He *Advances in Difference Equations* (2017) 2017:264
DOI 10.1186/s13662-017-1304-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence results and the monotone iterative technique for nonlinear fractional differential systems involving fractional integral boundary conditions

Ying He*

*Correspondence:
heying65332015@163.com
School of Mathematics and Statistics, Northeast Petroleum University, Daqing, 163318, P.R. China

Abstract

By establishing a comparison result and using the monotone iterative technique combined with the method of upper and lower solutions, we have investigated the existence of extremal solutions for nonlinear fractional differential systems with integral boundary conditions. As an example, an application is presented to demonstrate the accuracy of the new approach.

MSC: 34B15

Keywords: fractional differential system; upper and lower solutions; monotone iterative technique; integral boundary conditions

1 Introduction

In this paper, we consider the following differential equations with integral boundary conditions:

$$\begin{cases} -D^{\alpha}x(t) = f(t,x(t)), \quad t \in [0,1], \\ x(0) = 0, \\ D^{\alpha-1}x(1) = I^{\beta}g(\eta,x(\eta)) + k = \frac{1}{\Gamma(\beta)}\int_0^{\eta}(\eta-s)^{\beta-1}g(s,x(s))\,ds + k, \end{cases} \tag{1.1}$$

where D^{α} are the standard Riemann-Liouville fractional derivatives, I^{β} is the Riemann-Liouville fractional integral.

Throughout this paper, we always suppose that

(s_1) $1 < \alpha < 2$, $\beta > 1$, $0 < \eta < 1$, $k \in \mathbb{R}$, and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $g \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Recently, much attention has been focused on the study of the existence of solutions for fractional differential systems with initial or two-point boundary value conditions, by using the monotone iterative technique, combined with the method of upper and lower solutions; for details, see [1–7]. But up to now, three-point and fractional integral boundary value problems for fractional differential systems have seldom been considered. The aim of this paper is to investigate the existence of extremal solutions for fractional equation

(1.1), involving Riemann-Liouville fractional integral boundary conditions. To the best of our knowledge, in most of the papers and books considered to deal with fractional derivatives of order $\alpha \in (1,2)$, the nonlinear term f is required to satisfy monotonicity conditions on the unknown function x or their derivatives. These monotonicity type conditions are not required in this paper.

The paper is organized as follows: Preliminaries are in Section 2. Then in Section 3 we construct the monotone sequences of solutions and prove their uniform convergence to the solutions of the systems. Finally, an example is presented to demonstrate the accuracy of the new approach.

2 Preliminaries

In this section, we deduce some preliminary results which will be used in the next section.

Denote $C_\alpha[0,1] = \{x : x \in C[0,1], D^\alpha x(t) \in C[0,1]\}$ and endowed with the norm $\|x\|_\alpha = \|x\| + \|D^\alpha x\|$, where $\|x\| = \max_{0\le t\le 1} |x(t)|$ and $\|D^\alpha x\| = \max_{0\le t\le 1} |D^\alpha x(t)|$. Then $(C_\alpha[0,1], \|\cdot\|_\alpha)$ is a Banach space.

Definition 2.1 We say that $x(t) \in C_\alpha[0,1]$ is a lower solution of problem (1.1) if

$$\begin{cases} -D^\alpha x(t) \le f(t,x(t)), & t \in [0,1], \\ x(0) = 0, \\ D^{\alpha-1}x(1) \le I^\beta g(\eta, x(\eta)) + k, \end{cases}$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

For the sake of convenience, we now present some assumptions as follows:

(H_1) Assume that $x_0, y_0 \in C_\alpha[0,1]$ are lower and upper solutions of problem (1.1), respectively, and $x_0(t) \le y_0(t)$, $t \in [0,1]$.

(H_2) There exists $M(t) \in C[0,1]$ such that

$$f(t,y) - f(t,x) \ge -M(t)(y-x),$$

for $x_0(t) \le x(t) \le y(t) \le y_0(t)$, $t \in [0,1]$.

(H_3) There exists a constant $\lambda \ge 0$, such that

$$g(t,y) - g(t,x) \ge \lambda(y-x),$$

for $x_0(t) \le x(t) \le y(t) \le y_0(t)$, $t \in [0,1]$.

(H_4) $\Gamma(\alpha+\beta) > \lambda\eta^{\alpha+\beta-1}$.

