Liu et al. *Advances in Difference Equations* (2017) 2017:168
DOI 10.1186/s13662-017-1221-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Multiple positive solutions for a coupled system of fractional multi-point BVP with p-Laplacian operator

Yang Liu[1], Dapeng Xie[1*], Chuanzhi Bai[2] and Dandan Yang[2]

*Correspondence: dpxhfnu@163.com
[1]School of Mathematics and Statistics, Hefei Normal University, Hefei, Anhui 230061, P.R. China
Full list of author information is available at the end of the article

Abstract

We investigate the existence of positive solutions for a system of fractional multi-point BVP with p-Laplacian operator. Our main tool is the fixed point theorem due to Leggett-Williams. The result obtained in this paper corrects some mistakes in (Al-Hossain in Differ. Equ. Dyn. Syst., 2016, doi:10.1007/s11590-013-0708-4) and essentially improves and extends some well-known results.

Keywords: fractional differential equation; positive solutions; p-Laplacian operator; Green's function; cone

1 Introduction

In this paper, we are concerned with the existence of multiple positive solutions for the following p-Laplacian fractional operator multi-point BVP:

$$D_{a^+}^{\beta_1}\big(\varphi_p\big(D_{a^+}^{\alpha_1}u(t)\big)\big) = f_1\big(t, u(t), v(t)\big), \quad t \in (a,b), \tag{1.1}$$

$$D_{a^+}^{\beta_2}\big(\varphi_p\big(D_{a^+}^{\alpha_2}v(t)\big)\big) = f_2\big(t, u(t), v(t)\big), \quad t \in (a,b), \tag{1.2}$$

with the boundary conditions

$$\begin{cases} u^{(j)}(a) = 0, \quad j = 0,1,2,\ldots,n-2, \qquad u^{(\mu_1)}(b) = \sum_{i=1}^{\infty}\gamma_i u^{(\mu_1)}(\xi_i), \\ \varphi_p(D_{a^+}^{\alpha_1}u(a)) = 0, \qquad D_{a^+}^{\mu_2}(\varphi_p(D_{a^+}^{\alpha_1}u(b))) = \sum_{i=1}^{\infty}\delta_i(D_{a^+}^{\mu_2}(\varphi_p(D_{a^+}^{\alpha_1}u(\eta_i)))), \end{cases} \tag{1.3}$$

$$\begin{cases} v^{(k)}(a) = 0, \quad k = 0,1,2,\ldots,m-2, \qquad v^{(\mu_3)}(b) = \sum_{l=1}^{\infty}\gamma_l v^{(\mu_3)}(\xi_l), \\ \varphi_p(D_{a^+}^{\alpha_2}v(a)) = 0, \qquad D_{a^+}^{\mu_4}(\varphi_p(D_{a^+}^{\alpha_2}v(b))) = \sum_{l=1}^{\infty}\delta_l(D_{a^+}^{\mu_4}(\varphi_p(D_{a^+}^{\alpha_2}v(\eta_l)))), \end{cases} \tag{1.4}$$

where $a,b \in R$ with $a < b$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$ and $D_{a^+}^{\alpha_1}$, $D_{a^+}^{\alpha_2}$, $D_{a^+}^{\beta_1}$, $D_{a^+}^{\beta_2}$, $D_{a^+}^{\mu_2}$, $D_{a^+}^{\mu_4}$ are the standard Riemann-Liouville fractional derivatives. We make the following assumptions:

(H_0) $n-1 < \alpha_1 \le n$, $m-1 < \alpha_2 \le m$, for $n,m \ge 3$, $\beta_1,\beta_2 \in (1,2]$, $\mu_1 \in [1,\alpha_1-1)$, $\mu_3 \in [1,\alpha_2-1)$, $\mu_2,\mu_4 \in (0,1]$ and $\mu_2 \le \beta_1 - 1$, $\mu_4 \le \beta_2 - 1$, $f_i(t,u,v) \in C([a,b] \times R^2, R^+)$, for $i = 1,2$,

(H_1) $\gamma_i \geq 0$ $(i = 1, 2, \ldots)$ and $a < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi_i < \cdots < b$ satisfy that

$$\Phi_1 = (b-a)^{\alpha_1-\mu_1-1} - \sum_{i=1}^{\infty} \gamma_i (\xi_i - a)^{\alpha_1-\mu_1-1} > 0,$$

$$\Phi_2 = (b-a)^{\alpha_2-\mu_3-1} - \sum_{l=1}^{\infty} \gamma_l (\xi_l - a)^{\alpha_2-\mu_3-1} > 0,$$

(H_2) $\delta_i \geq 0$ $(i = 1, 2, \ldots)$ and $a < \eta_1 < \eta_2 < \cdots < \eta_{i-1} < \eta_i < \cdots < b$ satisfy that

$$\Delta_1 = (b-a)^{\beta_1-\mu_2-1} - \sum_{i=1}^{\infty} \delta_i (\eta_i - a)^{\beta_1-\mu_2-1} > 0,$$

$$\Delta_2 = (b-a)^{\beta_2-\mu_4-1} - \sum_{l=1}^{\infty} \delta_l (\eta_l - a)^{\beta_2-\mu_4-1} > 0.$$

Recently, the existence of solutions or positive solutions of p-Laplacian fractional differential equations at nonresonance or resonance have been of great interest, readers can see [1–14] and the references cited therein. In particular, if $a = 0$, $b = 1$, $n = m = 3$, $\delta_i = \gamma_i = 0$ $(i = 1, 2, \ldots)$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, $f_1(t, u) = f_2(t, u) = \lambda f(u(t))$ with $u(t) = v(t)$, the problem becomes the problem studied in [14]. Han, Lu and Zhang investigated the existence of positive solutions for the eigenvalue problems of the fractional differential equation with generalized p-Laplacian

$$\begin{cases} D_{0^+}^{\beta}(\varphi_p(D_{0^+}^{\alpha} u(t))) = \lambda f(u(t)), & t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0, & \varphi_p(D_{0^+}^{\alpha} u(0)) = (\varphi_p(D_{0^+}^{\alpha} u(1)))' = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $\lambda > 0$, $\varphi_p(s) = |s|^{p-2}s$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $f : (0, +\infty) \to (0, +\infty)$ is continuous. By using the properties of Green's function and Guo-Krasnosel'skii's fixed point theorem, some existence results of at least one or two positive solutions in terms of a different eigenvalue interval are obtained.

