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Existence of solutions for a mixed fractional boundary value problem

A Guezane Lakoud[1], R Khaldi[1] and Adem Kılıçman[2*]

*Correspondence:
akilic@upm.edu.my
[2]Department of Mathematics, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
Full list of author information is available at the end of the article

Abstract
In this paper, we prove the existence of solutions for a boundary value problem involving both left Riemann-Liouville and right Caputo-type fractional derivatives. For this, we convert the posed problem to a sum of two integral operators, then we apply Krasnoselskii's fixed point theorem to conclude the existence of nontrivial solutions.

Keywords: boundary value problem; fractional derivative; fixed point theorem; existence of solution; integral equation

1 Introduction

The study of differential equations with forward and backward fractional derivatives is interesting since they can model some physical phenomena such as the fractional oscillator equations and the fractional Euler-Lagrange equations. Recently, a linear boundary value problem involving both the right Caputo and the left Riemann-Liouville fractional derivatives have been studied by many authors [1–9]. Blaszczyk and Ciesielski [3–6] solved numerically a class of fractional Euler-Lagrange equations by transforming them into integral equations. Guezane-Lakoud, Khaldi and Torres [7] used an upper and lower solutions method combined with the monotonicity of the right Caputo derivative to prove the existence of solutions for a nonlinear fractional oscillator equation and the unbounded solutions were studied in [10] by Guezane-Lakoud and Kılıçman.

The aim of this work is to study of existence of solutions for a nonlinear boundary value problem involving both the right Caputo and the left Riemann-Liouville fractional derivatives:

$$-{}^{C}D_{1^-}^{\alpha}D_{0^+}^{\beta}u(t)+f\bigl(t,u(t)\bigr)=0,\quad 0<t<1, \tag{1}$$

$$u(0)=u'(0)=u(1)=0, \tag{2}$$

where $0<\alpha\leq 1$, $1<\beta\leq 2$, ${}^{C}D_{1^-}^{\alpha}$ denotes the right Caputo derivative, $D_{0^+}^{\beta}$ denotes the left Riemann-Liouville, u is the unknown function and $f\colon [0,1]\times R\to R$ satisfies some conditions that will be specified later. For this end, we transform problem (1)-(2) to an integral equation that we write as a sum of a contraction and a completely continuous operator; then we use Krasnoselskii's fixed point theorem to prove the existence of nontrivial solutions.

Note that, according to boundary conditions (2), the Caputo derivatives ${}^{C}D_{0^+}^{\beta}$ and ${}^{C}D_{1^-}^{\alpha}$ coincide respectively with the Riemann-Liouville derivatives $D_{0^+}^{\beta}$ and $D_{1^-}^{\alpha}$. So, equation (1) is reduced to the one containing only Caputo derivatives or only Riemann-Liouville derivatives, i.e.,

$$-{}^{C}D_{1^-}^{\alpha}{}^{C}D_{0^+}^{\beta}u(t)+f\bigl(t,u(t)\bigr)=0,\quad 0<t<1,$$

or

$$-D_{1^-}^{\alpha}D_{0^+}^{\beta}u(t)+f\bigl(t,u(t)\bigr)=0,\quad 0<t<1.$$

Note that this study can be extended to a similar boundary value problem as follows:

$$\begin{aligned}&{}^{C}D_{0^+}^{\alpha}{}^{C}D_{1^-}^{\beta}u(t)+f\bigl(t,u(t)\bigr)=0,\\&u(1)=u'(1)=u(0)=0,\end{aligned}$$

see Remark 2.7.

Let us mention some interesting papers where fractional boundary value problems have been studied by different methods such as the upper and lower solutions method, fixed point theorems, successive approximations method, Mawhin coincidence degree theory and many more; see, for example, [8, 11–17].

We recall some essential definitions on fractional calculus, we refer the reader to [15, 18, 19] for more details.

