u_2)$ *for any* $t \in [0,1]$, $u_2 \ge u_1 \ge 0$;
($H_3$) *there exist* $\varphi_1, \varphi_2 \in \Phi$ *such that*

$$f(t,\lambda u) \ge \varphi_1(\lambda) f(t,u), \qquad g(t,\lambda u) \ge \varphi_2(\lambda) g(t,u)$$

*for* $t \in [0,1]$, $u \in [0,+\infty)$, *where* $\lambda \in (0,1)$.

Then the problem (1.1) has a unique positive solution $(u^*, v^*)$ in $K_h$, where

$$h(t) = \bigl(h_1(t), h_2(t)\bigr) = \bigl(t^{\alpha-1}, t^{\beta-1}\bigr), \quad t \in [0,1].$$

Moreover, for $(u_0, v_0) \in K_h$, the sequences $\{u_n\}$ and $\{v_n\}$ converge to $u^*$ and $v^*$, respectively, where

$$u_{n+1}(t) = \int_0^1 G_{1\alpha}(t,s) f\bigl(s, v_n(s)\bigr)\, ds,$$
$$v_{n+1}(t) = \int_0^1 G_{1\beta}(t,s) g\bigl(s, u_n(s)\bigr)\, ds, \quad n = 1, 2, \ldots,$$

we have $u_{n+1}(t) \to u^*(t)$, $v_{n+1}(t) \to v^*(t)$ as $n \to \infty$.

**Lemma 3.2** $K_h = P_{h_1} \times P_{h_2}$, *where* $h = (h_1, h_2)$.

*Proof* From Section 2, we know that

$$K_h = \bigl\{(x,y) : \text{there exist } \lambda(x,y), \mu(x,y) > 0 \text{ such that } \lambda(h_1,h_2) \le (x,y) \le \mu(h_1,h_2)\bigr\}.$$

For $(x,y) \in P_{h_1} \times P_{h_2}$, we know $x \in P_{h_1}$, $y \in P_{h_2}$. Then there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$, such that $\lambda_1 h_1 \le x \le \mu_1 h_1$, $\lambda_2 h_2 \le y \le \mu_2 h_2$. Let $\lambda = \min\{\lambda_1, \lambda_2\}$, $\mu = \max\{\mu_1, \mu_2\}$. Then

$$\lambda(h_1,h_2) = (\lambda h_1, \lambda h_2) \le (x,y) \le (\mu_1 h_1, \mu_2 h_2) \le (\mu h_1, \mu h_2) = \mu(h_1,h_2).$$

That is, $(x,y) \in K_h$ and thus $P_{h_1} \times P_{h_2} \subset K_h$.

Conversely, for $(x,y) \in K_h$, there exist $\lambda, \mu > 0$ such that $\lambda h \le (x,y) \le \mu h$. That is,

$$(\lambda h_1, \lambda h_2) = \lambda(h_1,h_2) \le (x,y) \le \mu(h_1,h_2) = (\mu h_1, \mu h_2).$$

So we have $\lambda h_1 \le x \le \mu h_1$, $\lambda h_2 \le y \le \mu h_2$. That is, $x \in P_{h_1}$, $y \in P_{h_2}$. Hence, $(x,y) \in P_{h_1} \times P_{h_2}$. Consequently, $K_h \subset P_{h_1} \times P_{h_2}$. Therefore, $K_h = P_{h_1} \times P_{h_2}$. □

*Proof of Theorem* 3.1 Consider the operator $T$ defined in (3.1). From Lemma 3.1, we know that $(u,v) \in K$ is a positive solution of (1.1) if and only if $(u,v)$ is a positive fixed point of $T$. By using Lemma 2.2 and ($H_1$), we get $T_1 : P \to P$, $T_2 : P \to P$. And thus $T : K \to K$.