If $\gamma_1 = \xi$, $\xi_1 = \eta$, $\gamma_i = 0$ $(i = 2, 3, \ldots)$, $\delta_i = 0$ $(i = 1, 2, 3, \ldots)$, the problem becomes the problem considered in [1]. Al-Hossain investigated the existence of at least one positive solution for the following system of fractional order differential equations with p-Laplacian operators:

$$D_{a^+}^{\beta_1}\big(\varphi_p\big(D_{a^+}^{\alpha_1} u(t)\big)\big) + f_1\big(t, u(t), v(t)\big) = 0, \quad t \in (a,b), \tag{1.5}$$

$$D_{a^+}^{\beta_2}\big(\varphi_p\big(D_{a^+}^{\alpha_2} v(t)\big)\big) + f_2\big(t, u(t), v(t)\big) = 0, \quad t \in (a,b), \tag{1.6}$$

with the boundary conditions

$$\begin{cases} u^{(j)}(a) = 0, \quad j = 0, 1, 2, \ldots, n-2, & u^{(\mu_1)}(b) = \xi u^{(\mu_1)}(\eta), \\ \varphi_p(D_{a^+}^{\alpha_1} u(a)) = 0, & D_{a^+}^{\mu_2}(\varphi_p(D_{a^+}^{\alpha_2} u(b))) = 0, \end{cases} \tag{1.7}$$

$$\begin{cases} v^{(k)}(a) = 0, \quad k = 0, 1, 2, \ldots, m-2, & v^{(\mu_3)}(b) = \xi v^{(\mu_3)}(\eta), \\ \varphi_p(D_{a^+}^{\alpha_2} v(a)) = 0, & D_{a^+}^{\mu_4}(\varphi_p(D_{a^+}^{\alpha_2} v(b))) = 0, \end{cases} \tag{1.8}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta_1, \beta_2 \in (1,2]$, $n-1 < \alpha_1 \leq n$, $m-1 < \alpha_2 \leq m$ for $n, m \geq 3$, $\mu_2, \mu_4 \in (0,1]$, $\mu_1 \in [1, \alpha_1 - 2]$, $\mu_3 \in [1, \alpha_2 - 2]$ are fixed integers, $\xi \in (0,\infty)$, $\eta \in (a,b)$ are constants with $0 < \xi(\eta-a)^{\alpha_1-\mu_1-1} < (b-a)^{\alpha_1-\mu_1-1}$, $f_i(t,u,v) \in C([a,b] \times R^2, R^+)$, for $i = 1,2$, and $D_{a^+}^{\alpha_1}, D_{a^+}^{\alpha_2}, D_{a^+}^{\beta_1}, D_{a^+}^{\beta_2}, D_{a^+}^{\mu_2}, D_{a^+}^{\mu_4}$ are the standard Riemann-Liouville fractional derivatives. By using a fixed point theorem of the cone expansion and compression of functional type due to Avery, Henderson and O'Regan, the author established at least one existence result for positive solutions to the problem.

Unfortunately, there are some mistakes in paper [1]. First and foremost, the given Green's function $H_1(t,s)$ is not to problem (1.5)-(1.8). In fact, the Green's function $H_1(t,s)$ in [1] is the Green's function of the following system of fractional order differential equations with p-Laplacian operators:

$$D_{a^+}^{\beta_1}\bigl(\varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr)\bigr) = f_1\bigl(t,u(t),v(t)\bigr), \quad t \in (a,b), \tag{1.9}$$

$$D_{a^+}^{\beta_2}\bigl(\varphi_p\bigl(D_{a^+}^{\alpha_2}v(t)\bigr)\bigr) = f_2\bigl(t,u(t),v(t)\bigr), \quad t \in (a,b), \tag{1.10}$$

with the boundary conditions (1.7), (1.8). In [1], even if the author studied the above problem, there are still the following two mistakes in [1], which are very essential to the proof of the main results.

First, Lemma 3.3 in [1], the author gave the following result: if $s \leq t$, then

$$\begin{aligned} G_{11}(t,s) &= \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (t-s)^{\alpha_1-1}\right] \\ &= \frac{(t-a)^{\alpha_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-1}}\left[\frac{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (b-s)^{\alpha_1-1}\right] \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (t-s)^{\alpha_1-1}\right]. \end{aligned}$$

If $t \leq s$, we have

$$\begin{aligned} G_{11}(t,s) &= \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}}\right] \\ &= \frac{(t-a)^{\alpha_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-1}}\left[\frac{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (b-s)^{\alpha_1-1}\right] \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}}\right]. \end{aligned}$$

The above result is one of the mistakes. In fact, by

$$\begin{aligned} &\frac{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (b-s)^{\alpha_1-1} \\ &\quad = \frac{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1} - (b-s)^{\alpha_1-1}(b-a)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} \\ &\quad = \frac{(b-a)^{\alpha_1-\mu_1-1}(b-s)^{\alpha_1-\mu_1-1}[(b-a)^{\mu_1} - (b-s)^{\mu_1}]}{(b-a)^{\alpha_1-\mu_1-1}} \geq 0, \end{aligned}$$

then, for $s \le t$, we have

$$\begin{aligned} G_{11}(t,s) &= \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (t-s)^{\alpha_1-1}\right] \\ &\neq \frac{(t-a)^{\alpha_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-1}}\left[\frac{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (b-s)^{\alpha_1-1}\right] \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (t-s)^{\alpha_1-1}\right], \end{aligned}$$

and for $t \le s$, we have

$$\begin{aligned} G_{11}(t,s) &= \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}}\right] \\ &\neq \frac{(t-a)^{\alpha_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-1}}\left[\frac{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}} - (b-s)^{\alpha_1-1}\right] \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\left[\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-\mu_1-1}}\right]. \end{aligned}$$

Thus, as the above wrong conclusion, the proof of the inequality of Lemma 3.3 is not correct. In this paper, we give a method to deal with this problem in the proof of Lemma 2.4.