Let $p>0$, then the left and right Riemann-Liouville fractional integral of a function g are defined, respectively, by

$$\begin{aligned}I_{0^+}^{p}g(t)&=\frac{1}{\Gamma(p)}\int_0^t\frac{g(s)}{(t-s)^{1-p}}\,ds,\\I_{1^-}^{p}g(t)&=\frac{1}{\Gamma(p)}\int_t^1\frac{g(s)}{(s-t)^{1-p}}\,ds.\end{aligned}$$

The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order $p>0$, of a function g are, respectively,

$$\begin{aligned}D_{0^+}^{p}g(t)&=\frac{d^n}{dt^n}\bigl(I_{0^+}^{n-p}g\bigr)(t),\\{}^{C}D_{1^-}^{p}g(t)&=(-1)^nI_{1^-}^{n-p}g^{(n)}(t),\end{aligned}$$

where $n-1<p<n$. For the properties of Riemann-Liouville and Caputo fractional derivatives, we mention the following.

Let $n-1<p<n$ and $f\in L_1[0,1]$. Then

(1) $I_{0^+}^{p}{}^{C}D_{0^+}^{p}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k$;

(2) $I_{1^-}^{p}{}^{C}D_{1^-}^{p}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{(-1)^kf^{(k)}(1)}{k!}(1-t)^k$.

Next, we state Krasnoselskii's fixed point theorem that can be found with its proof in [20].

Theorem 1.1 (Krasnoselskii) *Let M be a closed bounded convex nonempty subset of a Banach space E. Suppose that A and B map M into E such that*

(i) *A is completely continuous,*
(ii) *B is a contraction mapping,*
(iii) *$x, y \in M$ implies $Ax + By \in M$.*

Then there exists $z \in M$ with $z = Az + Bz$.

2 Existence of nontrivial solutions

We begin by solving the following linear problem:

$$-{}^{C}D_{1^-}^{\alpha} D_{0^+}^{\beta} u(t) + y(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(0) = u(1) = 0. \tag{4}$$

Lemma 2.1 *Assume that $y \in L_1[0,1]$, then u is a solution to the linear boundary value problem* (3)-(4) *if and only if u satisfies the integral equation*

$$u(t) = \int_0^1 G(t,r)y(r)\,dr - t^{\beta} \int_0^1 g(r)y(r)\,dr, \tag{5}$$

where

$$G(t,r) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \begin{cases} \int_0^r (t-s)^{\beta-1}(r-s)^{\alpha-1}\,ds, & 0 \le r \le t \le 1, \\ \int_0^t (t-s)^{\beta-1}(r-s)^{\alpha-1}\,ds, & 0 \le t \le r \le 1. \end{cases} \tag{6}$$

$$g(r) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^r (1-s)^{\beta-1}(r-s)^{\alpha-1}\,ds.$$

Proof Firstly, we apply the right-hand side fractional integral $I_{1^-}^{\alpha}$ to equation (3), then the fractional integral $I_{0^+}^{\beta}$ to the resultant equation and take into account that $D_{0^+}^{\beta} u(t) = {}^{C}D_{0^+}^{\beta} u(t)$ and the properties of Caputo fractional derivatives. We get

$$u(t) = I_{0^+}^{\beta} I_{1^-}^{\alpha} y(t) + \frac{c_0 t^{\beta}}{\Gamma(\beta+1)} + u(0) + u'(0)t. \tag{7}$$

Using the boundary conditions $u(0) = u'(0) = 0$, then $u(1) = 0$, we get

$$c_0 = -\Gamma(\beta+1) I_{0^+}^{\beta} I_{1^-}^{\alpha} y(t)|_{t=1}.$$

Substituting c_0 in (7) yields

$$\begin{aligned} u(t) &= I_{0^+}^{\beta} I_{1^-}^{\alpha} y(t) - t^{\beta} \left(I_{0^+}^{\beta} I_{1^-}^{\alpha} y(t)|_{t=1} \right) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\int_s^1 (r-s)^{\alpha-1} y(r)\,dr \right) ds \\ &\quad - \frac{t^{\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \left(\int_s^1 (r-s)^{\alpha-1} y(r)\,dr \right) ds. \end{aligned}$$

Finally, by using the Fubini theorem, we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left(\int_0^r (t-s)^{\beta-1}(r-s)^{\alpha-1}\,ds \right) y(r)\,dr \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^1 \left(\int_0^t (t-s)^{\beta-1}(r-s)^{\alpha-1}\,ds \right) y(r)\,dr \\ &\quad - \frac{t^\beta}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \left(\int_0^r (1-s)^{\beta-1}(r-s)^{\alpha-1}\,ds \right) y(r)\,dr, \end{aligned}$$

this achieves the proof. □

In the next lemma, we give the properties of the functions G and g.