(H_5) $2\Gamma(\alpha+\beta)\int_0^1 |M(s)|\,ds < \Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]$.

(H_6) For any $t \in (0,1)$, we have

$$\Gamma(2-\alpha)t^\alpha M(t) > 1-\alpha$$

and

$$\Gamma(2-\alpha)\lambda\eta^\beta < \Gamma(\beta).$$

Lemma 2.1 ([8]) *Let* $h \in C[0,1]$, $b \in \mathbb{R}$, *and* $\Gamma(\alpha+\beta) \neq \lambda\eta^{\alpha+\beta-1}$; *then the fractional boundary value problem*

$$\begin{cases} -D^{\alpha}x(t) = h(t), \quad t \in [0,1], \\ x(0) = 0, \\ D^{\alpha-1}x(1) = \lambda I^{\beta}x(\eta) + b = \frac{\lambda}{\Gamma(\beta)} \int_0^{\eta} (\eta - s)^{\beta-1} x(s)\,ds + b, \end{cases} \tag{2.1}$$

has the following integral representation of the solution:

$$x(t) = \int_0^1 G(t,s)h(s)\,ds + \frac{b\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]},$$

where

$$G(t,s) = \frac{1}{\Delta} \begin{cases} [\Gamma(\alpha+\beta) - \lambda(\eta-s)^{\alpha+\beta-1}]t^{\alpha-1} \\ \quad - [\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}](t-s)^{\alpha-1}, & s \le t, s \le \eta; \\ \Gamma(\alpha+\beta)t^{\alpha-1} - \lambda(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}, & t \le s \le \eta; \\ \Gamma(\alpha+\beta)[t^{\alpha-1} - (t-s)^{\alpha-1}] + \lambda\eta^{\alpha+\beta-1}(t-s)^{\alpha-1}, & \eta \le s \le t; \\ \Gamma(\alpha+\beta)t^{\alpha-1}, & s \ge t, s \ge \eta, \end{cases}$$

and $\Delta = \Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]$.

Lemma 2.2 ([8]) *If* (H_4) *holds, then Green's function* $G(t,s)$ *satisfies*

$$0 \le G(t,s) \le \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]}\bigl(1 + t^{\alpha-1}\bigr).$$

Lemma 2.3 *Let* $b \in \mathbb{R}$, $\sigma(t) \in C[0,1]$ *and* (H_4), (H_5) *hold*; *then the following boundary problem*:

$$\begin{cases} -D^{\alpha}x(t) = \sigma(t) - M(t)x(t), \quad t \in [0,1], \\ x(0) = 0, \\ D^{\alpha-1}x(1) = \lambda I^{\beta}x(\eta) + b, \end{cases} \tag{2.2}$$

has a unique solution $x(t) \in C[0,1]$.

Proof It follows from Lemma 2.1 that problem (2.2) is equivalent to the following integral equation:

$$x(t) = \int_0^1 G(t,s)\bigl[\sigma(s) - M(s)x(s)\bigr]\,ds + \frac{b\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]}, \quad \forall t \in [0,1].$$

Let

$$Ax(t) = \int_0^1 G(t,s)\bigl[\sigma(s) - M(s)x(s)\bigr]\,ds + \frac{b\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]}, \quad \forall t \in [0,1].$$

For any $u, v \in C[0,1]$, by (H_4) and Lemma 2.2, we have

$$\begin{aligned} \left|Ax(t) - Ay(t)\right| &\leq \int_0^1 G(t,s)\left|M(s)\right| \cdot \left|x(s) - y(s)\right| ds \\ &\leq \frac{\Gamma(\alpha+\beta)(1+t^{\alpha-1})\|x-y\|}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]} \int_0^1 \left|M(s)\right| ds \\ &\leq \frac{2\Gamma(\alpha+\beta)\|x-y\|}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \lambda\eta^{\alpha+\beta-1}]} \int_0^1 \left|M(s)\right| ds. \end{aligned}$$

Noting that we have (H_5), which implies $\frac{2\Gamma(\alpha+\beta)\int_0^1 |M(s)|\, ds}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\lambda\eta^{\alpha+\beta-1}]} < 1$, $|Ax(t) - Ay(t)| < \|x - y\|$. Consequently,

$$\|Ax - Ay\| < \|x - y\|.$$

By the Banach fixed point theorem, the operator A has a unique fixed point. That is, (2.2) has a unique solution. □

