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A posteriori truncated regularization method for identifying unknown heat source on a spherical symmetric domain

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Abstract

In this paper, we mainly consider the inverse problem for identifying the unknown heat source in spherical symmetric domain. We propose a truncation regularization method combined with an *a posteriori* regularization parameter choice rule to deal with this problem. The Hölder type convergence estimate is obtained. Numerical results are presented to illustrate the accuracy and efficiency of this method.

Keywords: identifying the unknown source; ill-posed problem; regularization method; *a posteriori* parameter choice rule; spherical symmetric domain

1 Introduction

Identifying the unknown heat source in a parabolic partial differential equation from the over-specified data plays an important role in applied mathematics, physics and engineering. These problems are widely encountered in the modeling of physical phenomena. A typical example is groundwater pollutant source estimation in cities with large population [1]. Now many scholars have used different methods to identify various types of heat sources. In [2, 3], the authors used the method of fundamental solutions and radial basis functions to identify the unknown heat source. In [4, 5], the authors used the Fourier truncation method and the wavelet dual least squares method to identify the spatial variable heat source. In [6], the authors used the simplified Tikhonov method to identify the spatial variable heat source. In [7, 8], the authors determined the heat source which depends on one variable in a bounded domain using the boundary-element method and an iterative algorithm. In [9], the authors identified the heat source which depends only on time variable using the Lie-group shooting method (LGSM). In [10], the authors used the truncation method based on Hermite expansion to identify the unknown source in a space fractional diffusion equation. In [11], the authors identified the point source with some point measurement data. In [12], the authors proved the existence and uniqueness for identifying the heat source which depends only on time variable. In [13], the authors used the variational method to identify the heat source which has the form $F(x,t)$. In [14], the authors used the variational method to identify the heat source which has the form of $F(x,t) = F(x)H(t)$ for the variable coefficient heat conduction equation. As far as we know, most of the researches on heat source identification problem mainly concentrated on one-dimensional case. But for a high dimensional case, there are few research results. In [15], the authors used the spectral method to identify the heat source in a columnar symmetric domain. In [16], the authors used the spectral method to identify the heat source in a spherically symmetric parabolic equation. But the regularization parameters is selected by the *a priori* rule. There is a defect for any *a priori* method, *i.e.*, the *a priori* choice of the regularization parameter depends seriously on the *a priori* bound E of the unknown solution. However, the *a priori* bound E cannot be known exactly in practice, and working with a wrong constant E may lead to a badly regularized solution. In this paper, we not only give the *a posteriori* choice of the regularization parameter which depends only on the measurable data, but also we give some different examples to compare the effectiveness between the *posterior* choice rule and the *priori* choice rule. Moreover, we find the truncation regularization method is better than the other regularization methods, such as Tikhonov regularization and the quasi-boundary value regularization method for solving this problem. To the best of the authors' knowledge, there are few papers to choose the regularization parameter under the *a posteriori* rule for this problem.

In this paper, we consider the following heat source identification problem in spherical symmetric domain:

$$\begin{cases} u_t - \frac{2}{r}u_r - u_{rr} = f(r), & 0 < t < T, 0 < r < r_0, \\ u(r,0) = 0, & 0 \le r \le r_0, \\ u(r_0,t) = 0, & 0 \le t \le T, \\ \lim_{r\to 0} u(r,t) \text{ is bounded}, & 0 < t < T, \\ u(r,T) = \varphi(r), & 0 \le r \le r_0, \end{cases} \tag{1}$$

where r_0 is the radius, $f(r)$ is the unknown heat source. Our purpose is to identify $f(r)$ from the additional data $u(r,T) = \varphi(r)$. Since the data $\varphi(r)$ is based on (physical) observation, there must be measurement errors, and we assume the measured data function $\varphi^\delta(r) \in L^2[0, r_0; r^2]$, and it satisfies

$$\left\| \varphi(\cdot) - \varphi^\delta(\cdot) \right\| \le \delta, \tag{2}$$

where $\delta > 0$ is the measurable error level.

