Stević *Advances in Difference Equations* (2017) 2017:169
DOI 10.1186/s13662-017-1227-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of a unique bounded solution to a linear second-order difference equation and the linear first-order difference equation

Stevo Stević*

*Correspondence: sstevic@ptt.rs
Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, Beograd, 11000, Serbia
Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia

Abstract

We present some interesting facts connected with the following second-order difference equation:

$$x_{n+2} - q_n x_n = f_n, \quad n \in \mathbb{N}_0,$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are given sequences of numbers. We give some sufficient conditions for the existence of a unique bounded solution to the difference equation and present an elegant proof based on a combination of theory of linear difference equations and the Banach fixed point theorem. We also deal with the equation by using theory of solvability of difference equations. A global convergence result of solutions to a linear first-order difference equation is given. Some comments on an abstract version of the linear first-order difference equation are also given.

MSC: Primary 39A06; secondary 39A45

Keywords: second-order difference equation; unique bounded solutions; fixed point theorem; linear first-order difference equation

1 Introduction

Difference equations and systems of difference equations have been of a great interest in the last several decades. For some classical results in the research area, see, for example, [1–8], whereas some recent results can be found, for example, in [9–22] (see also the references therein). Book [23] contains a lot of results obtained up to 2,000.

It is a frequent situation that during some investigations in various areas of mathematics and science naturally appeared special cases of the following nonhomogeneous linear second-order difference equation:

$$x_{n+2} - q_n x_n = f_n, \quad n \in \mathbb{N}_0, \tag{1}$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are sequences of real or complex numbers. Recall that equations of type (1) with $f_n = 0$, $n \in \mathbb{N}_0$, appeared in many problem books on classical analysis ([7, 24]), especially in those dealing with sequences and integrals (by using the integration in parts an integral I_n is frequently presented in terms of a linear function of I_{n-2}). However, this case is very simple, so the case $f_n \not\equiv 0$, is more interesting.

Although equation (1) is of second order and resembles the following differential equation:

$$x''(t) - q(t)x(t) = f(t), \quad t \geq 0, \tag{2}$$

it is not the one that corresponds quite well to it. The following second-order difference equation:

$$\Delta^2 x_n - q_n x_n = f_n, \quad n \in \mathbb{N}_0,$$

where $\Delta x_n = x_{n+1} - x_n$ is the forward difference, is one of the discrete cousins of equation (2), and is much more interesting for investigation than equation (1). Nevertheless, equation (1) has some nice properties which seem are not so known and will be presented here.

As a further motivation, recall that if q is a positive number, then the difference equation

$$x_{n+2} - qx_n = 0, \quad n \in \mathbb{N}_0, \tag{3}$$

where x_0, x_1 are given numbers, is easily solved and it has the following general solution:

$$x_n = \hat{c}_1(\sqrt{q})^n + \hat{c}_2(-\sqrt{q})^n, \quad n \in \mathbb{N}_0, \tag{4}$$

where $\hat{c}_1$ and $\hat{c}_2$ are arbitrary constants ([3, 6, 8, 23]).

By using equation (4) it is immediately seen that the following three statements hold.

(a) If $q \in (0,1)$, then all the solutions to equation (3) converge to zero.

(b) If $q = 1$ then all the solutions to equation (3) are bounded.

(c) If $q > 1$, then all the solutions to equation (3) are unbounded, except the trivial solution

$$x_n = 0, \quad n \in \mathbb{N}_0,$$

which is obtained for the initial conditions $x_0 = x_1 = 0$.

In terms of the boundedness these statements claim that all the solutions to equation (3) are bounded if and only if $q \in (0,1]$, while for the case $q > 1$ there is only one bounded solution to the equation.

It is a classical problem to see how bounded perturbations of the right-hand side of equation (3), as well as of coefficient q influence on the boundedness character of the solutions of such obtained equations. We will present two interesting ways how the problem of the existence of a unique bounded solution can be solved for the case of the difference equation (1), where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are two given bounded sequences.