Lemma 2.4 ([9]) *Assume that* $x(t) \in C[0,1]$ *satisfies the following conditions*:
(i) $D^\alpha x(t) \in C[0,1]$, *for* $\alpha \in (1,2)$;
(ii) $x(t)$ *attains its global minimum at* $t_0 \in (0,1)$.
Then

$$D^\alpha x(t)|_{t=t_0} \geq \frac{1-\alpha}{\Gamma(2-\alpha)} t_0^{-\alpha} x(t_0).$$

Lemma 2.5 ([9]) *Assume that* $x(t) \in C[0,1]$ *satisfies the following conditions*:
(i) $D^\delta x(t) \in C[0,1]$, *for* $\delta \in (0,1)$;
(ii) $x(t)$ *attains its global minimum at* $t_0 \in (0,1]$.
Then

$$D^\delta x(t)|_{t=t_0} \leq \frac{t_0^{-\delta}}{\Gamma(1-\delta)} x(t_0).$$

Lemma 2.6 *Assume that* (H_6) *holds*, $x(t) \in C[0,1]$, *satisfying* $D^\alpha x(t) \in C[0,1]$ *and*

$$\begin{cases} -D^\alpha x(t) \geq -M(t)x(t), & t \in [0,1], \\ x(0) = 0, \\ D^{\alpha-1}x(1) \geq \lambda I^\beta x(\eta), \end{cases} \tag{2.3}$$

then $x(t) \geq 0$, $\forall t \in [0,1]$.

Proof Suppose that $x(t) \geq 0$, $t \in [0,1]$ is not true. From the continuity of $x(t)$ it follows that there exists some $t_1 \in (0,1]$ such that $x(t_1) = \min_{t\in[0,1]} x(t) < 0$.

Case (i). If $t_1 \in (0,1)$, by Lemma 2.4 and (H_6), we have

$$0 \geq D^\alpha x(t)|_{t=t_1} - M(t_1)x(t_1) \geq \left[\frac{1-\alpha}{\Gamma(2-\alpha)} t_1^{-\alpha} - M(t_1)\right] x(t_1) > 0,$$

which is a contradiction.

Case (ii). If $t_1 = 1$, by Lemma 2.5, one gets

$$D^{\alpha-1}x(t)|_{t=1} \leq \frac{x(1)}{\Gamma(2-\alpha)}.$$

On the other hand, from the boundary condition of (2.3) and (H_6), we obtain

$$\begin{aligned} D^{\alpha-1}x(1) \geq \lambda I^{\beta}x(\eta) &= \frac{\lambda}{\Gamma(\beta)}\int_0^{\eta}(\eta-s)^{\beta-1}x(s)\,ds \\ &= \frac{\lambda}{\Gamma(\beta)}\cdot(\eta-\xi)^{\beta-1}\cdot x(\xi)\cdot\eta, \quad 0<\xi<\eta<1 \\ &\geq \frac{\lambda}{\Gamma(\beta)}\cdot(\eta-\xi)^{\beta-1}\cdot x(1)\cdot\eta \\ &\geq \frac{\lambda}{\Gamma(\beta)}\cdot\eta^{\beta-1}\cdot x(1)\cdot\eta \\ &> \frac{x(1)}{\Gamma(2-\alpha)}, \end{aligned}$$

which is a contradiction. Therefore, we obtain $x(t) \geq 0$, $\forall t \in [0,1]$. The proof is complete. □

3 Main results

In this section, we present the main result of our paper, which ensures the existence of extremal solutions for problem (1.1).

Theorem 3.1 *Suppose that conditions* (H_1)-(H_6) *hold. Then problem* (1.1) *has extremal solutions* $x^*, y^* \in [x_0, y_0]$. *Moreover, there exist monotone iterative sequences* $\{x_n\}, \{y_n\} \subset C_\alpha[0,1]$ *such that* $x_n \to x^*$, $y_n \to y^*$ *uniformly on* $t \in [0,1]$, *as* $n \to \infty$ *and*

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq x^* \leq y^* \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0.$$