Lemma 2.2 *The functions G and g satisfy the following properties*:

(1) *The functions $G(t,r)$ and $g(r)$ are nonnegative.*

(2) $G(t,r) \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)}$ *and* $g(r) \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)}$ *for all* $r,t \in [0,1]$.

Proof It is obvious that $G(t,r) \geq 0$. Set

$$g_1(t,r) = \int_0^r (t-s)^{\beta-1}(r-s)^{\alpha-1}\,ds, \quad 0 \leq r \leq t \leq 1,$$

and

$$g_2(t,r) = \int_0^t (t-s)^{\beta-1}(r-s)^{\alpha-1}\,ds, \quad 0 \leq t \leq r \leq 1.$$

Then

$$\begin{aligned} g_1(t,r) &\leq \int_0^r (r-s)^{\alpha-1}\,ds = \frac{r^\alpha}{\alpha} \leq \frac{1}{\alpha}, \quad 0 \leq r \leq t \leq 1. \\ g_2(t,r) &\leq \int_0^t (t-s)^{\beta+\alpha-2}\,ds = \frac{t^{\beta+\alpha-1}}{\beta+\alpha-1} \leq \frac{1}{\alpha}, \quad 0 \leq t \leq r \leq 1. \end{aligned}$$

Consequently, $G(t,r) \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)}$ for all $r,t \in [0,1]$. Similarly, we prove that $g(r) \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)}$ for all $r \in [0,1]$. The proof is complete.

Define the Banach space $E = C([0,1],\mathbb{R})$ and the operators A and B on E by

$$\begin{aligned} Au(t) &= \int_0^1 G(t,r) f\bigl(r,u(r)\bigr)\,dr, \\ Bu(t) &= -t^\beta \int_0^1 g(r) f\bigl(r,u(r)\bigr)\,dr. \end{aligned}$$

Obviously, problem (1)-(2) has a solution if and only if $A + B$ has a fixed point, i.e.,

$$Au(t) + Bu(t) = u(t), \quad t \in [0,1].$$

To prove this end, we make the following hypothesis:

(H) The function $f(\cdot,0)$ is continuous and not identically null on $[0,1]$, and there exists a nonnegative function $k \in L_1([0,1],\mathbb{R}_+)$ such that

$$\begin{aligned}&\bigl|f(t,x)-f(t,y)\bigr| \leq k(t)|x-y|, \quad 0 \leq t \leq 1, x,y \in R,\\&\|k\|_{L_1} < \frac{\Gamma(\alpha+1)\Gamma(\beta)}{2}.\end{aligned}$$

Let

$$M = \bigl\{u \in E, \|u\| \leq R\bigr\},$$

where R is chosen such that

$$R \geq \frac{2L}{\Gamma(\alpha+1)\Gamma(\beta) - 2\|k\|_{L_1}}, \tag{8}$$

with $L = \max\{|f(t,0)|, 0 \leq t \leq 1\}$. Clearly, M is a nonempty, bounded and convex subset of the Banach space E. □

Theorem 2.3 *Under the hypothesis* (H), *problem* (1)-(2) *has nontrivial solutions in* M.

To prove Theorem 4, we have to prove that all the assumptions of Krasnoselskii's fixed point theorem are satisfied, for this we need the following lemmas.

Lemma 2.4 *Under the hypothesis* (H), *the mapping* A *is completely continuous on* M.

Proof The proof will be done in three steps.

Step 1. The mapping A is continuous on M. Consider the sequence $(u_n)_n \in M$ such that $u_n \to u$ in M, then from Lemma 3 and the hypothesis (H) we get

$$\begin{aligned}\bigl|Au_n(t) - Au(t)\bigr| &\leq \int_0^1 G(t,r)\bigl|f\bigl(r,u_n(r)\bigr) - f\bigl(r,u(r)\bigr)\bigr|\,dr\\&\leq \frac{\|k\|_{L_1}\|u_n-u\|}{\Gamma(\alpha+1)\Gamma(\beta)} \leq \frac{\|u_n-u\|}{2}.\end{aligned}$$

Consequently, $\|Au_n - Au\| \to 0$, when n tends to ∞.

