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Sums of finite products of Genocchi functions

Taekyun Kim[1,2*], Dae San Kim[3], Lee Chae Jang[4] and Gwan-Woo Jang[2]

*Correspondence: tkkim@kw.ac.kr
[1]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300160, China
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In a previous work, it was shown that Faber-Pandharipande-Zagier and Miki's identities can be derived from a polynomial identity which in turn follows from a Fourier series expansion of sums of products of Bernoulli functions. Motivated by this work, we consider three types of sums of finite products of Genocchi functions and derive Fourier series expansions for them. Moreover, we will be able to express each of them in terms of Bernoulli functions.

MSC: 11B83; 42A16

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1 Introduction

As is well known, the *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}.$$

The *Genocchi polynomials* $G_m(x)$ are given by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!}\quad\text{(see [1–12])}.\tag{1.1}$$

The first few Genocchi polynomials are as follows:

$$\begin{aligned}&G_0(x)=0,\qquad G_1(x)=1,\qquad G_2(x)=2x-1,\\&G_3(x)=3x^2-3x,\qquad G_4(x)=4x^3-6x^2+1,\\&G_5(x)=5x^4-10x^3+5x,\qquad G_6(x)=6x^5-15x^4+15x^2-3.\end{aligned}\tag{1.2}$$

From the relation $G_m(x)=mE_{m-1}(x)(m\geq 1)$, the following facts are obtained:

$$\begin{aligned}&\deg G_m(x)=m-1\quad(m\geq 1),\qquad G_m=mE_{m-1}\quad(m\geq 1),\\&G_0=0,\qquad G_1=1,\qquad G_{2m+1}=0\quad(m\geq 1),\quad\text{and}\quad G_{2m}\neq 0\quad(m\geq 1).\end{aligned}\tag{1.3}$$

In addition, we have

$$\begin{aligned} &\frac{d}{dx}G_m(x) = mG_{m-1}(x) \quad (m \geq 1), \\ &G_m(x+1) + G_m(x) = 2mx^{m-1} \quad (m \geq 0). \end{aligned} \tag{1.4}$$

From these, we immediately obtain

$$G_m(1) + G_m(0) = 2\delta_{m,1} \quad (m \geq 0) \tag{1.5}$$

and

$$\begin{aligned} \int_0^1 G_m(x)\,dx &= \frac{1}{m+1}\big(G_{m+1}(1) - G_{m+1}(0)\big) \\ &= \frac{2}{m+1}\big(-G_{m+1}(0) + \delta_{m,0}\big) \\ &= \begin{cases} 0 & \text{if } m \text{ is even}, \\ -\frac{2}{m+1}G_{m+1} & \text{if } m \text{ is odd}. \end{cases} \end{aligned} \tag{1.6}$$

For any real number x, we let $\langle x \rangle = x - [x] \in [0,1)$ denote the fractional part of x.

In this paper, we will consider three types of sums of finite products of Genocchi functions and derive the Fourier series expansions for them. Moreover, we will be able to express each of them in terms of Bernoulli functions $B_m(\langle x \rangle)$:

(1) $\alpha_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r \geq 1} G_{i_1}(\langle x \rangle) \cdots G_{i_r}(\langle x \rangle) (m > r \geq 1)$;
(2) $\beta_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r \geq 1} \frac{1}{i_1! \cdots i_r!} G_{i_1}(\langle x \rangle) \cdots G_{i_r}(\langle x \rangle) (m > r \geq 1)$;
(3) $\gamma_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r \geq 1} \frac{1}{i_1 \cdots i_r} G_{i_1}(\langle x \rangle) \cdots G_{i_r}(\langle x \rangle) (m > r \geq 1)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [13, 14]).

As to $\gamma_m(\langle x \rangle)$, we note that the polynomial identity (1.7) follows immediately from Theorems 4.1 and 4.2, which are in turn derived from the Fourier series expansion of $\gamma_m(\langle x \rangle)$.

$$\begin{aligned} &\sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r \geq 1} \frac{1}{i_1 i_2 \cdots i_r} G_{i_1}(x) G_{i_2}(x) \cdots G_{i_r}(x) \\ &\quad = \frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=1}^{m-r}\binom{m}{j}\Lambda_{m-j+1}B_j(x), \end{aligned} \tag{1.7}$$

where, for $l > r$,

$$\begin{aligned} \Lambda_l = {} &\sum_{0 \leq a \leq r}\binom{r}{a}(-1)^a 2^{r-a} \sum_{i_1+\cdots+i_a=l+a-r, i_1,\ldots,i_a \geq 1} \frac{G_{i_1}G_{i_2}\cdots G_{i_a}}{i_1 i_2 \cdots i_a} \\ &- \sum_{i_1+\cdots+i_r=l, i_1,\ldots,i_r \geq 1} \frac{G_{i_1}G_{i_2}\cdots G_{i_r}}{i_1 i_2 \cdots i_r}. \end{aligned} \tag{1.8}$$

The obvious polynomial identities can be derived also for $\alpha_m(\langle x \rangle)$ and $\beta_m(\langle x \rangle)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is noteworthy that from the

Fourier series expansion of the function

$$\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(\langle x\rangle)B_{m-k}(\langle x\rangle) \tag{1.9}$$

we can derive the Faber-Pandharipande-Zagier identity (see [1, 15–20]) and the Miki identity (see [17–23]). In case of $r = 2$, $\gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}G_k(\langle x\rangle)G_{m-k}(\langle x\rangle)$, and hence our problem here is a natural extension of the previous work, which leads to a simple proof for the important Faber-Pandharipande-Zagier and Miki identities (see [16, 22]). We will give an outline below, and this may be viewed as the main motivation for the present study.