Firstly, we prove that $T : K \to K$ is increasing. For $(u_1, v_1), (u_2, v_2) \in K$ with $(u_1, v_1) \leq (u_2, v_2)$, we know that $u_1(t) \leq u_2(t)$, $v_1(t) \leq v_2(t)$, and by using Lemma 2.2 and $(H_2)$,

$$T_1u_1(t) = \int_0^1 G_{1\alpha}(t,s)f\bigl(s, v_1(s)\bigr)\,ds \leq \int_0^1 G_{1\alpha}(t,s)f\bigl(s, v_2(s)\bigr)\,ds = T_1u_2(t),$$

$$T_2v_1(t) = \int_0^1 G_{1\beta}(t,s)g\bigl(s, u_1(s)\bigr)\,ds \leq \int_0^1 G_{1\beta}(t,s)g\bigl(s, u_2(s)\bigr)\,ds = T_2v_2(t).$$

Thus,

$$T(u_1, v_1)(t) = \bigl(T_1u_1(t), T_2v_1(t)\bigr) \leq \bigl(T_1u_2(t), T_2v_2(t)\bigr) = T(u_2, v_2)(t).$$

So that $T : K \to K$ is increasing.

In the sequel, we show that $T$ satisfies the two conditions of Lemma 2.5. From $(H_3)$, for any $\lambda \in (0, 1)$ and $(u, v) \in K$, we have

$$T_1(\lambda u)(t) = \int_0^1 G_{1\alpha}(t,s)f\bigl(s, \lambda v(s)\bigr)\,ds \geq \varphi_1(\lambda)\int_0^1 G_{1\alpha}(t,s)f\bigl(s, v(s)\bigr)\,ds = \varphi_1(\lambda)T_1u(t),$$

$$T_2(\lambda v)(t) = \int_0^1 G_{1,\beta}(t,s)g\bigl(s, \lambda u(s)\bigr)\,ds \geq \varphi_2(\lambda)\int_0^1 G_{1\beta}(t,s)g\bigl(s, u(s)\bigr)\,ds = \varphi_2(\lambda)T_2v(t),$$

and thus

$$T\bigl(\lambda(u, v)\bigr)(t) = T(\lambda u, \lambda v)(t) = \bigl(T_1(\lambda u)(t), T_2(\lambda v)(t)\bigr) \geq \bigl(\varphi_1(\lambda)T_1u(t), \varphi_2(\lambda)T_2v(t)\bigr).$$

Let $\varphi(t) = \min\{\varphi_1(t), \varphi_2(t)\}, t \in (0, 1)$. Then $\varphi \in \mathbf{\Phi}$ and

$$T\bigl(\lambda(u, v)\bigr) \geq \bigl(\varphi(\lambda)T_1u, \varphi(\lambda)T_2v\bigr) = \varphi(\lambda)(T_1u, T_2v) = \varphi(\lambda)T(u, v), \quad \lambda \in (0, 1).$$

Hence, the second condition of Lemma 2.5 holds.

Next we prove that the first condition of Lemma 2.5 also holds. To this aim, we take $h_0(t) = h(t) = (h_1(t), h_2(t))$, where $h_1(t) = t^{\alpha-1}$, $h_2(t) = t^{\beta-1}$, $t \in [0, 1]$. From Lemma 3.2, we only need prove $T_1h_1 \in P_{h_1}$, $T_2h_2 \in P_{h_2}$. By using Lemma 2.4 and $(H_1)$, $(H_2)$,

$$\begin{aligned} T_1h_1(t) &= \int_0^1 G_{1\alpha}(t,s)f\bigl(s, h_2(s)\bigr)\,ds \\ &\geq \int_0^1 \frac{(\alpha-1)\sigma_3 s(1-s)^{\alpha-1}}{(1-\sigma_1)\Gamma(\alpha)}t^{\alpha-1}f\bigl(s, s^{\beta-1}\bigr)\,ds \\ &\geq \frac{(\alpha-1)\sigma_3}{(1-\sigma_1)\Gamma(\alpha)}h_1(t)\int_0^1 s(1-s)^{\alpha-1}f(s, 0)\,ds \end{aligned}$$

and

$$\begin{aligned} T_1h_1(t) &\leq \int_0^1 \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)(1-\sigma_1)}f(s, 1)\,ds \\ &= \frac{1}{\Gamma(\alpha)(1-\sigma_1)}h_1(t)\int_0^1 (1-s)^{\alpha-1}f(s, 1)\,ds. \end{aligned}$$