Step 2. (Au) is uniformly bounded on M. Let $u \in M$, then by condition (H) it yields

$$\begin{aligned}\bigl|Au(t)\bigr| &\leq \int_0^1 G(t,r)\bigl|f\bigl(r,u(r)\bigr)\bigr|\,dr\\&\leq \int_0^1 G(t,r)\bigl[\bigl|f\bigl(r,u(r)\bigr) - f(r,0)\bigr| + \bigl|f(r,0)\bigr|\bigr]\,dr\\&\leq \frac{\|k\|_{L_1}\|u\|}{\Gamma(\alpha+1)\Gamma(\beta)} + \frac{L}{\Gamma(\alpha+1)\Gamma(\beta)} \leq \frac{R\|k\|_{L_1}+L}{\Gamma(\alpha+1)\Gamma(\beta)}.\end{aligned} \tag{9}$$

Step 3. (Au) is equicontinuous on M. We have, for $u \in M$, $0 \leq t_1 < t_2 \leq 1$,

$$\begin{aligned}\bigl|Au(t_1) - Au(t_2)\bigr| \leq& \int_0^{t_1} \bigl|G(t_1,r) - G(t_2,r)\bigr|\bigl|f\bigl(r,u(r)\bigr)\bigr|\,dr\\&+ \int_{t_1}^{t_2} \bigl|G(t_1,r) - G(t_2,r)\bigr|\bigl|f\bigl(r,u(r)\bigr)\bigr|\,dr\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_2}^{1}\left|G(t_1,r)-G(t_2,r)\right|\left|f\left(r,u(r)\right)\right|dr\\
&\le \frac{(R\|k\|_{L_1}+L)}{\Gamma(\alpha)\Gamma(\beta)}\left(\int_0^{t_1}\left(\int_0^{r}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)(r-s)^{\alpha-1}\,ds\right)dr\right.\\
&+\left(\int_{t_1}^{1}\int_0^{t_1}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)(r-s)^{\alpha-1}\,ds\right)dr\\
&+\int_{t_1}^{t_2}\left(\int_{t_1}^{r}(t_2-s)^{\beta-1}(r-s)^{\alpha-1}\,ds\right)dr\\
&\left.+\int_{t_2}^{1}\left(\int_{t_1}^{t_2}(t_2-s)^{\beta-1}(r-s)^{\alpha-1}\,ds\right)dr\right)\\
&\le \frac{(R\|k\|_{L_1}+L)}{\Gamma(\alpha)\Gamma(\beta)}\left[(\beta-1)(t_2-t_1)\left(\frac{1-(1-t_1)^{\alpha+1}}{\alpha(\alpha+1)}\right)\right.\\
&\left.+\left(\frac{(1-t_1)^{\alpha+1}-(1-t_2)^{\alpha+1}}{\alpha(\alpha+1)}\right)\right].
\end{aligned}
$$

So, $|Au(t_1)-Au(t_2)|$ tends to zero when $t_1 \to t_2$, thus (Au) is equicontinuous. Finally, by Arzela-Ascoli's theorem, it follows that A is a completely continuous mapping on M. The proof is complete. □

Lemma 2.5 *Under the hypothesis* (H), *the mapping B is a contraction on M.*

Proof Let $u, v \in M$, then

$$
\begin{aligned}
\left|Bu(t)-Bv(t)\right| &\le \int_0^1 g(r)\left|f\left(r,u(r)\right)-f\left(r,v(r)\right)\right|dr\\
&\le \frac{\|k\|_{L_1}\|u-v\|}{\Gamma(\alpha+1)\Gamma(\beta)} \le \frac{\|u-v\|}{2},
\end{aligned}
$$

thus B is a contraction on M. The proof is complete. □

Lemma 2.6 *Under the hypothesis* (H), *$Au + Bv \in M$ for all $u, v \in M$.*

Proof From (8) and (9), we get

$$
\left|Au(t)\right| \le \frac{(R\|k\|_{L_1}+L)}{\Gamma(\alpha+1)\Gamma(\beta)} \le \frac{R}{2}, \quad u \in M.
$$