The following polynomial identity follows immediately from the Fourier series expansion of the function in (1.9):

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(x)B_{m-k}(x)\\&\quad=\frac{2}{m^2}\left(B_m+\frac{1}{2}\right)+\frac{2}{m}\sum_{k=1}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k(x)+\frac{2}{m}H_{m-1}B_m(x)\quad(m\geq 2),\end{aligned} \tag{1.10}$$

where $H_m = \sum_{j=1}^{m}\frac{1}{j}$ are the harmonic numbers.

Simple modification of (1.10) yields

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k(2m-2k)}B_{2k}(x)B_{2m-2k}(x)+\frac{2}{2m-1}B_1(x)B_{2m-1}(x)\\&\quad=\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}(x)+\frac{1}{m}H_{2m-1}B_{2m}(x)\\&\qquad+\frac{2}{2m-1}B_1(x)B_{2m-1}\quad(m\geq 2).\end{aligned} \tag{1.11}$$

Letting $x = 0$ in (1.11) gives a slightly different version of the well-known Miki identity (see [22]):

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k(2m-2k)}B_{2k}B_{2m-2k}\\&\quad=\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}B_{2m-2k}+\frac{1}{m}H_{2m-1}B_{2m}\quad(m\geq 2).\end{aligned} \tag{1.12}$$

Setting $x = \frac{1}{2}$ in (1.12) with $\overline{B}_m = (\frac{1-2^{m-1}}{2^{m-1}})B_m = (2^{1-m}-1)B_m = B_m(\frac{1}{2})$, we have

$$\begin{aligned}&\sum_{k=1}^{m-1}\frac{1}{2k(2m-2k)}\overline{B}_{2k}\overline{B}_{2m-2k}\\&\quad=\frac{1}{m}\sum_{k=1}^{m}\frac{1}{2k}\binom{2m}{2k}B_{2k}\overline{B}_{2m-2k}+\frac{1}{m}H_{2m-1}\overline{B}_{2m}\quad(m\geq 2),\end{aligned} \tag{1.13}$$

which is the Faber-Pandharipande-Zagier identity (see [16]). Some of the different proofs of Miki's identity can be found in [15, 21–23]. Miki in [22] exploits a formula for the Fermat quotient $\frac{a^p-a}{p}$ modulo p^2, Shiratani-Yokoyama in [23] employs p-adic analysis, Gessel in [21] bases his work on two different expressions for Stirling numbers of the second kind $S_2(n,k)$, and Dunne-Schubert in [15] uses the asymptotic expansion of some special polynomials coming from the quantum field theory computations. As we can see, all of these proofs are quite involved. On the other hand, our proof of Miki's and Faber-Pandharipande-Zagier's identities follow from the polynomial identity (1.10), which in turn follows immediately from the Fourier series expansion of (1.9), together with the elementary manipulations outlined in (1.11)-(1.13). Some related recent work can be found in [10, 24–26].

2 The first type of sums of finite products

In this section, we will derive the Fourier series of the first type of sums of products of Genocchi functions. Let us denote

$$\alpha_m(x) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} G_{i_1}(x)G_{i_2}(x)\cdots G_{i_r}(x) \quad (m > r \geq 1). \tag{2.1}$$

Here the sum runs over all positive integers $i_1,\ldots,i_r$ with $i_1+\cdots+i_r = m, (m > r \geq 1)$. Note here that $\deg \alpha_m(x) = m - r \geq 1$. Then we will consider the function

$$\alpha_m\bigl(\langle x\rangle\bigr) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} G_{i_1}\bigl(\langle x\rangle\bigr)G_{i_2}\bigl(\langle x\rangle\bigr)\cdots G_{i_r}\bigl(\langle x\rangle\bigr) \quad (m > r \geq 1), \tag{2.2}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx}, \tag{2.3}$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m\bigl(\langle x\rangle\bigr)e^{-2\pi inx}\,dx = \int_0^1 \alpha_m(x)e^{-2\pi inx}\,dx. \tag{2.4}$$

Before proceeding, we note the following:

$$\begin{aligned}
\alpha_m'(x) &= \sum_{i_1+\cdots+i_r=m, i_1\geq 2, i_2,\ldots,i_r\geq 1} i_1 G_{i_1-1}(x)G_{i_2}(x)\cdots G_{i_r}(x) \\
&\quad + \cdots + \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_{r-1}\geq 1, i_r\geq 2} i_r G_{i_1}(x)\cdots G_{i_{r-1}}G_{i_r-1}(x) \\
&= \sum_{i_1+\cdots+i_r=m-1, i_1,\ldots,i_r\geq 1} (i_1+1)G_{i_1}(x)G_{i_2}(x)\cdots G_{i_r}(x) \\
&\quad + \cdots + \sum_{i_1+\cdots+i_r=m-1, i_1,\ldots,i_r\geq 1} (i_r+1)G_{i_1}(x)G_{i_2}(x)\cdots G_{i_r}(x) \\
&= (m+r-1) \sum_{i_1+\cdots+i_r=m-1, i_1,\ldots,i_r\geq 1} G_{i_1}(x)G_{i_2}(x)\cdots G_{i_r}(x) \\
&= (m+r-1)\alpha_{m-1}(x).
\end{aligned} \tag{2.5}$$

So, $\alpha_m'(x) = (m+r-1)\alpha_{m-1}(x)$, and from this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+r}\right)' = \alpha_m(x) \tag{2.6}$$

and

$$\int_0^1 \alpha_m(x)\,dx = \frac{1}{m+r}\bigl(\alpha_{m+1}(1) - \alpha_{m+1}(0)\bigr). \tag{2.7}$$