Using the separation variable method, we get the solution of problem (1) as follows:

$$u(r,t) = \sum_{n=1}^{\infty} f_n \left(\int_0^t e^{-(\frac{n\pi}{r_0})^2(T-\tau)}\, d\tau \right) \psi_n(r), \tag{3}$$

where $\psi_n(r)$ defined as follows are the characteristic functions:

$$\psi_n(r) = \frac{\sqrt{2}n\pi}{\sqrt{r_0^3}} j_0\left(\frac{n\pi r}{r_0}\right) = \frac{\sqrt{2}n\pi}{\sqrt{r_0^3}} \frac{\sin(\frac{n\pi r}{r_0})}{\frac{n\pi r}{r_0}}, \quad n = 1,2,3,\ldots, \tag{4}$$

$j_0(x)$ is the zero Bessel function. $\psi_n(r) \in L^2[0, r_0; r^2]$ is an orthonormal system in the *Hilbert* space $L^2[0, r_0; r^2]$. f_n is the Fourier coefficient of $f(r)$, which is defined by

$$f_n = \int_0^{r_0} r^2 f(r) \psi_n(r)\, dr. \tag{5}$$

Using $u(r,T) = \varphi(r)$, we obtain

$$\varphi(r) = \sum_{n=1}^{\infty} f_n \left(\int_0^T e^{-(\frac{n\pi}{r_0})^2 (T-\tau)} \, d\tau \right) \psi_n(r). \tag{6}$$

Due to the mean value theorem of integrals, we obtain

$$\varphi(r) = \sum_{n=1}^{\infty} f_n \bigl(Te^{-(\frac{n\pi}{r_0})^2 (T-t_n)} \bigr) \psi_n(r), \quad 0 < t_n < T. \tag{7}$$

Define the operator $K : f(\cdot) \to \varphi(\cdot)$, then we have

$$\varphi(r) = Kf(r) = \sum_{n=1}^{\infty} \bigl(Te^{-(\frac{n\pi}{r_0})^2 (T-t_n)} \bigr) (f, \psi_n) \psi_n. \tag{8}$$

It is easy to see that K is a linear compact operator, and the singular values $\{\sigma_n\}_{n=1}^{\infty}$ of K satisfy

$$\sigma_n = Te^{-(\frac{n\pi}{r_0})^2 (T-t_n)} \tag{9}$$

and

$$(\varphi, \psi_n) = (f, \psi_n) Te^{-(\frac{n\pi}{r_0})^2 (T-t_n)}, \tag{10}$$

i.e.,

$$(f, \psi_n) = \sigma_n^{-1} (\varphi, \psi_n). \tag{11}$$

So

$$f(r) = K^{-1} \varphi(r) = \sum_{n=1}^{\infty} \sigma_n^{-1} \bigl(\varphi(r), \psi_n(r) \bigr) \psi_n(r). \tag{12}$$

From equation (12), we can see $\sigma_n^{-1} \to \infty$ $(n \to \infty)$. Thus, the exact data function $\varphi(r)$ must decrease rapidly. But the measured data function $\varphi^\delta(r)$ only belongs to $L^2[0, r_0; r^2]$, we cannot expect it has the same decay rate in $L^2[0, r_0; r^2]$. Thus the problem (1) is ill-posed. It is impossible to solve this problem using a classical method. We will use the truncated regularization method to deal with the ill-posed problem. Before doing that, we impose an *a priori* bound on the unknown heat source, *i.e.*,

$$\bigl\| f(\cdot) \bigr\|_{H^p(0,\pi)} \le E, \quad p > 0, \tag{13}$$

where $E > 0$ is a constant and $\| \cdot \|_{H^p(0,\pi)}$ denotes the norm in Sobolev space which is defined as follows:

$$\bigl\| f(\cdot) \bigr\|_{H^p(0,\pi)} := \left(\sum_{n=1}^{\infty} \bigl(1 + n^2\bigr)^p \bigl| \bigl(f(\cdot), \psi_n(\cdot) \bigr) \bigr|^2 \right)^{\frac{1}{2}}. \tag{14}$$

This paper is organized as follows. In Section 2, under the *a posteriori* parameter choice rule, we give the convergence error estimate. In Section 3, three numerical examples are used to verify the effectiveness for the proposed method. In Section 4, the conclusion of this paper is given.