By l^∞ we will denote the space consisting of all bounded sequences $u = (u_n)_{n\in\mathbb{N}_0}$ with the supremum norm

$$\|u\|_\infty = \sup_{n\in\mathbb{N}_0} |u_n|. \tag{5}$$

It is well known that l^∞ with norm (5) is a Banach space.

The paper is partially based on several comments and ideas presented by the author at several international conferences and invited talks during the last several years, which has not been published so far. Some of the results could be known, but we could not find specific references for them.

The paper is organized as follows. First, we consider the case when $q_n = q \neq 0$ for every $n \in \mathbb{N}_0$ and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence. Then we consider the case when the sequences $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are bounded. By a nice combination of the theory of linear difference equations and the Banach fixed point theorem we present some sufficient conditions for the existence of a unique bounded solution to equation (1). In the third section we will present another way for dealing with the unique existence problem by using only the theory of linear difference equations. We finish the paper by giving some comments on the form of general solution to an abstract version of the linear first-order difference equation.

2 Fixed point approach in dealing with equation (1)

In this section we deal with the problem of the existence of a unique bounded solution to the difference equation (1). To do this we use fixed point theory. First we deal with the case when $q_n = q \neq 0$ for every $n \in \mathbb{N}_0$ and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence.

Before we formulate and prove the main result in the section we need an auxiliary result which is incorporated into the following lemma. The lemma is essentially folklore, but we will give a proof of it for completeness, for the benefit of the reader and also as a good motivation and better understanding of some ideas appearing in the proof of the main result.

Lemma 1 *Consider the difference equation*

$$x_{n+2} - qx_n = f_n, \quad n \in \mathbb{N}_0, \tag{6}$$

where $q \in \mathbb{C} \setminus \{0\}$, x_0 and x_1 are given complex numbers, and $(f_n)_{n\in\mathbb{N}_0}$ is a given sequence of complex numbers. Then the general solution to equation (6) *is given by the following formula*:

$$x_n = (\sqrt{q})^n \left(c_0 + \sum_{k=0}^{n-1} \frac{f_k}{2(\sqrt{q})^{k+2}} \right) + (-\sqrt{q})^n \left(d_0 + \sum_{k=0}^{n-1} \frac{(-1)^k f_k}{2(\sqrt{q})^{k+2}} \right), \quad n \in \mathbb{N}_0, \tag{7}$$

where c_0 and d_0 are arbitrary numbers, and $\sqrt{q}$ is one of two possible roots of q.

Proof The difference equation (6) can be solved. We will demonstrate it by using the method of variation of constants ([8, 23]). Namely, based on the form of the general solution to the homogeneous equation (3) given in (4), we seek the general solution to equation (6) in the form

$$x_n = c_n(\sqrt{q})^n + d_n(-\sqrt{q})^n, \quad n \in \mathbb{N}_0, \tag{8}$$

where $(c_n)_{n\in\mathbb{N}_0}$ and $(d_n)_{n\in\mathbb{N}_0}$ are two (undetermined) sequences.

Further, we pose the following condition:

$$x_{n+1} = c_{n+1}(\sqrt{q})^{n+1} + d_{n+1}(-\sqrt{q})^{n+1} = c_n(\sqrt{q})^{n+1} + d_n(-\sqrt{q})^{n+1}, \tag{9}$$

for $n \in \mathbb{N}_0$, that is,

$$(c_{n+1} - c_n)(\sqrt{q})^{n+1} + (d_{n+1} - d_n)(-\sqrt{q})^{n+1} = 0, \tag{10}$$

for $n \in \mathbb{N}_0$.