Next, we want to point out, in [1], the following conclusion in Lemma 3.4(i) is also wrong: $H_1(t,s) \le H_1(b,s)$ for $(t,s) \in [a,b] \times [a,b]$. In fact, for $a \le s \le t \le b$,

Case 1: if $0 < \mu_2 \le \beta_1 - 1 \le 1$, we have

$$\begin{aligned} \frac{\partial}{\partial t}H_1(t,s) &= \frac{(\beta_1-1)[(t-a)^{\beta_1-2}(b-s)^{\beta_1-\mu_2-1} - (t-s)^{\beta_1-2}(b-a)^{\beta_1-\mu_2-1}]}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ &\le \frac{(\beta_1-1)(t-s)^{\beta_1-2}[(b-s)^{\beta_1-\mu_2-1} - (b-a)^{\beta_1-\mu_2-1}]}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \le 0. \end{aligned}$$

Case 2: if $0 < \beta_1 - 1 \le \mu_2 \le 1$, we have

$$\begin{aligned} \frac{\partial}{\partial t}H_1(t,s) &= \frac{(\beta_1-1)[(t-a)^{\beta_1-2}(b-s)^{\beta_1-\mu_2-1} - (t-s)^{\beta_1-2}(b-a)^{\beta_1-\mu_2-1}]}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ &= \frac{(\beta_1-1)[(t-a)^{\mu_2-1}((t-a)(b-s))^{\beta_1-\mu_2-1} - (t-s)^{\mu_2-1}((t-s)(b-a))^{\beta_1-\mu_2-1}]}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ &\le \frac{(\beta_1-1)[(t-s)(b-a)]^{\beta_1-\mu_2-1}[(t-a)^{\mu_2-1} - (t-s)^{\mu_2-1}]}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \le 0. \end{aligned}$$

Thus, $H_1(t,s)$ in decreasing in t, which implies $H_1(t,s) \ge H_1(b,s)$ for $a \le s \le t \le b$.

Because of the wrong conclusion of Lemma 3.4 in [1], Theorems 4.2 and 4.3 in [1] cannot be established. The aim of this paper is to correct and improve the wrong results in [1]. By using the fixed point theorem due to Leggett-Williams, we will study the existence of triple positive solutions for the system of p-Laplacian fractional operator multi-point BVP (1.1)-(1.4). The associated Green's function for the above problem is given at first, and some useful properties of the Green's function are also obtained. The result obtained in this paper essentially improves and extends some known results. As application, an example is presented to illustrate the main result.

2 Preliminaries and lemmas

Lemma 2.1 ([15]) *Assume that $D_{a^+}^{\alpha} \in L^1(a,b)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I_{a^+}^{\beta_1} D_{a^+}^{\beta_1} u(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n}$$

for some $c_i \in R$, $i = 1, 2, \dots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.2 *If $y_1 \in C[a,b]$ and (H_0), (H_1) hold, then the fractional differential equation BVPs*

$$\begin{cases} D_{a^+}^{\alpha_1} u(t) + y_1(t) = 0, \quad t \in (a,b) \\ u^{(j)}(a) = 0, \quad j = 0,1,2,\dots,n-2, \qquad u^{(\mu_1)}(b) = \sum_{i=1}^{\infty} \gamma_i u^{(\mu_1)}(\xi_i), \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_a^b G_1(t,s) y_1(s)\, ds,$$

where

$$G_1(t,s) = \begin{cases} \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1} - (t-s)^{\alpha_1-1}(b-a)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} \\ \quad + \frac{(t-a)^{\alpha_1-1}[\sum_{i=1}^{\infty}\gamma_i[(b-s)(\xi_i-a)]^{\alpha_1-\mu_1-1} - \sum_{i=1}^{\infty}\gamma_i[(\xi_i-s)(b-a)]^{\alpha_1-\mu_1-1}]}{\Phi_1\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, \\ \quad a \le s \le \min\{t, \xi_1\}, \\ \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1} - (t-s)^{\alpha_1-1}(b-a)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} \\ \quad + \frac{(t-a)^{\alpha_1-1}[\sum_{i=1}^{\infty}\gamma_i[(b-s)(\xi_i-a)]^{\alpha_1-\mu_1-1} - \sum_{i=j}^{\infty}\gamma_i[(\xi_i-s)(b-a)]^{\alpha_1-\mu_1-1}]}{\Phi_1\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, \\ \quad \xi_{j-1} < s \le \xi_j \le t, j = 2,3,\dots, \\ \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1} - (t-s)^{\alpha_1-1}(b-a)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} \\ \quad + \frac{(t-a)^{\alpha_1-1}\sum_{i=1}^{\infty}\gamma_i[(b-s)(\xi_i-a)]^{\alpha_1-\mu_1-1}}{\Phi_1\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, \\ \quad \lim_{i\to\infty}\xi_i \le s \le t, \\ \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} \\ \quad + \frac{(t-a)^{\alpha_1-1}[\sum_{i=1}^{\infty}\gamma_i[(b-s)(\xi_i-a)]^{\alpha_1-\mu_1-1} - \sum_{i=1}^{\infty}\gamma_i[(\xi_i-s)(b-a)]^{\alpha_1-\mu_1-1}]}{\Phi_1\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, \\ \quad t \le s \le \xi_1, \\ \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} \\ \quad + \frac{(t-a)^{\alpha_1-1}[\sum_{i=1}^{\infty}\gamma_i[(b-s)(\xi_i-a)]^{\alpha_1-\mu_1-1} - \sum_{i=j}^{\infty}\gamma_i[(\xi_i-s)(b-a)]^{\alpha_1-\mu_1-1}]}{\Phi_1\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, \\ \quad t \le \xi_{j-1} < s \le \xi_j, j = 2,3,\dots, \\ \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} + \frac{(t-a)^{\alpha_1-1}\sum_{i=1}^{\infty}\gamma_i[(b-s)(\xi_i-a)]^{\alpha_1-\mu_1-1}}{\Phi_1\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, \\ \quad \max\{t, \lim_{i\to\infty}\xi_i\} \le s \le b, \end{cases}$$

$$= G_{11}(t,s) + \frac{(t-a)^{\alpha_1-1}\sum_{i=1}^{\infty}\gamma_i G_{12}(\xi_i,s)}{\Phi_1},$$

$$G_{11}(t,s) = \begin{cases} \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1} - (t-s)^{\alpha_1-1}(b-a)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, & s \le t, \\ \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, & t \le s, \end{cases}$$

$$G_{12}(t,s)=\begin{cases}\frac{[(b-s)(t-a)]^{\alpha_1-\mu_1-1}-[(t-s)(b-a)]^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, & s\le t,\\ \frac{[(b-s)(t-a)]^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}, & t\le s.\end{cases}$$

Proof Assume that $u\in C^{[\alpha_1]+1}[a,b]$ is a solution of fractional order BVPs (2.1) and is uniquely expressed as

$$I_{a^+}^{\alpha_1}D_{a^+}^{\alpha_1}u(t)=-I_{a^+}^{\alpha_1}y_1(t)$$

such that

$$u(t)=-\int_a^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds+c_1(t-a)^{\alpha_1-1}+c_2(t-a)^{\alpha_1-2}+\cdots+c_n(t-a)^{\alpha_1-n}.$$