2 Main result

From (12), we define

$$f_N^\delta = P_N\bigl(f^\delta(r)\bigr) = \sum_{n=1}^{N} \sigma_n^{-1}\varphi_n^\delta \psi_n \tag{15}$$

as the regularized solution of (1), where $P_N : L^2[0,r_0;r^2] \to \operatorname{span}\{\psi_n | n \le N\}$ is the rectangular projection,

$$\varphi_n^\delta = \bigl(\varphi^\delta, \psi_n\bigr), \quad n = 1,2,\ldots \tag{16}$$

is the Fourier coefficient of $\varphi^\delta(r)$. Due to the discrepancy principle, we consider an *a posteriori* regularization parameter choice rule as follows:

$$\bigl\|(I - P_N)\varphi^\delta\bigr\| \le \tau\delta < \bigl\|(I - P_{N-1})\varphi^\delta\bigr\|, \tag{17}$$

where $\tau > 1$ is a constant, I is an identity operator in $L^2[0,r_0;r^2]$.

Let

$$\rho_N = \bigl\|(I - P_N)\varphi^\delta\bigr\|. \tag{18}$$

According to the following lemma, we know there exists an unique solution for (18).

Lemma 1 *For* $\delta > 0$, *the function* ρ_N *satisfies*:

(a) ρ_N *is a continuous function*;
(b) $\lim_{N\to 0^+} \rho_N = \|\varphi^\delta\|$;
(c) $\lim_{N\to +\infty} \rho_N = 0$;
(d) ρ_N *is a strictly decreasing function over* $(0,\infty)$.

Lemma 2 ([17, 18]) *As* $n \ge 1$, *we obtain*

$$\frac{c_1}{n\pi} \le \sigma_n \le \frac{c_2}{n\pi}, \tag{19}$$

where c_1, c_2 *are constants.*

Lemma 3 *Assume conditions* (2) *and* (13) *hold*. *N is taken as the solution of* (17). *Then we have*

$$N(\delta) \le c_3\left[\frac{(\tau-1)\delta}{E}\right]^{\frac{-2}{p+2}}, \tag{20}$$

where $c_3 := \pi^{\frac{-2}{p+2}} c_2^{\frac{2}{p+2}}$.

Proof Using (17), we have

$$\begin{aligned}\left\|(I-P_{N-1})\varphi\right\|^2 &= \sum_{n=N(\delta)}^{\infty}|\varphi_n|^2 = \sum_{n=N(\delta)}^{\infty}\left(1+n^2\right)^{\frac{p}{2}}\sigma_n^{-2}|\varphi_n|^2\left(1+n^2\right)^{\frac{-p}{2}}\sigma_n^2 \\ &\le E^2\sup_{n\ge N(\delta)}\left(1+n^2\right)^{\frac{-p}{2}}\sigma_n^2 \le E^2N(\delta)^{-p}\left(\frac{c_2}{N(\delta)\pi}\right)^2 \\ &= E^2c_2^2\pi^{-2}N(\delta)^{-p-2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|(I-P_{N-1})\varphi\right\| &= \left\|(I-P_{N-1})\varphi^\delta-(I-P_{N-1})\left(\varphi^\delta-\varphi\right)\right\| \\ &\ge \left\|(I-P_{N-1})\varphi^\delta\right\| - \left\|(I-P_{N-1})\left(\varphi^\delta-\varphi\right)\right\| \\ &\ge \tau\delta-\delta=(\tau-1)\delta.\end{aligned}$$

So

$$(\tau-1)\delta \le Ec_2\pi^{-1}N(\delta)^{\frac{-p-2}{2}}.$$

Thus

$$N(\delta)\le c_2^{\frac{2}{p+2}}\pi^{\frac{-2}{p+2}}\left[\frac{(\tau-1)\delta}{E}\right]^{\frac{-2}{p+2}}.$$

This completes the proof of Lemma 3. □

Lemma 4 *If the regularized solution is given by* (15), *we have*

$$\left\|f_{N(\delta)}^\delta(\cdot)-f_{N(\delta)}(\cdot)\right\|\le c_4(\tau-1)^{\frac{-2}{p+2}}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, \tag{21}$$

where $c_4 := c_1^{-1}c_2^{\frac{2}{p+2}}\pi^{\frac{2}{p+2}}$.