From $(H_2)$, we obtain $f(s,1) \geq f(s,0) \geq 0$, $s \in [0,1]$. Because $f(s,0) \not\equiv 0$, we have

$$s(1-s)^{\alpha-1}f(s,0) \not\equiv 0, \qquad (1-s)^{\alpha-1}f(s,1) \not\equiv 0.$$

Hence,

$$\int_0^1 (1-s)^{\alpha-1}f(s,1)\,ds \geq \int_0^1 s(1-s)^{\alpha-1}f(s,0)\,ds > 0.$$

Note that $\sigma_3 \leq \sigma_1 < 1$ and $\alpha - 1 \leq 1$, we get

$$l_1 := \frac{(\alpha-1)\sigma_3}{(1-\sigma_1)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}f(s,0)\,ds \leq l_2 := \frac{1}{(1-\sigma_1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}f(s,1)\,ds.$$

So we have $l_2 \geq l_1 > 0$ and $l_1 h_1(t) \leq T_1 h_1(t) \leq l_2 h_1(t), t \in [0,1]$; and thus $T_1 h_1 \in P_{h_1}$. Similarly, by using Lemma 2.4 and $(H_1)$, $(H_2)$, we also can prove $T_2 h_2 \in P_{h_2}$. Therefore,

$$Th = T(h_1,h_2) = (T_1h_1, T_2h_2) \in P_{h_1} \times P_{h_2} = K_h.$$

Consequently, by Lemma 2.5, there exists a unique $x^* \in K_h$ such that $Tx^* = x^*$, and for any $x_0 \in K_h$, construct a sequence $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$, we have $x_n \to x^*$ as $n \to \infty$.

Set $x^* = (u^*, v^*)$, $x_0 = (u_0, v_0)$. Then we see that $(u^*, v^*)$ is the unique positive solution of the system (1.1) in $K_h$, and the sequences

$$u_{n+1}(t) = \int_0^1 G_{1\alpha}(t,s) f\bigl(s, v_n(s)\bigr)\,ds \to u^*(t),$$
$$v_{n+1}(t) = \int_0^1 G_{1\beta}(t,s) g\bigl(s, u_n(s)\bigr)\,ds \to v^*(t),$$

as $n \to \infty$. □

## 4 An example

**Example 4.1** Consider the following coupled system of fractional differential equations:

$$\begin{cases} D^{\frac{3}{2}}u(t) + [v(t)]^{\tau_1} + a_1(t) = 0, \qquad D^{\frac{4}{3}}v(t) + [u(t)]^{\tau_2} + a_2(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \int_0^1 t^2 u(t)\,dt, \qquad v(0) = 0, \qquad v(1) = \int_0^1 t v(t)\,dt, \end{cases} \tag{4.1}$$

where $\tau_1, \tau_2 \in (0,1)$, $a_1, a_2 : [0,1] \to [0,+\infty)$ are continuous with $a_i \not\equiv 0$. In this example, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$ and

$$f(t,x) = x^{\tau_1} + a_1(t), \qquad g(t,x) = x^{\tau_2} + a_2(t), \qquad \phi(t) = t^2, \qquad \psi(t) = t.$$