Using (9) with $n \to n+1$ along with (8) into equation (6) is obtained

$$c_{n+1}(\sqrt{q})^{n+2} + d_{n+1}(-\sqrt{q})^{n+2} - q\big(c_n(\sqrt{q})^n + d_n(-\sqrt{q})^n\big) = f_n,$$

that is,

$$(c_{n+1} - c_n)(\sqrt{q})^{n+2} + (d_{n+1} - d_n)(-\sqrt{q})^{n+2} = f_n, \tag{11}$$

for $n \in \mathbb{N}_0$.

For each fixed $n \in \mathbb{N}_0$, equalities (10) and (11) can be regarded as a two-dimensional linear system in the variables $c_{n+1} - c_n$ and $d_{n+1} - d_n$.

By solving the system the following is easily obtained:

$$c_{n+1} - c_n = \frac{f_n}{2(\sqrt{q})^{n+2}} \quad \text{and} \quad d_{n+1} - d_n = \frac{(-1)^n f_n}{2(\sqrt{q})^{n+2}}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$c_n = c_0 + \sum_{k=0}^{n-1} \frac{f_k}{2(\sqrt{q})^{k+2}} \tag{12}$$

and

$$d_n = d_0 + \sum_{k=0}^{n-1} \frac{(-1)^k f_k}{2(\sqrt{q})^{k+2}}, \tag{13}$$

for $n \in \mathbb{N}_0$.

Using (12) and (13) into (8) we get equation (7). That (7) represents the general solution to (6) follows from the facts that the sequence

$$x_n^p := (\sqrt{q})^n \sum_{k=0}^{n-1} \frac{f_k}{2(\sqrt{q})^{k+2}} + (-\sqrt{q})^n \sum_{k=0}^{n-1} \frac{(-1)^k f_k}{2(\sqrt{q})^{k+2}}$$

is a particular solution to equation (6), which is easily verified, while

$$x_n^h = c_0(\sqrt{q})^n + d_0(-\sqrt{q})^n,$$

is the general solution to the corresponding homogeneous equation ([8, 23]). □

The following result solves the problem of existence of a unique bounded solution to equation (1) for the case $q_n = q$, $n \in \mathbb{N}_0$, when $|q| > 1$.

Theorem 1 *Consider the difference equation* (6), *where q is a complex number such that $|q| > 1$ and $f := (f_n)_{n\in\mathbb{N}_0}$ is a given bounded sequence of complex numbers. Then there is a unique bounded solution to the difference equation.*

Proof By Lemma 1 we know that the general solution to the difference equation (6) is given by equation (7). By using equation (7) for even and odd indices it follows that

$$x_{2n} = q^n\left(c_0 + d_0 + \sum_{k=0}^{2n-1} \frac{(1+(-1)^k)f_k}{2(\sqrt{q})^{k+2}}\right) \tag{14}$$

and

$$x_{2n+1} = (\sqrt{q})^{2n+1}\left(c_0 - d_0 + \sum_{k=0}^{2n} \frac{(1-(-1)^k)f_k}{2(\sqrt{q})^{k+2}}\right), \tag{15}$$

for every $n \in \mathbb{N}_0$.

Now note that from the assumptions of the theorem we have

$$\left|\sum_{k=0}^{\infty} \frac{(1+(-1)^k)f_k}{2(\sqrt{q})^{k+2}}\right| \le \sum_{k=0}^{\infty} \frac{\|f\|_\infty}{(\sqrt{|q|})^{k+2}} = \frac{\|f\|_\infty}{\sqrt{|q|}(\sqrt{|q|}-1)}$$

and

$$\left|\sum_{k=0}^{\infty} \frac{(1-(-1)^k)f_k}{2(\sqrt{q})^{k+2}}\right| \le \sum_{k=0}^{\infty} \frac{\|f\|_\infty}{(\sqrt{|q|})^{k+2}} = \frac{\|f\|_\infty}{\sqrt{|q|}(\sqrt{|q|}-1)}$$

so that these two series are absolutely convergent.