Proof Due to (20), we obtain

$$\begin{aligned}\left\|f_{N(\delta)}^\delta(\cdot)-f_{N(\delta)}(\cdot)\right\|^2 &= \left\|\sum_{n=1}^{N(\delta)}\sigma_n^{-1}\left(\varphi(\cdot)-\varphi^\delta(\cdot)\right)\psi_n\right\|^2 \\ &= \sum_{n=1}^{N(\delta)}\sigma_n^{-2}\left|\varphi_n-\varphi_n^\delta\right|^2 \le \delta^2\cdot\sup_{1\le n\le N(\delta)}\sigma_n^{-2} \\ &\le \delta^2\left(\frac{N(\delta)\pi}{c_1}\right)^2 \le c_1^{-2}c_2^{\frac{4}{p+2}}\pi^{\frac{4}{p+2}}(\tau-1)^{\frac{-4}{p+2}}\delta^{\frac{2p}{p+2}}E^{\frac{4}{p+2}}.\end{aligned}$$

So

$$\left\|f_{N(\delta)}^\delta(\cdot)-f_{N(\delta)}(\cdot)\right\|\le c_1^{-1}c_2^{\frac{2}{p+2}}\pi^{\frac{2}{p+2}}(\tau-1)^{\frac{-2}{p+2}}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}.$$

This completes the proof of Lemma 4. □

Lemma 5 *Suppose conditions* (2) *and* (13) *hold.* $f(r)$ *given by* (12) *is the exact solution of* (1), *then we obtain*

$$\|f(\cdot)-f_{N(\delta)}(\cdot)\| \le c_5(\tau+1)^{\frac{p}{p+2}}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, \tag{22}$$

where $c_5 := c_1^{\frac{-p}{p+2}}\pi^{\frac{p}{p+2}}$.

Proof

$$\begin{aligned}
\|f(\cdot)-f_{N(\delta)}(\cdot)\|^2 &= \left\|\sum_{n=N(\delta)+1}^{\infty}\sigma_n^{-1}(\varphi,\psi_n)\psi_n\right\|^2 = \sum_{n=N(\delta)+1}^{\infty}\sigma_n^{-2}|\varphi_n|^2 \\
&= \sum_{n=N(\delta)+1}^{\infty}\left(\sigma_n^{-p}\left(1+n^2\right)^{\frac{-p}{2}}\left(1+n^2\right)^{\frac{p}{2}}\sigma_n^{-2}|\varphi_n|^2\right)^{\frac{2}{p+2}}\left(|\varphi_n|^2\right)^{\frac{p}{p+2}} \\
&\le \left(\sum_{n=N(\delta)+1}^{\infty}\left(\frac{c_1}{n\pi}\right)^{-p}n^{-p}\left(1+n^2\right)^{\frac{p}{2}}\sigma_n^{-2}|\varphi_n|^2\right)^{\frac{2}{p+2}}\left(\sum_{n=N(\delta)+1}^{\infty}|\varphi_n|^2\right)^{\frac{p}{p+2}} \\
&\le c_1^{\frac{-2p}{p+2}}\pi^{\frac{2p}{p+2}}E^{\frac{4}{p+2}}\left\|(I-P_N)\varphi\right\|^{\frac{2p}{p+2}} \\
&\le c_1^{\frac{-2p}{p+2}}\pi^{\frac{2p}{p+2}}E^{\frac{4}{p+2}}\left[\left\|(I-P_N)\varphi^\delta\right\| + \left\|(I-P_N)\left(\varphi^\delta-\varphi\right)\right\|\right]^{\frac{2p}{p+2}} \\
&\le c_1^{\frac{-2p}{p+2}}\pi^{\frac{2p}{p+2}}E^{\frac{4}{p+2}}\left[(\tau+1)\delta\right]^{\frac{2p}{p+2}}.
\end{aligned}$$

So

$$\|f(\cdot)-f_{N(\delta)}(\cdot)\| \le c_1^{\frac{-p}{p+2}}\pi^{\frac{p}{p+2}}(\tau+1)^{\frac{p}{p+2}}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}. \tag{23}$$

This completes the proof of Lemma 5. □

Now we give the convergent error estimate between the exact solution and the regularized solution.