Proof For $n = 0,1,2,\ldots$, we define

$$\begin{cases} -D^{\alpha}x_{n+1}(t) = f(t,x_n(t)) - M(t)[x_{n+1}(t) - x_n(t)], \quad t \in [0,1], \\ x_{n+1}(0) = 0, \\ D^{\alpha-1}x_{n+1}(1) = I^{\beta}\{g(\eta,x_n(\eta)) + \lambda[x_{n+1}(\eta) - x_n(\eta)]\} + k \\ \qquad = \lambda I^{\beta}x_{n+1}(\eta) + I^{\beta}[g(\eta,x_n(\eta)) - \lambda x_n(\eta)] + k, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -D^{\alpha}y_{n+1}(t) = f(t,y_n(t)) - M(t)[y_{n+1}(t) - y_n(t)], \quad t \in [0,1], \\ y_{n+1}(0) = 0, \\ D^{\alpha-1}y_{n+1}(1) = I^{\beta}\{g(\eta,y_n(\eta)) + \lambda[y_{n+1}(\eta) - y_n(\eta)]\} + k \\ \qquad = \lambda I^{\beta}y_{n+1}(\eta) + I^{\beta}[g(\eta,y_n(\eta)) - \lambda y_n(\eta)] + k. \end{cases} \tag{3.2}$$

In view of Lemma 2.3, for any $n \in \mathbb{N}$, problems (3.1) and (3.2) have a unique solution $x_{n+1}(t)$, $y_{n+1}(t)$ respectively, which are well defined. First, we show that

$$x_0(t) \le x_1(t) \le y_1(t) \le y_0(t), \quad t \in [0,1].$$

Let $w(t) = x_1(t) - x_0(t)$. The definitions of $x_1(t)$ and (H_1) yield

$$\begin{cases} -D^\alpha w(t) \ge -M(t)w(t), & t \in [0,1], \\ w(0) = 0, \\ D^{\alpha-1}w(1) \ge \lambda I^\beta w(\eta). \end{cases}$$

According to Lemma 2.6, we have $w(t) \ge 0$, $t \in [0,1]$, that is, $x_1(t) \ge x_0(t)$. Using the same reasoning, we can show that $y_0(t) \ge y_1(t)$, for all $t \in [0,1]$.

Now, we put $p(t) = y_1(t) - x_1(t)$. From (H_2) and (H_3), we get

$$\begin{aligned} -D^\alpha p(t) &= f\big(t, y_0(t)\big) - M(t)\big[y_1(t) - y_0(t)\big] - f\big(t, x_0(t)\big) + M(t)\big[x_1(t) - x_0(t)\big] \\ &\ge -M(t)\big[y_0(t) - x_0(t)\big] - M(t)\big[y_1(t) - y_0(t)\big] + M(t)\big[x_1(t) - x_0(t)\big] \\ &= -M(t)p(t). \end{aligned}$$

Also $p(0) = 0$, and

$$\begin{aligned} D^{\alpha-1}p(1) &= I^\beta\big\{g\big(\eta, y_0(\eta)\big) + \lambda\big[y_1(\eta) - y_0(\eta)\big]\big\} - I^\beta\big\{g\big(\eta, x_0(\eta)\big) + \lambda\big[x_1(\eta) - x_0(\eta)\big]\big\} \\ &= I^\beta\big\{g\big(\eta, y_0(\eta)\big) - g\big(\eta, x_0(\eta)\big) + \lambda\big[y_1(\eta) - y_0(\eta)\big] - \lambda\big[x_1(\eta) - x_0(\eta)\big]\big\} \\ &\ge I^\beta\big\{\lambda\big[y_0(\eta) - x_0(\eta)\big] + \lambda\big[y_1(\eta) - y_0(\eta)\big] - \lambda\big[x_1(\eta) - x_0(\eta)\big]\big\} \\ &= \lambda I^\beta p(\eta). \end{aligned}$$

These results and Lemma 2.6 imply that $y_1(t) \ge x_1(t)$, $t \in [0,1]$.