By $u^{(j)}(a)=0$, $j=0,1,2,\dots,n-2$, we get that $c_i=0$, for $i=2,3,\dots,n$, hence

$$u^{(\mu_1)}(t)=-\prod_{j=1}^{\mu_1}(\alpha_1-j)\int_a^t\frac{(t-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds+c_1\prod_{j=1}^{\mu_1}(\alpha_1-j)(t-a)^{\alpha_1-\mu_1-1}.$$

By $u^{(\mu_1)}(b)=\sum_{i=1}^{\infty}\gamma_i u^{(\mu_1)}(\xi_i)$, we have

$$\begin{aligned}&-\prod_{j=1}^{\mu_1}(\alpha_1-j)\int_a^b\frac{(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds+c_1\prod_{j=1}^{\mu_1}(\alpha_1-j)(b-a)^{\alpha_1-\mu_1-1}\\&\quad=\sum_{i=1}^{\infty}\gamma_i\left[-\prod_{j=1}^{\mu_1}(\alpha_1-j)\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds+c_1\prod_{j=1}^{\mu_1}(\alpha_1-j)(\xi_i-a)^{\alpha_1-\mu_1-1}\right].\end{aligned}$$

Then

$$\begin{aligned}c_1&=\frac{1}{\Phi_1}\left[\int_a^b\frac{(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds-\sum_{i=1}^{\infty}\gamma_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds\right]\\&=\left[\frac{1}{(b-a)^{\alpha_1-\mu_1-1}}+\frac{\sum_{i=1}^{\infty}\gamma_i(\xi_i-a)^{\alpha_1-\mu_1-1}}{\Phi_1(b-a)^{\alpha_1-\mu_1-1}}\right]\int_a^b\frac{(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds\\&\quad-\frac{1}{\Phi_1}\sum_{i=1}^{\infty}\gamma_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds\\&=\int_a^b\frac{(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}y_1(s)\,ds+\frac{\sum_{i=1}^{\infty}\gamma_i\int_a^b\frac{(b-s)^{\alpha_1-\mu_1-1}(\xi_i-a)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}y_1(s)\,ds}{\Phi_1}\\&\quad-\frac{\sum_{i=1}^{\infty}\gamma_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds}{\Phi_1}.\end{aligned}$$

Hence

$$\begin{aligned}u(t)&=-\int_a^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}y_1(s)\,ds+\int_a^b\frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}y_1(s)\,ds\\&\quad+\frac{\sum_{i=1}^{\infty}\gamma_i\int_a^b\frac{(b-s)^{\alpha_1-\mu_1-1}(\xi_i-a)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}y_1(s)\,ds}{\Phi_1}+\frac{\sum_{i=1}^{\infty}\gamma_i\int_a^b G_{12}(\xi_i,s)y_1(s)\,ds}{\Phi_1}\end{aligned}$$

$$= \int_a^b G_{11}(t,s)y_1(s)\,ds + \frac{(t-a)^{\alpha_1-1}\sum_{i=1}^{\infty}\gamma_i\int_a^b G_{12}(\xi_i,s)y_1(s)\,ds}{\Phi_1}$$
$$= \int_a^b G_1(t,s)y_1(s)\,ds. \qquad \square$$

Lemma 2.3 *Suppose that* (H_0)-(H_2) *are satisfied for* $y_2 \in C[a,b]$, *then the fractional order BVP*

$$D_{a^+}^{\beta_1}\bigl(\varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr)\bigr) = y_2(t), \quad t\in(a,b), \tag{2.2}$$

$$\begin{cases} u^{(j)}(a) = 0, \quad j = 0,1,2,\ldots,n-2, \qquad u^{(\mu_1)}(b) = \sum_{i=1}^{\infty}\gamma_i u^{(\mu_1)}(\xi_i), \\ \varphi_p(D_{a^+}^{\alpha_1}u(a)) = 0, \qquad D_{a^+}^{\mu_2}(\varphi_p(D_{a^+}^{\alpha_1}u(b))) = \sum_{i=1}^{\infty}\delta_i(D_{a^+}^{\mu_2}(\varphi_p(D_{a^+}^{\alpha_1}u(\eta_i)))), \end{cases} \tag{2.3}$$

has a unique solution

$$u(t) = \int_a^b G_1(t,s)\varphi_q\biggl(\int_a^b H_1(s,\tau)y_2(\tau)\,d\tau\biggr)ds,$$

where

$$H_1(t,s) = \begin{cases} \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}-(t-s)^{\beta_1-1}(b-a)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ \quad + \frac{(t-a)^{\beta_1-1}[\sum_{i=1}^{\infty}\delta_i[(b-s)(\eta_i-a)]^{\beta_1-\mu_2-1}-\sum_{i=1}^{\infty}\delta_i[(\eta_i-s)(b-a)]^{\beta_1-\mu_2-1}]}{\Delta_1\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, \\ \quad a\le s\le \min\{t,\eta_1\}, \\ \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}-(t-s)^{\beta_1-1}(b-a)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ \quad + \frac{(t-a)^{\beta_1-1}[\sum_{i=1}^{\infty}\delta_i[(b-s)(\eta_i-a)]^{\beta_1-\mu_2-1}-\sum_{i=k}^{\infty}\delta_i[(\eta_i-s)(b-a)]^{\beta_1-\mu_2-1}]}{\Delta_1\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, \\ \quad \eta_{k-1}< s\le \eta_k\le t, k=2,3,\ldots, \\ \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}-(t-s)^{\beta_1-1}(b-a)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ \quad + \frac{(t-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i[(b-s)(\eta_i-a)]^{\beta_1-\mu_2-1}}{\Delta_1\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, \\ \quad \lim_{i\to\infty}\eta_i\le s\le t, \\ \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ \quad + \frac{(t-a)^{\beta_1-1}[\sum_{i=1}^{\infty}\delta_i[(b-s)(\eta_i-a)]^{\beta_1-\mu_2-1}-\sum_{i=1}^{\infty}\delta_i[(\eta_i-s)(b-a)]^{\beta_1-\mu_2-1}]}{\Delta_1\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, \\ \quad t\le s\le \eta_1, \\ \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ \quad + \frac{(t-a)^{\beta_1-1}[\sum_{i=1}^{\infty}\delta_i[(b-s)(\eta_i-a)]^{\beta_1-\mu_2-1}-\sum_{i=k}^{\infty}\delta_i[(\eta_i-s)(b-a)]^{\beta_1-\mu_2-1}]}{\Delta_1\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, \\ \quad t\le \eta_{k-1}< s\le \eta_k, k=2,3,\ldots, \\ \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} + \frac{(t-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i[(b-s)(\eta_i-a)]^{\beta_1-\mu_2-1}}{\Delta_1\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, \\ \quad \max\{t,\lim_{i\to+\infty}\eta_i\}\le s\le b, \end{cases}$$

$$= H_{11}(t,s) + \frac{(t-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i H_{12}(\eta_i,s)}{\Delta_1},$$

$$H_{11}(t,s) = \begin{cases} \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}-(t-s)^{\beta_1-1}(b-a)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, & s\le t, \\ \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, & t\le s, \end{cases}$$

$$H_{12}(t,s) = \begin{cases} \frac{[(b-s)(t-a)]^{\beta_1-\mu_2-1}-[(t-s)(b-a)]^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, & s \le t, \\ \frac{[(b-s)(t-a)]^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, & t \le s. \end{cases}$$