Theorem 6 $f(r)$ *given by* (12) *is the exact solution of* (1), f_N^δ *given by* (15) *is the regularized solution of* (1). *The regularization parameter is given by* (17). *So we have*

$$\|f(\cdot)-f_{N(\delta)}^\delta(\cdot)\| \le \left[c_5(\tau+1)^{\frac{2}{p+2}} + c_4(\tau-1)^{\frac{-2}{p+2}}\right]E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{24}$$

Proof Using the triangle inequality, (21) and (22), we have

$$\begin{aligned}
\|f(\cdot)-f_{N(\delta)}^\delta(\cdot)\| &= \|f(\cdot)-f_{N(\delta)}(\cdot)+f_{N(\delta)}(\cdot)-f_{N(\delta)}^\delta(\cdot)\| \\
&\le \|f(\cdot)-f_{N(\delta)}(\cdot)\| + \|f_{N(\delta)}(\cdot)-f_{N(\delta)}^\delta(\cdot)\| \\
&\le c_5(\tau+1)^{\frac{2}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}} + c_4(\tau-1)^{\frac{-2}{p+2}}\delta^{\frac{P+4}{p+2}}E^{\frac{-2}{p+2}} \\
&\le \left[c_5(\tau+1)^{\frac{2}{p+2}} + c_4(\tau-1)^{\frac{-2}{p+2}}\right]E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.
\end{aligned}$$

This completes the proof of Theorem 6. □

3 Numerical experiments

In this section, three numerical examples are used to illustrate the usefulness of proposed method. Moreover, the comparisons of numerical effectiveness between the *a posteriori* parameter choice (17) and the *a priori* parameter choice rule which is obtained by $N = [(\frac{E}{\delta})^{\frac{2}{p+2}}]$ in [16] are also considered. The measurable data is given as follows:

$$\varphi^\delta(r) = \varphi + \varepsilon \operatorname{rand} n\bigl(\operatorname{size}(\varphi)\bigr), \tag{25}$$

where

$$\varphi = \bigl(\varphi(r_1), \dots, \varphi(r_n)\bigr)^T, \quad r_i = (i-1)\Delta r, \Delta r = \frac{r_0}{n-1}, i = 1, 2, \dots, n. \tag{26}$$

The total noise level δ can be measured in the sense of the root mean square error (RMSE) as follows:

$$\delta = \bigl\|\varphi^\delta - \varphi\bigr\|_{L^2} = \left(\frac{1}{n}\sum_{i=1}^{n}\bigl(\varphi_i - \varphi_i^\delta\bigr)^2\right)^{\frac{1}{2}}. \tag{27}$$

To show the accuracy of numerical solution, the approximate L^2 error is computed as follows:

$$e_a = \bigl\|f(r) - f_N^\delta(r)\bigr\|_{L^2},$$

and the approximate relative error in L^2 norm is denoted by

$$e_r = \frac{\|f(r) - f_N^\delta(r)\|}{\|f(r)\|}.$$

It is difficult to find an exact solution for problem (1) in our numerical experiment. We first give the heat source $f(r)$ and solve the following direct problem:

$$\begin{cases} u_t - \frac{2}{r}u_r - u_{rr} = f(r), & 0 < t < T, 0 < r < r_0, \\ u(r,0) = 0, & 0 \le r \le r_0, \\ u(r_0,t) = 0, & 0 \le t \le T, \\ u(0,t) = 0, & 0 < t < T. \end{cases} \tag{28}$$

Then we use $u(r,T) = \varphi(r)$ and (18) to obtain the exact data $\varphi(r)$ and the noise data $\varphi^\delta(r)$, respectively. Finally, we solve the inverse problem to obtain the regularization solution $f_{N(\delta)}^\delta(r)$. In the following three numerical examples, we take $T = 1$ and $r_0 = \pi$.

Example 1 Consider a smooth heat source: $f(r) = r \sin r$.