We put $\Delta_m = \alpha_m(1) - \alpha_m(0)$, for $m > r$. Then we have

$$\begin{aligned}\Delta_m &= \alpha_m(1) - \alpha_m(0) \\ &= \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \bigl(G_{i_1}(1)G_{i_2}(1)\cdots G_{i_r}(1) - G_{i_1}G_{i_2}\cdots G_{i_r}\bigr) \\ &= \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \bigl((-G_{i_1}+2\delta_{i_1,1})\cdots(-G_{i_r}+2\delta_{i_r,1}) - G_{i_1}\cdots G_{i_r}\bigr) \\ &= \sum_{0\leq a\leq r}\binom{r}{a}\sum_{i_1+\cdots+i_a=m+a-r, i_1,\ldots,i_a\geq 1}(-1)^a 2^{r-a} G_{i_1}\cdots G_{i_a} \\ &\quad - \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} G_{i_1}G_{i_2}\cdots G_{i_r},\end{aligned} \tag{2.8}$$

where we understand that, for $a = 0$, the inner sum is $2^r\delta_{m,r}$. Observe here that the sums over all $i_1+\cdots+i_r = m$ $(i_1,\ldots,i_r\geq 1)$ of any term with a of $-G_{i_e}$ and b of $2\delta_{i_f,1}$ $(1\leq e,f\leq r, a+b=r)$ all give

$$\begin{aligned}&\sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1}(-G_{i_1})\cdots(-G_{i_a})(2\delta_{i_{a+1},1})\cdots(2\delta_{i_{a+b},1}) \\ &\quad = \sum_{i_1+\cdots+i_a=m+a-r, i_1,\ldots,i_a\geq 1}(-1)^a 2^{r-a}G_{i_1}\cdots G_{i_a}.\end{aligned} \tag{2.9}$$

Note that, as $i_1+\cdots+i_a = m+a-r > a$, the above sum is not empty. From the definition of Δ_m, we have

$$\begin{aligned}&\alpha_m(0) = \alpha_m(1) \quad \Longleftrightarrow \quad \Delta_m = 0, \\ &\int_0^1 \alpha_m(x)\,dx = \frac{1}{m+r}\Delta_{m+1}.\end{aligned} \tag{2.10}$$

Now, we want to determine the Fourier coefficients $A_n^{(m)}$.

Case 1: $n \neq 0$. We have

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\bigl[\alpha_m(x)e^{-2\pi inx}\bigr]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\bigl(\alpha_m(1) - \alpha_m(0)\bigr) + \frac{m+r-1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}\,dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{m+r-1}{2\pi in} A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m \\
&= \frac{m+r-1}{2\pi in}\left(\frac{m+r-2}{2\pi in} A_n^{(m-2)} - \frac{1}{2\pi in}\Delta_{m-1}\right) - \frac{1}{2\pi in}\Delta_m \\
&= \frac{(m+r-1)_2}{(2\pi in)^2} A_n^{(m-2)} - \sum_{j=1}^{2} \frac{(m+r-1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} \\
&= \cdots \\
&= \frac{(m+r-1)_{m-r}}{(2\pi in)^{m-r}} A_n^{(r)} - \sum_{j=1}^{m-r} \frac{(m+r-1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} \\
&= -\frac{1}{m+r}\sum_{j=1}^{m-r} \frac{(m+r)_j}{(2\pi in)^j}\Delta_{m-j+1}, \qquad (2.11)
\end{aligned}
$$

where

$$A_n^{(r)} = \int_0^1 e^{-2\pi inx}\,dx = 0. \qquad (2.12)$$

Case 2: $n = 0$. We have

$$A_0^{(m)} = \int_0^1 \alpha_m(x)\,dx = \frac{1}{m+r}\Delta_{m+1}. \qquad (2.13)$$

We recall the following facts about Bernoulli functions $B_m(\langle x\rangle)$:

(a) for $m \geq 2$,

$$B_m\bigl(\langle x\rangle\bigr) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \qquad (2.14)$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x\rangle) & \text{for } x \in \mathbb{Z}^c, \\ 0 & \text{for } x \in \mathbb{Z}, \end{cases} \qquad (2.15)$$

where $\mathbb{Z}^c = \mathbb{R} - \mathbb{Z}$. $\alpha_m(\langle x\rangle)$ $(m > r \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those positive integers $m > r$ with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers $m > r$ with $\Delta_m \neq 0$. Assume first that $\Delta_m = 0$, for a positive integer $m > r$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise C^∞, and continuous. Thus, the Fourier series of $\alpha_m(\langle x\rangle)$ converges uniformly to $\alpha_m(\langle x\rangle)$, and

$$
\begin{aligned}
&\alpha_m\bigl(\langle x\rangle\bigr) \\
&= \frac{1}{m+r}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m+r}\sum_{j=1}^{m-r}\frac{(m+r)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\
&= \frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=1}^{m-r}\binom{m+r}{j}\Delta_{m-j+1}\left(-j! \sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)
\end{aligned}
$$

$$= \frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=2}^{m-r}\binom{m+r}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr)$$

$$+ \Delta_m \times \begin{cases} B_1(\langle x\rangle) & \text{for } x \in \mathbb{Z}^c, \\ 0 & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.16}$$

Now, we can state our first result.