Proof It follows from Lemma 2.1 and $1 < \beta_1 \le 2$ that

$$I_{a^+}^{\beta_1} D_{a^+}^{\beta_1}\bigl(\varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr)\bigr) = \varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr) + c_1(t-a)^{\beta_1-1} + c_2(t-a)^{\beta_1-2} \quad \text{for some } c_1, c_2 \in R.$$

Since (2.2), we get

$$I_{a^+}^{\beta_1} D_{a^+}^{\beta_1}\bigl(\varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr)\bigr) = I_{a^+}^{\beta_1} y_2(t).$$

Then

$$\varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr) = I_{a^+}^{\beta_1} y_2(t) + C_1(t-a)^{\beta_1-1} + C_2(t-a)^{\beta_1-2} \quad \text{for some } C_1, C_2 \in R.$$

Note that $\varphi_p(D_{a^+}^{\alpha_1}u(a)) = 0$, we have $C_2 = 0$, then

$$\begin{aligned} D_{a^+}^{\mu_2}\varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr) &= D_{a^+}^{\mu_2} I_{a^+}^{\beta_1} y_2(t) + C_1 D_{a^+}^{\mu_2}(t-a)^{\beta_1-1} \\ &= I_{a^+}^{\beta_1-\mu_2} y_2(t) + C_1\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\mu_2)}(t-a)^{\beta_1-\mu_2-1} \\ &= \int_a^t \frac{(t-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1-\mu_2)} y_2(\tau)\,d\tau + C_1\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\mu_2)}(t-a)^{\beta_1-\mu_2-1}. \end{aligned}$$

By $D_{a^+}^{\mu_2}(\varphi_p(D_{a^+}^{\alpha_1}u(b))) = \sum_{i=1}^{\infty}\delta_i(D_{a^+}^{\mu_2}(\varphi_p(D_{a^+}^{\alpha_1}u(\eta_i))))$, we have

$$C_1 = -\frac{1}{\Delta_1}\Biggl[\int_a^b \frac{(b-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau - \sum_{i=1}^{\infty}\delta_i \int_a^{\eta_i}\frac{(\eta_i-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau\Biggr].$$

Thus, the solution $u(t)$ of the fractional order BVP (2.2)-(2.3) satisfies

$$\begin{aligned} \varphi_p\bigl(D_{a^+}^{\alpha_1}u(t)\bigr) &= \int_a^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau \\ &\quad - \frac{(t-a)^{\beta_1-1}}{\Delta_1} \\ &\quad \times \Biggl[\int_a^b \frac{(b-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau - \sum_{i=1}^{\infty}\delta_i \int_a^{\eta_i}\frac{(\eta_i-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau\Biggr] \\ &= \int_a^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau \\ &\quad - \biggl(\frac{1}{(b-a)^{\beta_1-\mu_2-1}} + \frac{\sum_{i=1}^{\infty}\delta_i(\eta_i-a)^{\beta_1-\mu_2-1}}{(b-a)^{\beta_1-\mu_2-1}\Delta_1}\biggr) \\ &\quad \times (t-a)^{\beta_1-1}\int_a^b \frac{(b-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau \\ &\quad + \frac{(t-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i\int_a^{\eta_i}\frac{(\eta_i-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau}{\Delta_1} \end{aligned}$$

$$= -\left(\int_a^b \frac{(b-\tau)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} y_2(\tau)\,d\tau - \int_a^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau\right)$$
$$- \frac{(t-a)^{\beta_1-1}}{\Delta_1}\sum_{i=1}^{\infty}\delta_i\left(\int_a^b \frac{(b-\tau)^{\beta_1-\mu_2-1}(\eta_i-a)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} y_2(\tau)\,d\tau\right.$$
$$\left.- \int_a^{\eta_i} \frac{(\eta_i-\tau)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)} y_2(\tau)\,d\tau\right)$$
$$= -\left(\int_a^b H_{11}(t,\tau)y_2(\tau)\,d\tau + \frac{(t-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i\int_a^b H_{12}(\eta_i,\tau)y_2(\tau)\,d\tau}{\Delta_1}\right)$$
$$= -\int_a^b H_1(t,\tau)y_2(\tau)\,d\tau.$$

Then the fractional order BVP (2.2)-(2.3) is equivalent to the following problem:

$$D_{a^+}^{\alpha_1}u(t) + \varphi_q\left(\int_a^b H_1(t,\tau)y_2(\tau)\,d\tau\right) = 0 \quad \text{for } t \in (a,b),$$
$$u^{(j)}(a) = 0, \quad j = 0,1,2,\ldots,n-2, \qquad u^{(\mu_1)}(b) = \sum_{i=1}^{\infty}\gamma_i u^{(\mu_1)}(\xi_i).$$

In view of Lemma 2.2, we get

$$u(t) = \int_a^b G_1(t,s)\varphi_q\left(\int_a^b H_1(s,\tau)y_2(\tau)\,d\tau\right) ds.$$ □

Lemma 2.4 *Suppose* (H_0)-(H_2) *hold. Then* $G_1(t,s)$ *has the following properties*:

(i) $0 \le G_1(t,s) \le G_1(b,s)$ *for all* $(t,s) \in [a,b] \times [a,b]$,

(ii) $G_1(t,s) \ge (\frac{1}{4})^{\alpha_1-1}G_1(b,s)$ *for all* $(t,s) \in I \times (a,b)$, *where* $I = [\frac{3a+b}{4}, \frac{a+3b}{4}]$.