Using this fact, (14), (15) and the assumption $|q| > 1$ we see that for a bounded solution $(x_n)_{n\in\mathbb{N}_0}$ to equation (6) we must have

$$c_0 + d_0 = -\sum_{k=0}^{\infty} \frac{(1+(-1)^k)f_k}{2(\sqrt{q})^{k+2}} = -\sum_{j=0}^{\infty} \frac{f_{2j}}{(\sqrt{q})^{2j+2}} \tag{16}$$

and

$$c_0 - d_0 = \sum_{k=0}^{\infty} \frac{((-1)^k-1)f_k}{2(\sqrt{q})^{k+2}} = -\sum_{j=0}^{\infty} \frac{f_{2j+1}}{(\sqrt{q})^{2j+3}} \tag{17}$$

(at the moment we do not know if a bounded solution to (6) exists, but if it does, then its subsequences (x_{2n}) and (x_{2n+1}) will be bounded, which will imply that (16) and (17) must hold).

From (16) and (17) it follows that

$$c_0 = -\frac{1}{2}\sum_{k=0}^{\infty} \frac{f_k}{(\sqrt{q})^{k+2}} \tag{18}$$

and

$$d_0 = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}f_k}{(\sqrt{q})^{k+2}}. \tag{19}$$

If we use (18) and (19) in (7), we get

$$\begin{aligned} x_n &= -(\sqrt{q})^n\frac{1}{2}\sum_{k=n}^{\infty}\frac{f_k}{(\sqrt{q})^{k+2}} + (-\sqrt{q})^n\frac{1}{2}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}f_k}{(\sqrt{q})^{k+2}} \\ &= (\sqrt{q})^n\frac{1}{2}\sum_{k=n}^{\infty}\frac{((-1)^{n+k+1}-1)f_k}{(\sqrt{q})^{k+2}}, \end{aligned} \tag{20}$$

for $n \in \mathbb{N}_0$.

A direct calculation shows that sequence $(x_n)_{n\in\mathbb{N}_0}$ defined by (20) is a solution to equation (6). On the other hand, by using the assumptions of the theorem we easily get

$$|x_n| \le \frac{\|f\|_\infty}{\sqrt{|q|}(\sqrt{|q|}-1)} < \infty, \quad n \in \mathbb{N}_0,$$

from which the boundedness of $(x_n)_{n\in\mathbb{N}_0}$, follows. From this and since (c_0, d_0) is a unique solution to the linear system (16)-(17), $(x_n)_{n\in\mathbb{N}_0}$ defined by (20) is a unique bounded solution to equation (6). □

Now we are in a position to formulate and prove the main result in the section in an elegant way.

Theorem 2 *Consider the difference equation* (1), *where*

$$1 < a \le q_n \le b, \quad n \in \mathbb{N}_0, \tag{21}$$

or

$$-b \le q_n \le -a < -1, \quad n \in \mathbb{N}_0, \tag{22}$$

for some positive numbers a and b, and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers. Then equation (1) *has a unique bounded solution.*

Proof We will prove the theorem under condition (21). The proof when (22) holds is similar/dual so is omitted.

Let q be a positive number such that

$$q \in \bigl(\max\{a, (b+1)/2\}, b\bigr). \tag{23}$$

Write equation (1) in the following form:

$$x_{n+2} - qx_n = (q_n - q)x_n + f_n, \tag{24}$$

for $n \in \mathbb{N}_0$.