Theorem 2.1 *For each positive integer l, with $l > r$, we let*

$$\Delta_l = \sum_{0\le a\le r}\binom{r}{a}\sum_{i_1+\cdots+i_a=l+a-r, i_1,\ldots,i_a\ge 1}(-1)^a 2^{r-a} G_{i_1}\cdots G_{i_a} - \sum_{i_1+\cdots+i_r=l, i_1,\ldots,i_r\ge 1} G_{i_1}\cdots G_{i_r}. \tag{2.17}$$

Assume that $\Delta_m = 0$, for a positive integer $m > r$. Then we have the following:

(a) $\sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} G_{i_1}(\langle x\rangle)\cdots G_{i_r}(\langle x\rangle)$ *has the Fourier series expansion*

$$\sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} G_{i_1}\bigl(\langle x\rangle\bigr)\cdots G_{i_r}\bigl(\langle x\rangle\bigr) = \frac{1}{m+r}\Delta_{m+1} + \sum_{n=-\infty, n\ne 0}^{\infty}\left(-\frac{1}{m+r}\sum_{j=1}^{m-r}\frac{(m+r)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}, \tag{2.18}$$

for all $x \in \mathbb{R}$, where the convergence is uniform,

(b)

$$\sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} G_{i_1}\bigl(\langle x\rangle\bigr)\cdots G_{i_r}\bigl(\langle x\rangle\bigr) = \frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=2}^{m-r}\binom{m+r}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr), \tag{2.19}$$

for all $x \in \mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.

Assume next that $\Delta_m \ne 0$, for a positive integer $m > r$. Then $\alpha_m(0) \ne \alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x\rangle)$ converges pointwise to $\alpha_m(\langle x\rangle)$, for $x \in \mathbb{Z}^c$, and it converges to

$$\frac{1}{2}\bigl(\alpha_m(0) + \alpha_m(1)\bigr) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.20}$$

for $x \in \mathbb{Z}$. Now, we can state our second result.

Theorem 2.2 *For each positive integer l, with $l > r$, we let*

$$\Delta_l = \sum_{0\le a\le r}\binom{r}{a}\sum_{i_1+\cdots+i_a=l+a-r, i_1,\ldots,i_a\ge 1}(-1)^a 2^{r-a} G_{i_1}\cdots G_{i_a} - \sum_{i_1+\cdots+i_r=l, i_1,\ldots,i_r\ge 1} G_{i_1}\cdots G_{i_r}. \tag{2.21}$$

Assume that $\Delta_m \neq 0$, *for a positive integer* $m > r$. *Then we have the following*:

(a)

$$\frac{1}{m+r}\Delta_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+r}\sum_{j=1}^{m-r}\frac{(m+r)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}$$
$$= \begin{cases} \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1} G_{i_1}(\langle x\rangle)\cdots G_{i_r}(\langle x\rangle) & \text{for } x\in\mathbb{Z}^c, \\ \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1} G_{i_1}\cdots G_{i_r} + \frac{1}{2}\Delta_m & \text{for } x\in\mathbb{Z}, \end{cases} \tag{2.22}$$

(b)

$$\frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=1}^{m-r}\binom{m+r}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr)$$
$$= \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1} G_{i_1}\bigl(\langle x\rangle\bigr)\cdots G_{i_r}\bigl(\langle x\rangle\bigr), \quad \text{for } x\in\mathbb{Z}^c, \tag{2.23}$$

$$\frac{1}{m+r}\Delta_{m+1} + \frac{1}{m+r}\sum_{j=2}^{m-r}\binom{m+r}{j}\Delta_{m-j+1}B_j\bigl(\langle x\rangle\bigr)$$
$$= \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1} G_{i_1}\cdots G_{i_r} + \frac{1}{2}\Delta_m, \quad x\in\mathbb{Z}. \tag{2.24}$$

3 The second type of sums of finite products

Let $\beta_m(x) = \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}(x)\cdots G_{i_r}(x)$ ($m > r \geq 1$). Here the sum runs over all positive integers $i_1,\ldots,i_r$ with $i_1+\cdots+i_r = m$ ($r \geq 1$). Then we will consider the function

$$\beta_m\bigl(\langle x\rangle\bigr) = \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}\bigl(\langle x\rangle\bigr)\cdots G_{i_r}\bigl(\langle x\rangle\bigr), \tag{3.1}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx}, \tag{3.2}$$

where

$$B_n^{(m)} = \int_0^1 \beta_m\bigl(\langle x\rangle\bigr)e^{-2\pi inx}\,dx = \int_0^1 \beta_m(x)e^{-2\pi inx}\,dx. \tag{3.3}$$

Before proceeding, we need to observe the following:

$$\beta_m'(x) = \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\left\{\frac{i_1}{i_1!\cdots i_r!}G_{i_1-1}(x)G_{i_2}(x)\cdots G_{i_r}(x) + \cdots\right.$$
$$\left. + \frac{i_r}{i_1!\cdots i_r!}G_{i_1}(x)\cdots G_{i_{r-1}}(x)G_{i_r-1}(x)\right\}$$
$$= \sum_{i_1+\cdots+i_r=m,i_1\geq 2,i_2,\ldots,i_r\geq 1}\frac{1}{(i_1-1)!i_2!\cdots i_r!}G_{i_1-1}(x)G_{i_2}(x)\cdots G_{i_r}(x)$$

$$
\begin{aligned}
&+\cdots+\sum_{i_1+\cdots+i_r=m,i_1,i_2,\ldots,i_{r-1}\geq 1,i_r\geq 2}\frac{1}{i_1!\cdots i_{r-1}!(i_r-1)!}G_{i_1}(x)\cdots G_{i_{r-1}}(x)G_{i_r-1}(x)\\
&=\sum_{i_1+\cdots+i_r=m-1,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}(x)\cdots G_{i_r}(x)\\
&+\cdots+\sum_{i_1+\cdots+i_r=m-1,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}(x)\cdots G_{i_r}(x)\\
&=r\beta_{m-1}(x).
\end{aligned}
\tag{3.4}
$$