Example 2 Consider a piecewise smooth heat source:

$$f(r)=\begin{cases}0, & 0\le r\le \frac{\pi}{4},\\ \frac{4}{\pi}(r-\frac{\pi}{4}), & \frac{\pi}{4}<r\le \frac{\pi}{2},\\ -\frac{4}{\pi}(r-\frac{3\pi}{4}), & \frac{\pi}{2}<r\le \frac{3\pi}{4},\\ 0, & \frac{3\pi}{4}<r\le \pi.\end{cases} \tag{29}$$

Example 3 Consider the following discontinuous function:

$$f(r)=\begin{cases}0, & 0\le r\le \frac{\pi}{3},\\ 1, & \frac{\pi}{3}<r\le \frac{2\pi}{3},\\ 0, & \frac{2\pi}{3}<r\le \pi.\end{cases} \tag{30}$$

Firstly, we use Examples 1 and Examples 2 to compare the numerical effects among the truncate regularization method, the Tikhonov regularization method and quasi-boundary value regularization method under the *a posteriori* choice rule. The numerical results is shown in Tables 1 and 2. The Tikhonov regularization solution of problem (1) is given as follows:

$$f_\alpha^\delta(r)=\sum_{n=1}^{\infty}\frac{\sigma_n^{-1}}{1+\alpha^2\sigma_n^{-2}}\varphi_n^\delta\psi_n(r),$$

where $0<\alpha<1$ is the regularization parameter.

Through modifying the final value condition $u(r,T)=\varphi(r)$, we solve the following problem:

$$\begin{cases}v_t-\frac{2}{r}v_r-v_{rr}=f(r), & 0<t<T, 0<r<r_0,\\ v(r,0)=0, & 0\le r\le r_0,\\ v(r_0,t)=0, & 0\le t\le T,\\ \lim_{r\to 0} v(r,t) \text{ is bounded}, & 0<t<T,\\ v(r,T)=\varphi(r)-\mu^2 f(r), & 0\le r\le r_0,\end{cases}$$

where μ is the regularization parameter. Then we obtain the quasi-boundary value solution of problem (1) as follows:

$$f_\mu^\delta(r)=\sum_{n=1}^{\infty}\frac{1}{\mu^2+\sigma_n}\varphi_n^\delta\psi_n(r).$$

Tables 1 and 2 gives the comparisons of the numerical results of the truncate regularization method, the Tikhonov regularization method and the quasi-boundary regularization method under the *a posteriori* choice rule for different ε. From Tables 1 and 2, we can see that the effectiveness of the truncate regularization method in the present paper is better than the other regularization methods.

Figure 1 shows the comparisons between the exact solution and its computed approximation with different noise levels for Examples 1. Figure 2 shows the comparisons between

Table 1 Numerical results for different ε under an *a posteriori* choice rule for three regularization methods about Examples 1

	ε	0.05	0.01	0.005	0.001	0.0005	0.0001
Truncate	e_a	0.0697	0.0376	0.0243	0.0118	0.0082	0.0035
	e_r	0.0593	0.0320	0.0207	0.0101	0.0070	0.0030
Tikhonov	e_a	0.1024	0.0407	0.0277	0.0152	0.0065	0.0034
	e_r	0.0871	0.0346	0.0236	0.0129	0.0056	0.0029
Quasi-boundary	e_a	0.6363	0.1518	0.0437	0.0160	0.0107	0.0043
	e_r	0.5414	0.1291	0.0372	0.0136	0.0091	0.0037

Table 2 Numerical results for different ε under an *a posteriori* choice rule for three regularization methods about Examples 2

	ε	0.05	0.01	0.005	0.001	0.0005	0.0001
Truncate	e_a	0.0972	0.0521	0.0304	0.0241	0.0226	0.0139
	e_r	0.2392	0.1283	0.0749	0.0594	0.0557	0.0341
Tikhonov	e_a	0.1295	0.0871	0.0361	0.0273	0.0243	0.0378
	e_r	0.3178	0.2143	0.0889	0.0672	0.0598	0.0930
Quasi-boundary	e_a	0.7800	0.2036	0.0566	0.0539	0.0482	0.0393
	e_r	1.9194	0.5011	0.1393	0.1327	0.1186	0.0967