Let A be the following operator defined on the class of all sequences:

$$A(u)=\left((\sqrt{q})^n\frac{1}{2}\sum_{k=n}^{\infty}\frac{((-1)^{n+k+1}-1)((q_k-q)u_k+f_k)}{(\sqrt{q})^{k+2}}\right)_{n\in\mathbb{N}_0}. \tag{25}$$

If $u\in l^\infty$, then from (25) it follows that

$$\begin{aligned}\|A(u)\|_\infty &= \sup_{n\in\mathbb{N}_0}\left|(\sqrt{q})^n\frac{1}{2}\sum_{k=n}^{\infty}\frac{((-1)^{n+k+1}-1)((q_k-q)u_k+f_k)}{(\sqrt{q})^{k+2}}\right| \\ &\le \sup_{n\in\mathbb{N}_0}\sum_{k=n}^{\infty}\frac{(q_k+q)|u_k|+|f_k|}{|\sqrt{q}|^{k+2-n}} \\ &\le \frac{(b+q)\|u\|_\infty+\|f\|_\infty}{\sqrt{q}(\sqrt{q}-1)}<\infty,\end{aligned}$$

which means that the operator A maps the Banach space l^∞ into itself.

On the other hand, for every $u,v\in l^\infty$ we have

$$\begin{aligned}\|A(u)-A(v)\|_\infty &= \sup_{n\in\mathbb{N}_0}\left|(\sqrt{q})^n\frac{1}{2}\sum_{k=n}^{\infty}\frac{((-1)^{n+k+1}-1)(q_k-q)(u_k-v_k)}{(\sqrt{q})^{k+2}}\right| \\ &= \sup_{n\in\mathbb{N}_0}\left|(\sqrt{q})^n\sum_{j=0}^{\infty}\frac{(q_{n+2j}-q)(u_{n+2j}-v_{n+2j})}{(\sqrt{q})^{n+2j+2}}\right| \\ &\le \sup_{n\in\mathbb{N}_0}\sum_{j=0}^{\infty}\frac{|q_{n+2j}-q||u_{n+2j}-v_{n+2j}|}{q^{j+1}} \\ &\le \frac{\max\{q-a,b-q\}}{q-1}\|u-v\|_\infty.\end{aligned} \tag{26}$$

Since $a>1$ we have $q-a<q-1$. On the other hand, from (23) we see that $0<b-q<q-1$. From this and (26), we have

$$\|A(u)-A(v)\|_\infty\le\hat{q}\|u-v\|_\infty, \tag{27}$$

for every $u,v\in l^\infty$, and for

$$\hat{q}:=\frac{\max\{q-a,b-q\}}{q-1}\in(0,1),$$

that is, $A:l^\infty\to l^\infty$ is a contraction.

By the Banach fixed point theorem we see that the operator has a unique fixed point, say $x^*=(x_n^*)_{n\in\mathbb{N}_0}\in l^\infty$, that is, $A(x^*)=x^*$ or equivalently

$$x_n^*=(\sqrt{q})^n\frac{1}{2}\sum_{k=n}^{\infty}\frac{((-1)^{n+k+1}-1)((q_k-q)x_k^*+f_k)}{(\sqrt{q})^{k+2}}, \tag{28}$$

for $n\in\mathbb{N}_0$.

A direct calculation shows that this bounded sequence satisfies the difference equation (1) for every $n\in\mathbb{N}_0$, from which the theorem follows. □

Remark 1 Beside the choice of operator (25), which is naturally imposed, a crucial point in the proof of Theorem 2 is the choice of constant q in (23) to get the contractivity of the operator. It is also expected that a modification of the above arguments can be applied to some other related difference equations.

3 Equation (1) versus the linear first-order difference equation

Another approach in dealing with equation (1) is to note that it is obtained from a linear first-order difference equation by scaling the indices. The *linear first-order difference equation* is of the form

$$x_{n+1} = q_n x_n + f_n, \quad n \in \mathbb{N}_0, \tag{29}$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are arbitrary real (or complex) sequences and $x_0 \in \mathbb{R}$ (or $x_0 \in \mathbb{C}$). The main feature of the equation is that it is solvable. How equation (29) is solved can be found, for example, in [2, 8, 23]. On periodic solutions to equation (29) see [9]. If $q_n \neq 0$ for every $n \in \mathbb{N}_0$, then general solution to equation (29) can be obtained, for example, as follows. Dividing both sides of (29) by $\prod_{j=0}^{n} q_j$, we obtain

$$\frac{x_{n+1}}{\prod_{j=0}^{n} q_j} = \frac{x_n}{\prod_{j=0}^{n-1} q_j} + \frac{f_n}{\prod_{j=0}^{n} q_j}, \quad n \in \mathbb{N}_0. \tag{30}$$