Thus $\beta'_m(x)=r\beta_{m-1}(x)$, and, from this, we obtain

$$
\left(\frac{\beta_{m+1}(x)}{r}\right)'=\beta_m(x)
\tag{3.5}
$$

and

$$
\int_0^1\beta_m(x)\,dx=\frac{1}{r}\bigl(\beta_{m+1}(1)-\beta_{m+1}(0)\bigr).
\tag{3.6}
$$

For $m>r$, let

$$
\begin{aligned}
\Omega_m&=\beta_m(1)-\beta_m(0)\\
&=\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}(1)\cdots G_{i_r}(1)\\
&\quad-\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}\cdots G_{i_r}\\
&=\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}(-G_{i_1}+2\delta_{i_1,1})\cdots(-G_{i_r}+2\delta_{i_r,1})\\
&\quad-\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}\cdots G_{i_r}\\
&=\sum_{0\leq a\leq r}\binom{r}{a}\sum_{i_1+\cdots+i_a=m+a-r,i_1,\ldots,i_a\geq 1}(-1)^a2^{r-a}\frac{G_{i_1}G_{i_2}\cdots G_{i_a}}{i_1!\cdots i_a!}\\
&\quad-\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}G_{i_1}G_{i_2}\cdots G_{i_r},
\end{aligned}
\tag{3.7}
$$

where we understand that, for $a=0$, the inner sum is $2^r\delta_{m,r}$. Observe that the sums over all $i_1+\cdots+i_r=m$ $(i_1,\ldots i_r\geq 1)$ of any term with a of $-G_{i_e}$ and b of $2\delta_{i_f,1}$ $(1\leq e,f\leq r, a+b=r)$ all give

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!\cdots i_r!}(-G_{i_1})\cdots(-G_{i_a})(2\delta_{i_{a+1},1})\cdots(2\delta_{i_{a+b},1})\\
&=\sum_{i_1+\cdots+i_a=m+a-r,i_1,\ldots,i_a\geq 1}\frac{(-1)^a2^{r-a}}{i_1!\cdots i_a!}G_{i_1}\cdots G_{i_a}.
\end{aligned}
\tag{3.8}
$$

From the definition of Ω_m, we have

$$\begin{aligned} &\beta_m(0) = \beta_m(1) \quad \Longleftrightarrow \quad \Omega_m = 0, \\ &\int_0^1 \beta_m(x)\,dx = \frac{1}{r}\Omega_{m+1}. \end{aligned} \tag{3.9}$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$. We have

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\bigl[\beta_m(x)e^{-2\pi inx}\bigr]_0^1 + \frac{1}{2\pi in}\int_0^1 \beta_m'(x)e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\bigl(\beta_m(1) - \beta_m(0)\bigr) \\ &\quad + \frac{r}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}\,dx \\ &= \frac{r}{2\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\Omega_m \\ &= \frac{r}{2\pi in}\left(\frac{r}{2\pi in}B_n^{(m-2)} - \frac{1}{2\pi in}\Omega_{m-1}\right) - \frac{1}{2\pi in}\Omega_m \\ &= \left(\frac{r}{2\pi in}\right)^2 B_n^{(m-2)} - \sum_{j=1}^{2}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\ &= \cdots \\ &= \left(\frac{r}{2\pi in}\right)^{m-r} B_n^{(r)} - \sum_{j=1}^{m-r}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \\ &= -\sum_{j=1}^{m-r}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}, \end{aligned} \tag{3.10}$$

where

$$B_n^{(r)} = \int_0^1 e^{-2\pi inx}\,dx = 0. \tag{3.11}$$

Case 2: $n = 0$. We have

$$B_0^{(m)} = \int_0^1 \beta_m(x)\,dx = \frac{1}{r}\Omega_{m+1}. \tag{3.12}$$

$\beta_m(\langle x\rangle)$ ($m > r \geq 1$) is piecewise C^∞. Moreover, $\beta_m(\langle x\rangle)$ is continuous for those positive integers $m > r$ with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those integers $m > r$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for a positive integer $m > r$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$\begin{aligned}\beta_m\big(\langle x\rangle\big) &= \frac{1}{r}\Omega_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{j=1}^{m-r}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}\\ &= \frac{1}{r}\Omega_{m+1} + \sum_{j=1}^{m-r}\frac{r^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\ &= \frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m-r}\frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j\big(\langle x\rangle\big) + \Omega_m \times \begin{cases} B_1(\langle x\rangle) & \text{for } x\in\mathbb{Z}^c,\\ 0 & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned} \tag{3.13}$$

Now, we can state our first result.