After a simple computation, we have

$$\sigma_1 = \int_0^1 t^{\alpha-1}\phi(t)\,dt = \int_0^1 t^{\frac{1}{2}} \cdot t^2\,dt = \int_0^1 t^{\frac{5}{2}}\,dt = \frac{2}{7},$$
$$\sigma_2 = \int_0^1 t^{\beta-1}\psi(t)\,dt = \int_0^1 t^{\frac{1}{3}} \cdot t\,dt = \int_0^1 t^{\frac{4}{3}}\,dt = \frac{3}{7},$$

$$\sigma_3 = \int_0^1 t^{\frac{1}{2}}(1-t)t^2\,dt = \int_0^1 \left(t^{\frac{5}{2}} - t^{\frac{7}{2}}\right)dt = \frac{4}{63},$$
$$\sigma_4 = \int_0^1 t^{\frac{1}{3}}(1-t)t\,dt = \int_0^1 \left(t^{\frac{4}{3}} - t^{\frac{7}{3}}\right)dt = \frac{9}{70}.$$

Evidently, $\sigma_1, \sigma_2 \in (0,1)$, $\sigma_3, \sigma_4 > 0. f, g \in C([0,1]\times[0,+\infty),[0,+\infty))$ and $f(t,0) = a_1(t) \not\equiv 0$, $g(t,0) = a_2(t) \not\equiv 0$. Note that $x^{\tau_i}$ $(i = 1,2)$ are increasing in $[0,+\infty)$, we have $f(t,x)$, $g(t,x)$ are increasing in $x \in [0,+\infty)$. Hence, $(H_1)$, $(H_2)$ are satisfied. In addition, let

$$\varphi_1(t) = t^{\tau_1}, \qquad \varphi_2(t) = t^{\tau_2}, \quad t \in (0,1).$$

Then $\varphi_1, \varphi_2 \in \mathbf{\Phi}$ and, for $\lambda \in (0,1)$ and $x \in [0,+\infty)$,

$$f(t,\lambda x) = \lambda^{\tau_1}x^{\tau_1} + a_1(t) \geq \lambda^{\tau_1}\left[x^{\tau_1} + a_1(t)\right] = \varphi_1(\lambda)f(t,x);$$
$$g(t,\lambda x) = \lambda^{\tau_2}x^{\tau_2} + a_2(t) \geq \lambda^{\tau_2}\left[x^{\tau_2} + a_2(t)\right] = \varphi_2(\lambda)g(t,x).$$

The condition $(H_3)$ in Theorem 3.1 also holds. Therefore, by Theorem 3.1, the boundary value problem (4.1) has a unique positive solution $(u^*, v^*)$ in $k_h$, where

$$h(t) = \left(h_1(t), h_2(t)\right) = \left(t^{\frac{1}{2}}, t^{\frac{1}{3}}\right), \quad t \in [0,1],$$

and for any given $(u_0, v_0) \in k_h$, making the sequences

$$u_{n+1}(t) = \int_0^1 G_{1\alpha}(t,s)\left\{\left[v_n(s)\right]^{\tau_1} + a_1(s)\right\}ds,$$
$$v_{n+1}(t) = \int_0^1 G_{1\beta}(t,s)\left\{\left[u_n(s)\right]^{\tau_2} + a_2(s)\right\}ds,$$

$n = 1,2,\ldots$, then we obtain $u_n(t) \to u^*(t)$, $v_n(t) \to v^*(t)$ as $n \to \infty$, where

$$G_{1\alpha}(t,s) = G_{2\alpha}(t,s) + G_{3\alpha}(t,s), G_{1\beta}(t,s) = G_{2\beta}(t,s) + G_{3\beta}(t,s),$$

$$G_{2\alpha}(t,s) = \begin{cases} \frac{t^{\frac{1}{2}}(1-s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}, & s \leq t, \\ \frac{t^{\frac{1}{2}}(1-s)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}, & t \leq s, \end{cases}$$

$$G_{2\beta}(t,s) = \begin{cases} \frac{t^{\frac{1}{3}}(1-s)^{\frac{1}{3}} - (t-s)^{\frac{1}{3}}}{\Gamma(\frac{4}{3})}, & s \leq t, \\ \frac{t^{\frac{1}{3}}(1-s)^{\frac{1}{3}}}{\Gamma(\frac{4}{3})}, & t \leq s, \end{cases}$$