Proceeding as in Step 2 of the proof of Lemma 2.5, we get

$$
\left|Bv(t)\right| \le \frac{(R\|k\|_{L_1}+L)}{\Gamma(\alpha+1)\Gamma(\beta)} \le \frac{R}{2}, \quad v \in M,
$$

thus

$$
\|Au+Bv\| \le \|Au\| + \|Bv\| \le R,
$$

so, $Au + Bv \in M$. The proof is complete. □

Proof of Theorem 4 Since the mapping A is completely continuous by Lemma 2.5, the mapping B is a contraction by Lemma 2.6 and $Au + Bv \in M$ for all $u, v \in M$ by Lemma 2.6. Then all the hypotheses of Theorem 1 are satisfied. Thus there exists a nontrivial solution $u^* \in M$ for problem (1)-(2) such that $u^* = Au^* + Bu^*$. The proof is complete. □

Remark 2.7 The present study can be extended to similar problems. For example, we can prove the existence of solutions for the following boundary value problem:

$$^{C}D_{0^+}^{\alpha}{}^{C}D_{1^-}^{\beta}u(t) + f\bigl(t, u(t)\bigr) = 0, \tag{10}$$

$$u(1) = u'(1) = u(0) = 0. \tag{11}$$

In fact, let Q be the reflection operator $(Qf)(t) = f(1-t)$. Since $(QQu)(t) = u(t)$, $^{C}D_{1^-}^{\beta}Q = Q^{C}D_{0^+}^{\beta}$ and $^{C}D_{0^+}^{\alpha}Q = Q^{C}D_{1^-}^{\alpha}$ (see [19]), then the boundary value problem (10)-(11) is equivalent to the following one:

$$Q^{C}D_{1^-}^{\alpha}{}^{C}D_{0^+}^{\beta}Qu(t) + f\bigl(t, QQu(t)\bigr) = 0,$$
$$u(1) = u'(1) = u(0) = 0.$$

Set $v(t) = Qu(t)$, then applying the operator Q, the boundary value problem (10)-(11) becomes

$$-{}^{C}D_{1^-}^{\alpha}{}^{C}D_{0^+}^{\beta}v(t) + F\bigl(t, v(t)\bigr) = 0,$$
$$v(0) = v'(0) = v(1) = 0,$$

where $F(t,x) = -f((1-t),x)$. Thus both f and F satisfy condition (H). Thanks to Theorem 4, we conclude that problem (10)-(11) has a nontrivial solution.

Example 2.8 If we consider problem (1)-(2) with $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$ and

$$f(t,x) = \frac{\sin t^2}{3}\left(x - \frac{t}{2(1+x^2)}\right),$$

then it has a solution u^* such that $\|u^*\| \leq 1$. Indeed, the hypothesis (H) holds:

$$\begin{aligned}\bigl|f(t,x) - f(t,y)\bigr| &= \frac{\sin t^2}{3}\left|x - y - \frac{t}{2}\left(\frac{1}{1+x^2} - \frac{1}{1+y^2}\right)\right| \\ &= \frac{\sin t^2}{3}|x-y|\left|1 + \frac{t}{2}\left(\frac{x+y}{(1+x^2)(1+y^2)}\right)\right| \\ &\leq \frac{1}{2}\sin t^2|x-y| = k(t)|x-y|, \quad 0 \leq t \leq 1, x, y \in R.\end{aligned}$$

Moreover, we have

$$\|k\|_{L_1} = \int_0^1 \frac{1}{2}\sin t^2\,dt = 0.15513 < 0.39270 = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{2}.$$

Since

$$L = \sup\left\{ \left| f(t,0) \right|, 0 \leq t \leq 1 \right\} = \frac{\sin 1}{6} = 0.14025$$

and

$$\frac{2L}{\Gamma(\alpha+1)\Gamma(\beta) - 2\|k\|_{L_1}} = 0.75464,$$

then R can be chosen as $R = 1 \geq 0.75464$. Thus, by Theorem 4, this problem has a nontrivial solution

$$u^* \in M = \left\{ u \in E, \|u\| \leq 1 \right\}.$$

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally, further read and approved the final manuscript.

Author details
[1]Laboratory of Advanced Materials, Department of Mathematics, Badji Mokhtar-Annaba University, P.O. Box 12, Annaba, 23000, Algeria. [2]Department of Mathematics, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.

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