Theorem 3.1 *For each positive integer l, with $l > r$, we let*

$$\begin{aligned}\Omega_l &= \sum_{0\leq a\leq r}\binom{r}{a}(-1)^a 2^{r-a}\sum_{i_1+\cdots+i_a=l+a-r,i_1,\ldots,i_a\geq 1}\frac{G_{i_1}G_{i_2}\cdots G_{i_a}}{i_1!i_2!\cdots i_a!}\\ &\quad - \sum_{i_1+\cdots+i_r=l,i_1,\ldots,i_r\geq 1}\frac{G_{i_1}G_{i_2}\cdots G_{i_r}}{i_1!i_2!\cdots i_r!}.\end{aligned} \tag{3.14}$$

Assume that $\Omega_m = 0$, for a positive integer $m > r$. Then we have the following:

(a) $\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!i_2!\cdots i_r}G_{i_1}(\langle x\rangle)G_{i_2}(\langle x\rangle)\cdots G_{i_r}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}&\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!i_2!\cdots i_r!}G_{i_1}\big(\langle x\rangle\big)G_{i_2}\big(\langle x\rangle\big)\cdots G_{i_r}\big(\langle x\rangle\big)\\ &\quad = \frac{1}{r}\Omega_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{j=1}^{m-r}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},\end{aligned} \tag{3.15}$$

for all $x\in\mathbb{R}$, where the convergence is uniform,

(b)

$$\begin{aligned}&\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!i_2!\cdots i_r}G_{i_1}\big(\langle x\rangle\big)G_{i_2}\big(\langle x\rangle\big)\cdots G_{i_r}\big(\langle x\rangle\big)\\ &\quad = \frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m-r}\frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j\big(\langle x\rangle\big),\end{aligned} \tag{3.16}$$

for all $x\in\mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.

Assume next that $\Omega_m \neq 0$, for a positive integer $m > r$. Then $\beta_m(0)\neq\beta_m(1)$. Thus $\beta_m(\langle x\rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x\in\mathbb{Z}^c$, and it converges to

$$\begin{aligned}\frac{1}{2}\big(\beta_m(0)+\beta_m(1)\big) &= \beta_m(0) + \frac{1}{2}\Omega_m\\ &= \sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\geq 1}\frac{1}{i_1!i_2!\cdots i_r}G_{i_1}G_{i_2}\cdots G_{i_r} + \frac{1}{2}\Omega_m,\end{aligned} \tag{3.17}$$

for $x\in\mathbb{Z}$. Now, we can state our second theorem.

Theorem 3.2 *For each positive integer l, with $l > r$, we let*

$$\Omega_l = \sum_{0\le a\le r} \binom{r}{a}(-1)^a 2^{r-a} \sum_{i_1+\cdots+i_a=l+a-r, i_1,\ldots,i_a\ge 1} \frac{G_{i_1}G_{i_2}\cdots G_{i_a}}{i_1!i_2!\cdots i_a!} - \sum_{i_1+\cdots+i_r=l, i_1,\ldots,i_r\ge 1} \frac{G_{i_1}G_{i_2}\cdots G_{i_r}}{i_1!i_2!\cdots i_r!}. \tag{3.18}$$

Assume that $\Omega_m \neq 0$, for a positive integer $m > r$. Then we have the following:

(a)

$$\frac{1}{r}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\sum_{j=1}^{m-r} \frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} \right) e^{2\pi inx} = \begin{cases} \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} \frac{1}{i_1!i_2!\cdots i_r!} G_{i_1}(\langle x\rangle)G_{i_2}(\langle x\rangle)\cdots G_{i_r}(\langle x\rangle) & \text{for } x\in\mathbb{Z}^c, \\ \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} \frac{1}{i_1!i_2!\cdots i_r!} G_{i_1}G_{i_2}\cdots G_{i_r} + \frac{1}{2}\Omega_m & \text{for } x\in\mathbb{Z}, \end{cases} \tag{3.19}$$

(b)

$$\frac{1}{r}\Omega_{m+1} + \sum_{j=1}^{m-r} \frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j\big(\langle x\rangle\big) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} \frac{1}{i_1!i_2!\cdots i_r!} G_{i_1}\big(\langle x\rangle\big)G_{i_2}\big(\langle x\rangle\big)\cdots G_{i_r}\big(\langle x\rangle\big), \tag{3.20}$$

for $x \in \mathbb{Z}^c$, and

$$\frac{1}{r}\Omega_{m+1} + \sum_{j=2}^{m-r} \frac{r^{j-1}}{j!}\Omega_{m-j+1}B_j\big(\langle x\rangle\big) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} \frac{1}{i_1!i_2!\cdots i_r!} G_{i_1}G_{i_2}\cdots G_{i_r} + \frac{1}{2}\Omega_m, \tag{3.21}$$

for $x \in \mathbb{Z}$.

4 The third type of sums of finite products

Let $\gamma_m(x) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\ge 1} \frac{1}{i_1 i_2\cdots i_r} G_{i_1}(x)\cdots G_{i_r}(x)$ $(m > r \ge 1)$. Here the sum runs over all positive integers $i_1,\ldots,i_r$, with $i_1+\cdots+i_r = m$. Before proceeding, we observe the following:

$$\begin{aligned} \gamma_m'(x) &= \sum_{i_1+\cdots+i_r=m, i_1\ge 2, i_2,\ldots,i_r\ge 1} \frac{1}{i_2\cdots i_r} G_{i_1-1}(x)G_{i_2}(x)\cdots G_{i_r}(x) \\ &\quad + \cdots \\ &\quad + \sum_{i_1+\cdots+i_r=m, i_r\ge 2, i_1,\ldots,i_{r-1}\ge 1} \frac{1}{i_1\cdots i_{r-1}} G_{i_1}(x)\cdots G_{i_{r-1}}(x)G_{i_r-1}(x) \\ &= \sum_{i_1+\cdots+i_r=m-1, i_1,\ldots,i_r\ge 1} \frac{1}{i_2\cdots i_r} G_{i_1}(x)G_{i_2}(x)\cdots G_{i_r}(x) \\ &\quad + \cdots \end{aligned}$$

$$+ \sum_{i_1+\cdots+i_r=m-1, i_1,\ldots,i_r\geq 1} \frac{1}{i_1\cdots i_{r-1}} G_{i_1}(x)\cdots G_{i_r}(x)$$
$$= \sum_{i_1+\cdots+i_r=m-1, i_1,\ldots,i_r\geq 1} \left\{ \frac{1}{i_2\cdots i_r} + \frac{1}{i_1 i_3\cdots i_r} + \cdots + \frac{1}{i_1\cdots i_{r-1}} \right\} G_{i_1}(x) G_{i_2}(x)\cdots G_{i_r}(x)$$
$$= (m-1) \sum_{i_1+\cdots+i_r=m-1, i_1,\ldots,i_r\geq 1} \frac{1}{i_1\cdots i_r} G_{i_1}(x) G_{i_2}(x)\cdots G_{i_r}(x)$$
$$= (m-1)\gamma_{m-1}(x). \tag{4.1}$$