$$\begin{aligned} G_{3\alpha}(t,s) &= \frac{t^{\frac{1}{2}}}{1-\sigma_1}\int_0^1 t^2 G_{2\alpha}(t,s)\,dt \\ &= \frac{t^{\frac{1}{2}}}{\frac{5}{7}\Gamma(\frac{3}{2})}\left\{\int_0^s t^2 \cdot t^{\frac{1}{2}}(1-s)^{\frac{1}{2}}\,dt + \int_s^1 t^2 \cdot \left[t^{\frac{1}{2}}(1-s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}\right]dt\right\} \\ &= \frac{7t^{\frac{1}{2}}}{5\Gamma(\frac{2}{3})}\left[\frac{2}{7}(1-s)^{\frac{1}{2}} - \frac{2}{3}(1-s)^{\frac{3}{2}} + \frac{8}{15}(1-s)^{\frac{5}{2}} - \frac{16}{105}(1-s)^{\frac{7}{2}}\right] \\ &= \frac{2t^{\frac{1}{2}}}{75\Gamma(\frac{3}{2})}\left[15(1-s)^{\frac{1}{2}} - 35(1-s)^{\frac{3}{2}} + 28(1-s)^{\frac{5}{2}} - 8(1-s)^{\frac{7}{2}}\right], \end{aligned}$$

and

$$\begin{aligned} G_{3\beta}(t,s) &= \frac{t^{\frac{1}{3}}}{1-\sigma_2}\int_0^1 tG_{2\beta}(t,s)\,dt \\ &= \frac{t^{\frac{1}{3}}}{\frac{4}{7}\Gamma(\frac{4}{3})}\left\{\int_0^s t\cdot t^{\frac{1}{3}}(1-s)^{\frac{1}{3}}\,dt + \int_s^1 t\big[t^{\frac{1}{3}}(1-s)^{\frac{1}{3}} - (t-s)^{\frac{1}{3}}\big]\,dt\right\} \\ &= \frac{7t^{\frac{1}{3}}}{4\Gamma(\frac{4}{3})}\left[\frac{3}{7}(1-s)^{\frac{1}{3}} - \frac{3}{4}(1-s)^{\frac{4}{3}} + \frac{9}{28}(1-s)^{\frac{7}{3}}\right] \\ &= \frac{3t^{\frac{1}{3}}}{16\Gamma(\frac{4}{3})}\big[4(1-s)^{\frac{1}{3}} - 7(1-s)^{\frac{4}{3}} + 3(1-s)^{\frac{7}{3}}\big]. \end{aligned}$$

**Remark 4.1** In Example 4.1, we replace $f$, $g$ by $f \equiv g \equiv 1$. Then the problem (4.1) has a unique solution $(u^*, v^*)$, where

$$\begin{aligned} u^*(t) &= \int_0^1 G_{1\alpha}(t,s)\,ds = \frac{2}{135\Gamma(\frac{3}{2})}\big(49t^{\frac{1}{2}} - 45t^{\frac{2}{3}}\big), \quad t \in [0,1], \\ v^*(t) &= \int_0^1 G_{1\beta}(t,s)\,ds = \frac{3}{160\Gamma(\frac{4}{3})}\big(49t^{\frac{1}{3}} - 40t^{\frac{4}{3}}\big), \quad t \in [0,1]. \end{aligned}$$

Further, we can easily obtain

$$\begin{aligned} &\frac{8}{135\Gamma(\frac{3}{2})}t^{\frac{1}{2}} \le u^*(t) \le \frac{98}{135\Gamma(\frac{3}{2})}t^{\frac{1}{2}}, \quad t \in [0,1], \\ &\frac{27}{160\Gamma(\frac{4}{3})}t^{\frac{1}{3}} \le v^*(t) \le \frac{147}{160\Gamma(\frac{4}{3})}t^{\frac{1}{3}}, \quad t \in [0,1]. \end{aligned}$$

So the unique solution $(u^*, v^*)$ is a positive solution and satisfies $(u^*, v^*) \in K_{(t^{\frac{1}{2}}, t^{\frac{1}{3}})}$.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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