Summing the equalities in (30) from 0 to $n-1$, we easily get the following formula for a general solution to equation (29):

$$x_n = \prod_{j=0}^{n-1} q_j \left(x_0 + \sum_{i=0}^{n-1} \frac{f_i}{\prod_{j=0}^{i} q_j} \right). \tag{31}$$

It is important to note that many nonlinear difference equations and systems can be reduced to some special cases of equation (29), which means that they are solvable too. Some interest in the area has been renewed recently. For some recent classes of solvable difference equations see, for example, [11, 13, 14, 16, 17, 19, 25], while some related systems of difference equations can be found in [10, 12, 16, 18]. For some recent results on solvable product-type systems of difference equations, see [15, 20–22] and the references therein. For some classical equations and systems which can be reduced to (29) or solved by some other methods; see, for example, [3, 5, 6, 8, 23].

How useful and powerful equation (31) is shows the following small but nice result, which is an old result by the author which has never been published so far but has been presented at several talks.

Theorem 3 *Consider equation* (29). *Assume that* $0 < q_n \le p < 1$, *for* $n \in \mathbb{N}_0$, $(f_n)_{n\in\mathbb{N}_0}$ *is a sequence of real numbers, and that there is a finite*

$$\lim_{n\to\infty} \frac{f_n}{1-q_n}. \tag{32}$$

Then every real solution $(x_n)_{n\in\mathbb{N}_0}$ *to the equation is convergent.*

Proof Let the limit in (32) be equal to l. Since $q_n \neq 0$ for every $n \in \mathbb{N}_0$ we see that every solution to equation (29) can be written in the form as in (31). Since

$$\frac{1}{p^n} \leq \frac{1}{\prod_{j=0}^{n-1} q_j} < \frac{1}{\prod_{j=0}^{n} q_j}, \quad n \in \mathbb{N}_0,$$

it follows that the sequence

$$Q_n = \frac{1}{\prod_{j=0}^{n-1} q_j}, \quad n \in \mathbb{N},$$

increasingly tends to $+\infty$ as $n \to \infty$.

Using the Stoltz theorem ([7, 24]) and condition (32) we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{x_0 + \sum_{i=0}^{n-1} \frac{f_i}{\prod_{j=0}^{i} q_j}}{\frac{1}{\prod_{j=0}^{n-1} q_j}} = \lim_{n\to\infty} \frac{\frac{f_{n-1}}{\prod_{j=0}^{n-1} q_j}}{\frac{1-q_{n-1}}{\prod_{j=0}^{n-1} q_j}} = l,$$

which implies that $\lim_{n\to\infty} x_n = l$, for every solution to equation (29), from which the theorem follows. □

Remark 2 Special cases of Theorem 3 frequently appear at problem books or students' competitions (see, for example, [24]), but in all the cases that we have met so far the existence of the limit $\lim_{n\to\infty} f_n$ is assumed. However, if (32) holds, then the limit need not exist. For example, if we assume that sequences $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are defined as follows:

$$q_{2n} = q_0, \qquad q_{2n+1} = q_1,$$
$$f_{2n} = f_0, \qquad f_{2n+1} = f_1, \quad n \in \mathbb{N}_0,$$

where the following conditions are satisfied:

$$\frac{f_0}{1-q_0} = \frac{f_1}{1-q_1},$$

$f_1 \neq 0 \neq f_0 \neq f_1$ and $q_1, q_2 \in (0,1)$, then it is clear that

$$\lim_{n\to\infty} \frac{f_n}{1-q_n} = \frac{f_0}{1-q_0},$$

but there is no $\lim_{n\to\infty} f_n$.