So, $\gamma'_m(x) = (m-1)\gamma_{m-1}(x)$, and from this, we get

$$\left(\frac{\gamma_{m+1}(x)}{m} \right)' = \gamma_m(x) \tag{4.2}$$

and

$$\int_0^1 \gamma_m(x)\,dx = \frac{1}{m}\bigl(\gamma_{m+1}(1) - \gamma_{m+1}(0)\bigr). \tag{4.3}$$

For $m > r$, we let

$$\Lambda_m = \gamma_m(1) - \gamma_m(0)$$
$$= \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \frac{1}{i_1\cdots i_r} \bigl(G_{i_1}(1)\cdots G_{i_r}(1) - G_{i_1}\cdots G_{i_r} \bigr)$$
$$= \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \frac{1}{i_1\cdots i_r} (-G_{i_1} + 2\delta_{i_1,1})\cdots(-G_{i_r} + 2\delta_{i_r,1})$$
$$- \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \frac{1}{i_1\cdots i_r} G_{i_1}\cdots G_{i_r}$$
$$= \sum_{0\leq a\leq r} \binom{r}{a} (-1)^a 2^{r-a} \sum_{i_1+\cdots+i_a=m+a-r, i_1,\ldots,i_a\geq 1} \frac{G_{i_1} G_2\cdots G_{i_a}}{i_1\cdots i_a}$$
$$- \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \frac{1}{i_1\cdots i_r} G_{i_1}\cdots G_{i_r}, \tag{4.4}$$

where we understand that, for $a = 0$, the inner sum is $2^r\delta_{m,r}$. Note that

$$\gamma_m(0) = \gamma_m(1) \iff \Lambda_m = 0 \tag{4.5}$$

and

$$\int_0^1 \gamma_m(x)\,dx = \frac{1}{m}\Lambda_{m+1}. \tag{4.6}$$

We are now going to consider the function

$$\gamma_m\bigl(\langle x\rangle\bigr) = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \frac{1}{i_1\cdots i_r} G_{i_1}\bigl(\langle x\rangle\bigr)\cdots G_{i_r}\bigl(\langle x\rangle\bigr), \tag{4.7}$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\gamma_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi inx}, \tag{4.8}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m\bigl(\langle x\rangle\bigr) e^{-2\pi inx}\,dx = \int_0^1 \gamma_m(x) e^{-2\pi inx}\,dx. \tag{4.9}$$

Now, we would like to determine the Fourier coefficients $C_n^{(m)}$.

Case 1: $n \neq 0$. We have

$$\begin{aligned} C_n^{(m)} &= -\frac{1}{2\pi in}\bigl[\gamma_m(x)e^{-2\pi inx}\bigr]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}\,dx \\ &= -\frac{1}{2\pi in}\bigl(\gamma_m(1) - \gamma_m(0)\bigr) + \frac{m-1}{2\pi in}\int_0^1 \gamma_{m-1}(x)e^{-2\pi inx}\,dx \\ &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m \\ &= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1}\right) - \frac{1}{2\pi in}\Lambda_m \\ &= \frac{(m-1)_2}{(2\pi in)^2}C_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} \\ &= \cdots \\ &= \frac{(m-1)_{m-r}}{(2\pi in)^{m-r}}C_n^{(r)} - \sum_{j=1}^{m-r}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} \\ &= -\frac{1}{m}\sum_{j=1}^{m-r}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}. \end{aligned} \tag{4.10}$$

Note that

$$C_n^{(r)} = \int_0^1 e^{-2\pi inx}\,dx = 0. \tag{4.11}$$

Case 2: $n = 0$. We have

$$C_0^{(m)} = \int_0^1 \gamma_m(x)\,dx = \frac{1}{m}\Lambda_{m+1}. \tag{4.12}$$

$\gamma_m(\langle x\rangle)$, $(m > r \geq 1)$ is piecewise C^∞. Moreover, $\gamma_m(\langle x\rangle)$ is continuous for those integers $m > r$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m > r$ with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$, for a positive integer $m > r$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(\langle x\rangle)$ is piecewise C^∞, and continuous. So the Fourier series of $\gamma_m(\langle x\rangle)$ converges uniformly to

$\gamma_m(\langle x\rangle)$, and

$$\begin{aligned}
&\gamma_m\big(\langle x\rangle\big)\\
&= \frac{1}{m}\Lambda_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-r}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\right)e^{2\pi inx}\\
&= \frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=1}^{m-r}\binom{m}{j}\Lambda_{m-j+1}\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
&= \frac{1}{m}\Lambda_{m+1} + +\frac{1}{m}\sum_{j=2}^{m-r}\binom{m}{j}\Lambda_{m-j+1}B_j\big(\langle x\rangle\big)\\
&\quad + \Lambda_m \times \begin{cases} B_1(\langle x\rangle) & \text{for } x\in\mathbb{Z}^c,\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned} \tag{4.13}$$

We are now ready to state our first theorem.