Proof (i) For $t \le s$, we have

$$\frac{\partial}{\partial t}G_{11}(t,s) = \frac{(\alpha_1-1)(t-a)^{\alpha_1-2}(b-s)^{\alpha_1-\mu_1-1}}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} \ge 0.$$

For $t \ge s$, we get

$$\frac{\partial}{\partial t}G_{11}(t,s) = \frac{(\alpha_1-1)[(t-a)^{\alpha_1-2}(b-s)^{\alpha_1-\mu_1-1} - (t-s)^{\alpha_1-2}(b-a)^{\alpha_1-\mu_1-1}]}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}$$
$$= \frac{(\alpha_1-1)[(t-a)^{\mu_1-1}[(t-a)(b-s)]^{\alpha_1-\mu_1-1} - (t-s)^{\mu_1-1}[(t-s)(b-a)]^{\alpha_1-\mu_1-1}]}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}}$$
$$\ge \frac{(\alpha_1-1)(t-s)^{\mu_1-1}[((t-a)(b-s))^{\alpha_1-\mu_1-1} - ((t-s)(b-a))^{\alpha_1-\mu_1-1}]}{\Gamma(\alpha_1)(b-a)^{\alpha_1-\mu_1-1}} \ge 0.$$

Thus, for all $(t,s) \in [a,b] \times [a,b]$, we get

$$\frac{\partial}{\partial t}G_1(t,s) = \frac{\partial}{\partial t}G_{11}(t,s) + \frac{(\alpha_1-1)(t-a)^{\alpha_1-2}\sum_{i=1}^{\infty}\gamma_i G_{12}(\xi_i,s)}{\Phi_1} \ge 0.$$

Then $0 = G_1(a,s) \le G_1(t,s) \le G_1(b,s)$, $\forall (t,s) \in [a,b] \times [a,b]$.

(ii) For $t \leq s$, we have

$$\frac{G_{11}(t,s)}{G_{11}(b,s)} = \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1}} = \left(\frac{t-a}{b-a}\right)^{\alpha_1-1} \quad \text{for } s \in (a,b).$$

For $t \geq s$, we get

$$\begin{aligned}\frac{G_{11}(t,s)}{G_{11}(b,s)} &= \frac{(t-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1} - (t-s)^{\alpha_1-1}(b-a)^{\alpha_1-\mu_1-1}}{(b-a)^{\alpha_1-1}(b-s)^{\alpha_1-\mu_1-1} - (b-s)^{\alpha_1-1}(b-a)^{\alpha_1-\mu_1-1}} \\ &\geq \frac{(t-a)^{\alpha_1-\mu_1-1}(b-s)^{\alpha_1-\mu_1-1}[(t-a)^{\mu_1} - (t-s)^{\mu_1}]}{(b-a)^{\alpha_1-\mu_1-1}(b-s)^{\alpha_1-\mu_1-1}[(b-a)^{\mu_1} - (b-s)^{\mu_1}]} \\ &\geq \left(\frac{t-a}{b-a}\right)^{\alpha_1-1} \quad \text{for } s \in (a,b).\end{aligned}$$

Thus,

$$G_{11}(t,s) \geq \left(\frac{t-a}{b-a}\right)^{\alpha_1-1} G_{11}(b,s) \quad \text{for all } (t,s) \in [a,b] \times (a,b).$$

Then

$$\begin{aligned}G_1(t,s) &= G_{11}(t,s) + \frac{(t-a)^{\alpha_1-1}\sum_{i=1}^{\infty}\gamma_i G_{12}(\xi_i,s)}{\Phi_1} \\ &\geq \left(\frac{t-a}{b-a}\right)^{\alpha_1-1} G_{11}(b,s) + \left(\frac{t-a}{b-a}\right)^{\alpha_1-1} \frac{(b-a)^{\alpha_1-1}\sum_{i=1}^{\infty}\gamma_i G_{12}(\xi_i,s)}{\Phi_1} \\ &= \left(\frac{t-a}{b-a}\right)^{\alpha_1-1} G_1(b,s) \geq \left(\frac{1}{4}\right)^{\alpha_1-1} G_1(b,s) \quad \text{for all } (t,s) \in I \times (a,b).\end{aligned}$$

□

Lemma 2.5 *Suppose* (H_0)-(H_2) *hold. Then* $H_1(t,s)$ *has the following properties*:

(i) $0 \leq H_1(t,s) \leq \omega_1(s)$ *for all* $(t,s) \in [a,b] \times [a,b]$, *where*

$$\omega_1(s) = H_{11}(s,s) + \frac{(b-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i H_{12}(\eta_i,s)}{\Delta_1},$$

(ii) $H_1(t,s) \geq h_1(s)H_{11}(s,s)$ *for all* $(t,s) \in I \times (a,b)$, *where* $I = [\frac{3a+b}{4}, \frac{a+3b}{4}]$, *and for* $\zeta_1 \in I$,

$$h_1(s) = \begin{cases} \frac{(b-s)^{\beta_1-\mu_2-1}(\frac{3}{4})^{\beta_1-1} - (\frac{a+3b}{4}-s)^{\beta_1-1}(b-a)^{-\mu_2}}{(b-s)^{\beta_1-\mu_2-1}}, & s \in (a,\zeta_1], \\ (\frac{1}{4})^{\beta_1-1}, & s \in [\zeta_1,b). \end{cases}$$

Proof (i) For $t \leq s$, it is easy to show that $\frac{\partial}{\partial t}H_{11}(t,s) \geq 0$ for all $(t,s) \in [a,b] \times [a,b]$, then

$$0 = H_{11}(a,s) \leq H_{11}(t,s) \leq H_{11}(s,s) \quad \text{for } t \leq s.$$

For $t \geq s$, we have

$$\begin{aligned}\frac{\partial}{\partial t}H_{11}(t,s) &= \frac{(\beta_1-1)[(t-a)^{\beta_1-2}(b-s)^{\beta_1-\mu_2-1} - (t-s)^{\beta_1-2}(b-a)^{\beta_1-\mu_2-1}]}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \\ &\leq \frac{(\beta_1-1)(t-s)^{\beta_1-2}[(b-s)^{\beta_1-\mu_2-1} - (b-a)^{\beta_1-\mu_2-1}]}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}} \leq 0.\end{aligned}$$

Thus,

$$0 = H_{11}(a,s) \le H_{11}(t,s) \le H_{11}(s,s) \quad \text{for } t \ge s.$$

Hence, for all $(t,s) \in [a,b] \times [a,b]$, by $\mu_2 \le \beta_1 - 1$ and $(b-s)(t-a) \ge (t-s)(b-a)$, we can easily show that $H_{12}(t,s) \ge 0$, then

$$\begin{aligned} 0 &\le H_1(t,s) \\ &= H_{11}(t,s) + \frac{(t-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i H_{12}(\eta_i,s)}{\Delta_1} \\ &\le H_{11}(s,s) + \frac{(b-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i H_{12}(\eta_i,s)}{\Delta_1} \\ &= \omega_1(s). \end{aligned}$$