If we assume that

$$\inf_{n\in\mathbb{N}_0} |q_n| =: q > 1, \tag{33}$$

then

$$\left|\prod_{j=0}^{n-1} q_j\right| \to +\infty, \quad \text{as } n \to \infty$$

(we can assume here also $\liminf_{n\to\infty}|q_n| > 1$). Hence, in this case a solution to equation (29) will be bounded, only if

$$x_0 = -\sum_{i=0}^{\infty} \frac{f_i}{\prod_{j=0}^{i} q_j}, \tag{34}$$

and using (34) in (31), we see that it will be given by the following formula:

$$x_n = -\prod_{j=0}^{n-1} q_j \sum_{i=n}^{\infty} \frac{f_i}{\prod_{j=0}^{i} q_j}, \quad n \in \mathbb{N}_0. \tag{35}$$

In order that (35) holds the convergence of the series in (34) is necessary, which will be so, for example, if $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence.

Now note that if $(x_n)_{n\in\mathbb{N}_0}$ is a solution to equation (1), then its subsequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n+1})_{n\in\mathbb{N}_0}$ are solutions to the equations

$$y_{n+1} = q_{2n} y_n + f_{2n}, \quad n \in \mathbb{N}_0, \tag{36}$$

and

$$y_{n+1} = q_{2n+1} y_n + f_{2n+1}, \quad n \in \mathbb{N}_0, \tag{37}$$

respectively, which are nothing but two difference equations of the form in (29), and consequently they are solvable.

By applying the arguments from (33) to (35), it is easy to see that the following result holds.

Theorem 4 *Consider equation* (1), *where condition* (33) *holds and* $(f_n)_{n\in\mathbb{N}_0}$ *is a bounded sequence. Then the equation has a unique bounded solution given by the formulas*

$$x_{2n} = -\prod_{j=0}^{n-1} q_{2j} \sum_{i=n}^{\infty} \frac{f_{2i}}{\prod_{j=0}^{i} q_{2j}},$$
$$x_{2n+1} = -\prod_{j=0}^{n-1} q_{2j+1} \sum_{i=n}^{\infty} \frac{f_{2i+1}}{\prod_{j=0}^{i} q_{2j+1}},$$

for every $n \in \mathbb{N}_0$.

Remark 3 Theorem 4 also shows that equation (1) has a unique bounded solution if (33) holds and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence, and gives it explicitly, although in not so nice way. Moreover, the conditions in Theorem 4 are somewhat weaker so that its result is somewhat stronger than the one in Theorem 2, which again shows the importance and usefulness of equation (31). Nevertheless, both approaches in dealing with equation (1) are interesting, each of them in its own way.

3.1 An abstract form of equation (29)

Equation (29) is not only 'solvable' for the case of real or complex initial values and coefficients. Namely, let S be a set equipped with two binary operations $\odot$ and $\oplus$ (two maps which send elements of the Cartesian product $S \times S$ to S), or in terminology of abstract algebra, let $(S,\odot)$ and $(S,\oplus)$ be two groupoids/magmas, such that operation $\odot$ is left-distributive over $\oplus$, that is,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z), \tag{38}$$

for every $x, y, z \in S$. Then the 'difference' equation

$$x_n = (a_{n-1} \odot x_{n-1}) \oplus b_{n-1}, \quad n \in \mathbb{N}, \tag{39}$$

where $x_0 \in S$, $a_n, b_n \in S$, $n \in \mathbb{N}_0$, is solvable.