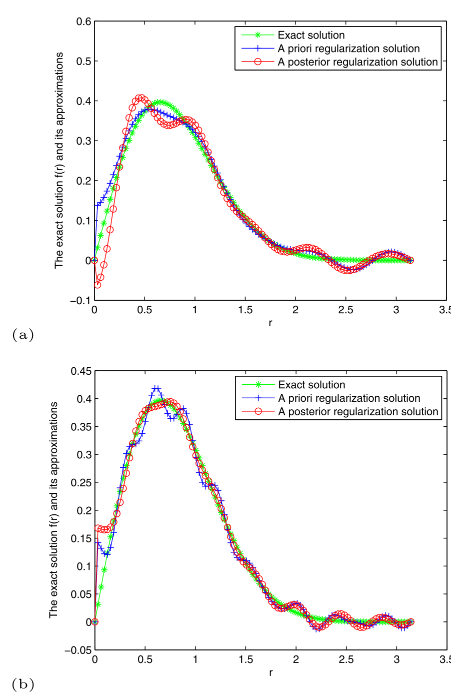

Figure 1 The comparison of numerical effects between the exact solution and its computed approximations for $p = 1$ with Examples 1: (a) $\varepsilon = 0.001$, (b) $\varepsilon = 0.0001$.

the exact solution and its computed approximation with different noise levels for Examples 2. Figure 3 indicates the comparisons between the exact solution and its computed approximation with different noise levels for Examples 3. From Figures 1-3, we can find that the smaller ε, the better the computed approximation is. Moreover, we can also see that the *a posteriori* parameter choice also works well.

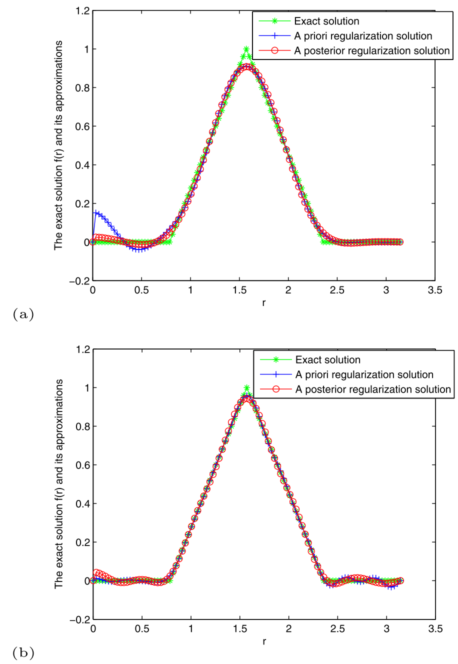

Figure 2 The comparison of numerical effects between the exact solution and its computed approximations for $p = 1$ with Examples 2: (a) $\varepsilon = 0.001$, (b) $\varepsilon = 0.0001$.

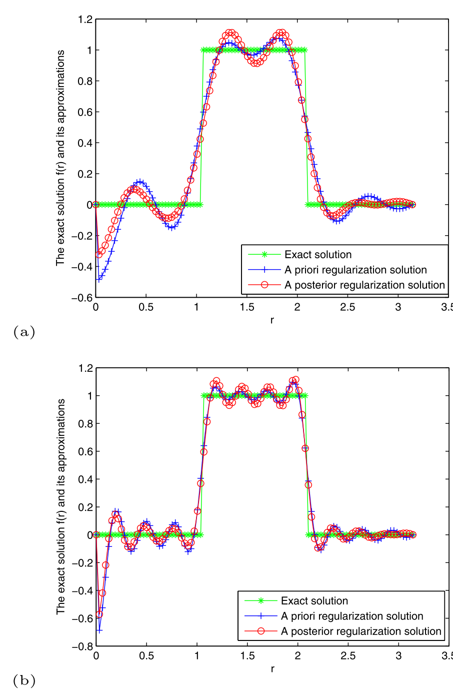

Figure 3 The comparison of numerical effects between the exact solution and its computed approximations for $p = 1$ with Examples 3: (a) $\varepsilon = 0.001$, (b) $\varepsilon = 0.0001$.

4 Conclusion

Using the Morozov discrepancy principle, we obtain an *a posteriori* parameter choice rule which only depends on the measured data. Under the *a posteriori* choices of the regularization parameter, the Hölder type error estimate which is order optimal is obtained. Meanwhile, several numerical examples verify the efficiency and accuracy of this method.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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