(ii) Setting:

$$\begin{aligned} h_1(t,s) &= \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1} - (t-s)^{\beta_1-1}(b-a)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, \\ h_2(t,s) &= \frac{(b-s)^{\beta_1-\mu_2-1}(t-a)^{\beta_1-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}. \end{aligned}$$

Then

$$\begin{aligned} \min_{t\in I} H_{11}(t,s) &= \begin{cases} h_1(\frac{a+3b}{4},s), & s \in (a, \frac{3a+b}{4}], \\ \min\{h_1(\frac{a+3b}{4},s), h_2(\frac{3a+b}{4},s)\}, & s \in [\frac{3a+b}{4}, \frac{a+3b}{4}], \\ h_2(\frac{3a+b}{4},s), & s \in [\frac{a+3b}{4}, b), \end{cases} \\ &= \begin{cases} h_1(\frac{a+3b}{4},s), & s \in (a,\zeta_1], \\ h_2(\frac{3a+b}{4},s), & s \in [\zeta_1,b), \end{cases} \\ &= \begin{cases} \frac{(b-s)^{\beta_1-\mu_2-1}(\frac{3}{4}(b-a))^{\beta_1-1} - (\frac{a+3b}{4}-s)^{\beta_1-1}(b-a)^{\beta_1-\mu_2-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, & s \in (a,\zeta_1], \\ \frac{(b-s)^{\beta_1-\mu_2-1}(\frac{1}{4}(b-a))^{\beta_1-1}}{\Gamma(\beta_1)(b-a)^{\beta_1-\mu_2-1}}, & s \in [\zeta_1,b), \end{cases} \\ &= \begin{cases} \frac{(b-s)^{\beta_1-\mu_2-1}(\frac{3}{4}(b-a))^{\beta_1-1} - (\frac{a+3b}{4}-s)^{\beta_1-1}(b-a)^{\beta_1-\mu_2-1}}{(b-s)^{\beta_1-\mu_2-1}(s-a)^{\beta_1-1}} H_{11}(s,s), & s \in (a,\zeta_1], \\ \frac{(b-s)^{\beta_1-\mu_2-1}(\frac{1}{4}(b-a))^{\beta_1-1}}{(b-s)^{\beta_1-\mu_2-1}(s-a)^{\beta_1-1}} H_{11}(s,s), & s \in [\zeta_1,b), \end{cases} \\ &\ge \begin{cases} \frac{(b-s)^{\beta_1-\mu_2-1}(\frac{3}{4})^{\beta_1-1} - (\frac{a+3b}{4}-s)^{\beta_1-1}(b-a)^{-\mu_2}}{(b-s)^{\beta_1-\mu_2-1}} H_{11}(s,s), & s \in (a,\zeta_1], \\ (\frac{1}{4})^{\beta_1-1} H_{11}(s,s), & s \in [\zeta_1,b), \end{cases} \\ &= h_1(s)H_{11}(s,s) \quad \text{for } s \in (a,b). \end{aligned}$$

Therefore, for all $(t,s) \in I \times (a,b)$, we have

$$H_1(t,s) = H_{11}(t,s) + \frac{(t-a)^{\beta_1-1}\sum_{i=1}^{\infty}\delta_i H_{12}(\eta_i,s)}{\Delta_1} \ge H_{11}(t,s) \ge h_1(s)H_{11}(s,s).$$

We complete the proof. □

Remark 2.6 In a similar manner, the results of the Green's function $G_2(t,s)$ and $H_2(t,s)$ for the homogeneous BVP corresponding to the fractional differential equation (1.2) and (1.4) are obtained. Consider the following conditions:

(i) $G_i(t,s) \geq \Lambda G_i(b,s)$ for all $(t,s) \in I \times (a,b)$, $i = 1,2$,

(ii) $H_i(t,s) \geq h(s)H_{i1}(s,s)$ for all $(t,s) \in I \times (a,b)$, $i = 1,2$,

where $I = [\frac{3a+b}{4}, \frac{a+3b}{4}]$, $\Lambda = \min\{(\frac{1}{4})^{\alpha_1-1}, (\frac{1}{4})^{\alpha_2-1}\}$, $h(s) = \min\{h_1(s), h_2(s)\}$.

Let c, d, e, r be positive real numbers, $K_r = \{x \in K : \|x\| < r\}$, $K(\psi,d,e) = \{x \in K : d \leq \psi(x), \|x\| \leq e\}$.

Lemma 2.7 ([16]) *Let K be a cone in a real Banach space E, $K_r = \{x \in K : \|x\| < r\}$, ψ be a nonnegative continuous concave functional on K such that $\psi(x) \leq \|x\|$, $\forall x \in \overline{K}_r$ and $K(\psi,d,e) = \{x \in K : d \leq \psi(x), \|x\| \leq e\}$. Suppose $T : \overline{K}_r \to \overline{K}_r$ is completely continuous and there exist constants $0 < c < d < e \leq r$ such that*

(i) *$\{x \in K(\psi,d,e) | \psi(x) > d\} \neq \emptyset$ and $\psi(Tx) > d$ for $x \in K(\psi,d,e)$;*

(ii) *$\|Tx\| < c$ for $x \leq c$;*

(iii) *$\psi(Tx) > d$ for $x \in K(\psi,d,r)$ with $\|Tx\| > e$.*

Then T has at least three fixed points x_1, x_2 and x_3 with $\|x_1\| < c$, $d < \psi(x_2)$, $c < \|x_3\|$ with $\psi(x_3) < d$.

3 The main result and proof

Let the Banach space $E = C[a,b] \times C[a,b]$ be endowed with the norm $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in E$ and $\|x\| = \max_{a\leq t\leq b} |x(t)|$. Define a cone $K \in E$ by

$$K = \Bigl\{(u,v) \in E : \min_{a\leq t\leq b} u(t) \geq 0, \min_{a\leq t\leq b} v(t) \geq 0 \text{ and } \min_{t\in I}\bigl\{u(t) + v(t)\bigr\} \geq \Lambda \bigl\|(u,v)\bigr\|\Bigr\},$$

where $I = [\frac{3a+b}{4}, \frac{a+3b}{4}]$.