In the next step, we show that x_1, y_1 are lower and upper solutions of problem (1.1), respectively. Note that

$$\begin{aligned} -D^\alpha x_1(t) &= f\big(t, x_0(t)\big) - f\big(t, x_1(t)\big) + f\big(t, x_1(t)\big) - M(t)\big[x_1(t) - x_0(t)\big] \\ &\le M(t)\big[x_1(t) - x_0(t)\big] + f\big(t, x_1(t)\big) - M(t)\big[x_1(t) - x_0(t)\big] \\ &= f\big(t, x_1(t)\big). \end{aligned}$$

Also $x_1(0) = 0$, and

$$\begin{aligned} D^{\alpha-1}x_1(1) &= I^\beta\big\{g\big(\eta, x_0(\eta)\big) - g\big(\eta, x_1(\eta)\big) + g\big(\eta, x_1(\eta)\big) + \lambda\big[x_1(\eta) - x_0(\eta)\big]\big\} + k \\ &\le I^\beta\big\{\lambda\big[x_0(\eta) - x_1(\eta)\big] + g\big(\eta, x_1(\eta)\big) + \lambda\big[x_1(\eta) - x_0(\eta)\big]\big\} + k \\ &= I^\beta g\big(\eta, x_1(\eta)\big) + k \end{aligned}$$

by assumptions (H_2) and (H_3). This proves that x_1 is a lower solution of problem (1.1). Similarly, we can prove that y_1 is an upper solution of (1.1).

Using mathematical induction, we see that

$$x_0(t) \le x_1(t) \le \cdots \le x_n(t) \le x_{n+1}(t) \le y_{n+1}(t) \le y_n(t) \le \cdots \le y_1(t) \le y_0(t), \quad t \in [0,1],$$

since the space of solution is $C_\alpha[0,1]$. Using the standard arguments, it is easy to show $\{x_n\}$ and $\{y_n\}$ are uniformly bounded and equi-continuous. By the Arzela-Ascoli theorem, we have $\{x_n\}$ and $\{y_n\}$ converge, say to $x^*(t)$ and $y^*(t)$, uniformly on $[0,1]$, respectively. That is

$$\lim_{n\to\infty} x_n(t) = x^*(t), \qquad \lim_{n\to\infty} y_n(t) = x^*(t), \quad t \in [0,1].$$

Moreover, $x^*(t)$ and $y^*(t)$ are the solutions of problem (1.1) and $x_0 \le x^* \le y^* \le y_0$ on $[0,1]$.

To prove that $x^*(t)$, $y^*(t)$ are extremal solutions of (1.1), let $u \in [x_0, y_0]$ be any solution of problem (1.1). We suppose that $x_m(t) \le u(t) \le y_m(t)$, $t \in [0,1]$ for some m. Let $v(t) = u(t) - x_{m+1}(t)$, $z(t) = y_{m+1}(t) - u(t)$. Then by assumption (H_2) and (H_3), we see that

$$\begin{cases} -D^\alpha v(t) \ge -M(t)v(t), & t \in [0,1], \\ v(0) = 0, \\ D^{\alpha-1}v(1) \ge \lambda I^\beta v(\eta), \end{cases}$$

and

$$\begin{cases} -D^\alpha z(t) \ge -M(t)z(t), & t \in [0,1], \\ z(0) = 0, \\ D^{\alpha-1}z(1) \ge \lambda I^\beta z(\eta). \end{cases}$$

These and Lemma 2.6 imply that $x_{m+1}(t) \le u(t) \le y_{m+1}(t)$, $t \in [0,1]$, so by induction $x_n(t) \le u(t) \le y_n(t)$, on $[0,1]$ for all n. Taking the limit as $n \longrightarrow \infty$, we conclude $x^*(t) \le u(t) \le y^*(t)$, $t \in [0,1]$. The proof is complete. □

Example Consider the following problem:

$$\begin{cases} -D^{\frac{3}{2}}x(t) = -\frac{1}{16}t^2x^2(t) + \frac{1}{5}t^3, & t \in [0,1], \\ x(0) = 0, \\ D^{\frac{1}{2}}x(1) = I^{\frac{3}{2}}g(\frac{1}{4}, x(\frac{1}{4})) + 1.2 = \frac{1}{\Gamma(\frac{3}{2})}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{1}{2}}(s+1)x(s)\,ds + 1.2, \end{cases} \tag{3.3}$$

where $\alpha = \frac{3}{2}$, $\beta = \frac{3}{2}$, $\eta = \frac{1}{4}$, $k = 1.2$, and

$$\begin{cases} f(t,x) = -\frac{1}{16}t^2x^2(t) + \frac{1}{5}t^3, \\ g(t,x) = (t+1)x. \end{cases}$$

Take $x_0(t) = 0$, $y_0(t) = 2t^{\frac{1}{2}}$. It is not difficult to verify that x_0, y_0 are lower and upper solutions of (3.3), respectively, and $x_0 \le y_0$. So (H_1) holds.