Indeed, for $n = 1$, we get $x_1 = (a_0 \odot x_0) \oplus b_0$. From equality (39) with $n = 2$ along with (38), is obtained

$$x_2 = \bigl(a_1 \odot \bigl((a_0 \odot x_0) \oplus b_0\bigr)\bigr) \oplus b_1 = \bigl(a_1 \odot (a_0 \odot x_0)\bigr) \oplus (a_1 \odot b_0) \oplus b_1.$$

An inductive argument shows that

$$\begin{aligned} x_n &= \bigl(a_{n-1} \odot \bigl(a_{n-2} \odot \cdots \odot (a_0 \odot x_0) \cdots \bigr)\bigr) \\ &\quad \oplus \bigl(a_{n-1} \odot \bigl(a_{n-2} \odot \cdots \odot (a_1 \odot b_0) \cdots \bigr)\bigr) \oplus \cdots \oplus (a_{n-1} \odot b_{n-2}) \oplus b_{n-1}, \end{aligned} \tag{40}$$

for every $n \in \mathbb{N}$, which means that (40) is a formula for general solution to equation (39) under the posed conditions.

If operation $\odot$ has precedence over $\oplus$, that is, if

$$x \odot y \oplus z = (x \odot y) \oplus z,$$

for every $x, y, z \in S$, then equation (40) can be written in the following form:

$$\begin{aligned} x_n &= a_{n-1} \odot \bigl(a_{n-2} \odot \cdots \odot (a_0 \odot x_0) \cdots \bigr) \\ &\quad \oplus a_{n-1} \odot \bigl(a_{n-2} \odot \cdots \odot (a_1 \odot b_0) \cdots \bigr) \oplus \cdots \oplus a_{n-1} \odot b_{n-2} \oplus b_{n-1}, \quad n \in \mathbb{N}. \end{aligned}$$

If further $(S,\odot)$ is a semigroup; a groupoid where operation $\odot$ is associative, that is,

$$(x \odot y) \odot z = x \odot (y \odot z),$$

for every $x, y, z \in S$, then from the well known fact that in this case for every set $d_1, d_2, \ldots, d_k$ of elements from S the product

$$\bigodot_{j=1}^{k} d_j := d_1 \odot d_2 \odot \cdots \odot d_k \tag{41}$$

is unambiguous, that is, the same element is obtained regardless of how parentheses are inserted in product (41), equation (40) can be written in the following form:

$$\begin{aligned} x_n &= a_{n-1} \odot a_{n-2} \odot \cdots \odot a_0 \odot x_0 \\ &\quad \oplus a_{n-1} \odot a_{n-2} \odot \cdots \odot a_1 \odot b_0 \oplus \cdots \oplus a_{n-1} \odot b_{n-2} \oplus b_{n-1} \\ &= \left(\bigodot_{j=0}^{n-1} a_{n-1-j} \right) \odot x_0 \oplus \bigoplus_{j=0}^{n-1} \left(\bigodot_{i=0}^{n-2-j} a_{n-1-i} \right) \odot b_j, \quad n \in \mathbb{N}, \end{aligned}$$

where if $d_1, d_2, \ldots, d_k \in S$, then

$$\bigoplus_{j=1}^{k} d_j := d_1 \oplus d_2 \oplus \cdots \oplus d_k. \tag{42}$$

Note that in (41) and (42) the order of elements is important, since we do not assume commutativity of these operations on S.

If further the operation $\odot$ is commutative, that is,

$$x \odot y = y \odot x,$$

for every $x, y \in S$, then equation (40) can be written in the following form:

$$x_n = \left(\bigodot_{j=0}^{n-1} a_j \right) \odot x_0 \oplus \bigoplus_{j=0}^{n-1} \left(\bigodot_{i=j+1}^{n-1} a_i \right) \odot b_j, \quad n \in \mathbb{N},$$

from which if $S = \mathbb{R}$ or $S = \mathbb{C}$ and $\odot$ and $\oplus$ are multiplication and addition in $\mathbb{R}$ or $\mathbb{C}$, respectively, is obtained equation (30).

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author has contributed solely to the writing of this paper. He read and approved the manuscript.

Publisher's Note

Received: 13 March 2017 Accepted: 1 June 2017 Published online: 15 June 2017

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