Theorem 4.1 *For each positive integer l, with $l > r$, we let*

$$\begin{aligned}
\Lambda_l = {}& \sum_{0\le a\le r}\binom{r}{a}(-1)^a 2^{r-a}\sum_{i_1+\cdots+i_a=l+a-r,i_1,\ldots,i_a\ge 1}\frac{G_{i_1}G_{i_2}\cdots G_{i_a}}{i_1i_2\cdots i_a}\\
&- \sum_{i_1+\cdots+i_r=l,i_1,\ldots,i_r\ge 1}\frac{G_{i_1}G_{i_2}\cdots G_{i_r}}{i_1i_2\cdots i_r}.
\end{aligned} \tag{4.14}$$

Assume that $\Lambda_m = 0$, for a positive integer $m > r$. Then we have the following:

(a) $\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\ge 1}\frac{1}{i_1i_2\cdots i_r}G_{i_1}(\langle x\rangle)G_{i_2}(\langle x\rangle)\cdots G_{i_r}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\ge 1}\frac{1}{i_1i_2\cdots i_r}G_{i_1}\big(\langle x\rangle\big)G_{i_2}\big(\langle x\rangle\big)\cdots G_{i_r}\big(\langle x\rangle\big)\\
&\quad = \frac{1}{m}\Lambda_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-r}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\right)e^{2\pi inx},
\end{aligned} \tag{4.15}$$

for all $x \in \mathbb{R}$, where the convergence is uniform,

(b) *we have*

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r=m,i_1,\ldots,i_r\ge 1}\frac{1}{i_1i_2\cdots i_r}G_{i_1}\big(\langle x\rangle\big)G_{i_2}\big(\langle x\rangle\big)\cdots G_{i_r}\big(\langle x\rangle\big)\\
&\quad = \frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=2}^{m-r}\binom{m}{j}\Lambda_{m-j+1}B_j\big(\langle x\rangle\big),
\end{aligned} \tag{4.16}$$

for all $x \in \mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.

Assume next that $\Lambda_m \neq 0$, for a positive integer $m > r$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(\langle x\rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\gamma_m(\langle x\rangle)$ converges pointwise to $\gamma_m(\langle x\rangle)$, for $x \in \mathbb{Z}^c$, and it converges to

$$\begin{aligned}\frac{1}{2}\bigl(\gamma_m(0)+\gamma_m(1)\bigr) &= \gamma_m(0)+\frac{1}{2}\Lambda_m \\ &= \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1} \frac{1}{i_1 i_2\cdots i_r} G_{i_1}G_{i_2}\cdots G_{i_r} + \frac{1}{2}\Lambda_m,\end{aligned} \tag{4.17}$$

for $x \in \mathbb{Z}$. Now, we state our second result.

Theorem 4.2 *For each positive integer l, with $l > r$, we let*

$$\begin{aligned}\Lambda_l &= \sum_{0\leq a\leq r}\binom{r}{a}(-1)^a 2^{r-a}\sum_{i_1+\cdots+i_a=l+a-r, i_1,\ldots,i_a\geq 1}\frac{G_{i_1}G_{i_2}\cdots G_{i_a}}{i_1 i_2\cdots i_a} \\ &\quad - \sum_{i_1+\cdots+i_r=l, i_1,\ldots,i_r\geq 1}\frac{G_{i_1}G_{i_2}\cdots G_{i_r}}{i_1 i_2\cdots i_r}.\end{aligned} \tag{4.18}$$

Assume that $\Lambda_m \neq 0$, for a positive integer $m > r$. Then we have the following:

(a)

$$\begin{aligned}&\frac{1}{m}\Lambda_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-r}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\right)e^{2\pi inx} \\ &\quad = \begin{cases}\sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1}\frac{1}{i_1 i_2\cdots i_r}G_{i_1}(\langle x\rangle)G_{i_2}(\langle x\rangle)\cdots G_{i_r}(\langle x\rangle) & \text{for } x\in\mathbb{Z}^c, \\ \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1}\frac{1}{i_1 i_2\cdots i_r}G_{i_1}G_{i_2}\cdots G_{i_r} + \frac{1}{2}\Lambda_m & \text{for } x\in\mathbb{Z},\end{cases}\end{aligned} \tag{4.19}$$

(b)

$$\begin{aligned}&\frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=1}^{m-r}\binom{m}{j}\Lambda_{m-j+1}B_j\bigl(\langle x\rangle\bigr) \\ &\quad = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1}\frac{1}{i_1 i_2\cdots i_r}G_{i_1}\bigl(\langle x\rangle\bigr)G_{i_2}\bigl(\langle x\rangle\bigr)\cdots G_{i_r}\bigl(\langle x\rangle\bigr),\end{aligned} \tag{4.20}$$

for $x \in \mathbb{Z}^c$, and

$$\begin{aligned}&\frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=2}^{m-r}\binom{m}{j}\Lambda_{m-j+1}B_j\bigl(\langle x\rangle\bigr) \\ &\quad = \sum_{i_1+\cdots+i_r=m, i_1,\ldots,i_r\geq 1}\frac{1}{i_1 i_2\cdots i_r}G_{i_1}G_{i_2}\cdots G_{i_r} + \frac{1}{2}\Lambda_m,\end{aligned} \tag{4.21}$$

for $x \in \mathbb{Z}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300160, China. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea. [3]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea. [4]Graduate School of Education, Konkuk University, Seoul, 143-701, Republic of Korea.

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