In addition, we have

$$f(t,y)-f(t,x)=-\frac{1}{16}t^2x^2+\frac{1}{16}t^2y^2\geq-\frac{1}{4}t^{\frac{3}{2}}(y-x) \tag{3.4}$$

and

$$g(t,y)-g(t,x)=(t+1)(y-x)\geq(y-x), \tag{3.5}$$

where $x_0(t)\leq x(t)\leq y(t)\leq y_0(t)$.

Therefore (H_2) and (H_3) hold.

From (3.4) and (3.5), we have

$$M(t)=\frac{1}{4}t^{\frac{3}{2}},\quad \lambda=1.$$

Then

$$\begin{aligned}
&\Gamma(\alpha+\beta)=\Gamma(3)=2>\lambda\eta^{\alpha+\beta-1}=\left(\frac{1}{4}\right)^2,\\
&2\Gamma(\alpha+\beta)\int_0^1\bigl|M(s)\bigr|\,ds=2\cdot2\int_0^1\frac{1}{4}s^{\frac{3}{2}}\,ds=\frac{2}{5}<\Gamma(\alpha)\bigl[\Gamma(\alpha+\beta)-\lambda\eta^{\alpha+\beta-1}\bigr]\\
&\qquad=\Gamma\left(\frac{3}{2}\right)\left[2-\left(\frac{1}{4}\right)^2\right]\approx1.717,\\
&\Gamma(2-\alpha)\lambda\eta^{\beta}=\Gamma\left(2-\frac{3}{2}\right)\cdot1\cdot\left(\frac{1}{4}\right)^{\frac{3}{2}}=\frac{1}{4}\cdot\Gamma\left(\frac{3}{2}\right)<\Gamma(\beta)=\Gamma\left(\frac{3}{2}\right),\\
&\Gamma(2-\alpha)\cdot t^{\alpha}\cdot M(t)=\Gamma\left(\frac{1}{2}\right)\cdot t^{\frac{3}{2}}\cdot\frac{1}{4}\cdot t^{\frac{3}{2}}>1-\alpha=-\frac{1}{2},\quad\text{for } t\in(0,1).
\end{aligned}$$

It shows that (H_4), (H_5) and (H_6) hold. By Theorem 3.1, problem (3.3) has extremal solutions in $[x_0(t),y_0(t)]$.

Acknowledgements
Project was supported by the Guiding Innovation Foundation of Northeast Petroleum University (No. 2016YDL-02) and the Youth Scientific Research Fund of Northeast Petroleum University (No. NEPUQN2015-1-21).

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The whole work was carried out, read and approved by the author.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 29 March 2017 Accepted: 3 August 2017 Published online: 01 September 2017

References

1. Wang, G: Monotone iterative technique for boundary value problems of a nonlinear fractional equation with deviating arguments. J. Comput. Appl. Math. **236**, 2425-2430 (2012)
2. Wang, G, Agarwal, RP, Cabada, A: Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations. Appl. Math. Lett. **25**, 1019-1024 (2012)
3. Zhang, L, Ahmad, B, Wang, G: Explicit iterations and extremal solutions for fractional differential equations with nonlinear integral boundary conditions. Appl. Math. Comput. **268**, 388-392 (2015)

4. Zhang, L, Ahmad, B, Wang, G: The existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative. Appl. Math. Lett. **31**, 1-6 (2014)
5. Jankowski, T: Boundary problems for fractional differential equations. Appl. Math. Lett. **28**, 14-19 (2014)
6. Jian, H, Liu, B, Xie, S: Monotone iterative solutions for nonlinear fractional differential systems with deviating arguments. Appl. Math. Comput. **262**, 1-14 (2015)
7. Liu, X, Jia, M, Ge, W: The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator. Appl. Math. Lett. **65**, 56-62 (2017)
8. Wang, G: Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval. Appl. Math. Lett. **17**, 1-7 (2015)
9. Xie, W, Xiao, J, Luo, Z: Existence of solutions for Riemann-Liouville fractional boundary value problem. Abstr. Appl. Anal. **2014**, Article ID 540351 (2014)