It is well known that the system of fractional order BVP (1.1)-(1.4) is equivalent to

$$u(t) = \int_a^b G_1(t,s)\varphi_q\left(\int_a^b H_1(s,\tau)f_1\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\right) ds,$$
$$v(t) = \int_a^b G_2(t,s)\varphi_q\left(\int_a^b H_2(s,\tau)f_2\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\right) ds.$$

Define the operators $T_1, T_2 : K \to E$ by

$$T_1(u,v)(t) = \int_a^b G_1(t,s)\varphi_q\left(\int_a^b H_1(s,\tau)f_1\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\right) ds,$$
$$T_2(u,v)(t) = \int_a^b G_2(t,s)\varphi_q\left(\int_a^b H_2(s,\tau)f_2\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\right) ds,$$

and the operator $T : K \to E$ by

$$T(u,v) = \bigl(T_1(u,v), T_2(u,v)\bigr) \quad \text{for } (u,v) \in E.$$

It is clear that the existence of a positive solution to system (1.1)-(1.4) is equivalent to the existence of fixed points of the operator T.

Lemma 3.1 *$T: K \to K$ is completely continuous.*

Proof The continuity of functions $G_i(t,s)$, $H_i(t,s)$ and $f_i(t,u(t),v(t))$ for $i = 1,2$ implies that $T: K \to K$ is continuous. For all $(t,s) \in I \times [a,b]$ (where $I = [\frac{3a+b}{4}, \frac{a+3b}{4}]$), we have

$$\begin{aligned}\min_{t\in I}\bigl\{T_1(u,v)(t) + T_2(u,v)(t)\bigr\} &= \min_{t\in I}\biggl\{\int_a^b G_1(t,s)\varphi_q\biggl(\int_a^b H_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\quad + \int_a^b G_2(t,s)\varphi_q\biggl(\int_a^b H_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\biggr\} \\ &\geq \Lambda\biggl\{\int_a^b G_1(b,s)\varphi_q\biggl(\int_a^b H_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds \\ &\quad + \int_a^b G_2(b,s)\varphi_q\biggl(\int_a^b H_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\biggr\} \\ &\geq \Lambda\bigl(\|T_1(u,v)\| + \|T_2(u,v)\|\bigr) \\ &= \Lambda\bigl\|\bigl(T_1(u,v),T_2(u,v)\bigr)\bigr\| \\ &= \Lambda\bigl\|T(u,v)\bigr\|.\end{aligned}$$

Thus, $T(K) \subset K$. So, we can easily show that $T: K \to K$ is completely continuous by the Arzela-Ascoli theorem. The proof is completed. □

Let the nonnegative continuous concave functional ψ be defined on the cone K by

$$\psi(u,v) = \min_{t\in I}\bigl|u(t)+v(t)\bigr|, \quad \text{where } I = \biggl[\frac{3a+b}{4}, \frac{a+3b}{4}\biggr].$$

For convenience, we denote

$$\begin{aligned}N = \max\biggl\{&\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} G_1(b,s)\varphi_q\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} h(\tau)H_{11}(\tau,\tau)\,d\tau\biggr)\,ds\biggr)^{-1}, \\ &\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} G_2(b,s)\varphi_q\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} h(\tau)H_{21}(\tau,\tau)\,d\tau\biggr)\,ds\biggr)^{-1}\biggr\},\end{aligned}$$

and

$$\begin{aligned}M = \min\biggl\{&\biggl(\int_a^b G_1(b,s)\varphi_q\biggl(\int_a^b \omega_1(\tau)\,d\tau\biggr)\,ds\biggr)^{-1}, \\ &\biggl(\int_a^b G_2(b,s)\varphi_q\biggl(\int_a^b \omega_2(\tau)\,d\tau\biggr)\,ds\biggr)^{-1}\biggr\}.\end{aligned}$$

Theorem 3.2 *Suppose that* (H_0)-(H_2) *hold. If there exist positive real numbers* $0 < c < d < \Lambda r$ *such that the following conditions hold*:

(H_3) $f_i(t,u,v) < \varphi_p(\frac{rM}{2})$ *for* $i = 1,2$, *for all* $t \in [a,b]$, $(u,v) \in [0,r] \times [0,r]$;
(H_4) $f_i(t,u,v) > \varphi_p(\frac{dN}{2\Lambda})$ *for* $i = 1,2$, *for all* $t \in I = [\frac{3a+b}{4}, \frac{a+3b}{4}]$, $(u,v) \in [d, \frac{d}{\Lambda}] \times [d, \frac{d}{\Lambda}]$;
(H_5) $f_i(t,u,v) < \varphi_p(\frac{cM}{2})$ *for* $i = 1,2$, *for all* $t \in [a,b]$, $(u,v) \in [0,c] \times [0,c]$.

Then the system of fractional differential equations BVP (1.1)-(1.4) *has at least three positive solutions* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *with*

$$\max_{a\le t\le b}\bigl|u_1(t)+v_1(t)\bigr|<c, \qquad d<\min_{t\in I}\bigl|u_2(t)+v_2(t)\bigr|<\max_{a\le t\le b}\bigl|u_2(t)+v_2(t)\bigr|<r,$$

$$c<\max_{a\le t\le b}\bigl|u_3(t)+v_3(t)\bigr|<r, \qquad \min_{t\in I}\bigl|u_3(t)+v_3(t)\bigr|<d.$$

Proof Firstly, if $(u,v)\in\overline{K}_r$, then we may assert that $T:\overline{K}_r\to\overline{K}_r$ is a completely continuous operator. To see this, suppose $(u,v)\in\overline{K}_r$, then $\|(u,v)\|\le r$. It follows from Lemma 2.4, Lemma 2.5 and (H_3) that

$$\begin{aligned}\bigl\|T(u,v)\bigr\| &= \max_{a\le t\le b}\bigl|T(u,v)(t)\bigr|\\ &= \max_{a\le t\le b}\biggl(\int_a^b G_1(t,s)\varphi_q\biggl(\int_a^b H_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\\ &\quad+\int_a^b G_2(t,s)\varphi_q\biggl(\int_a^b H_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\biggr)\\ &\le \frac{rM}{2}\biggl(\int_a^b G_1(b,s)\varphi_q\biggl(\int_a^b \omega_1(\tau)\,d\tau\biggr)\,ds+\int_a^b G_2(b,s)\varphi_q\biggl(\int_a^b \omega_2(\tau)\,d\tau\biggr)\,ds\biggr)\\ &\le \frac{r}{2}+\frac{r}{2}=r.\end{aligned}$$

Therefore, $T:\overline{K}_r\to\overline{K}_r$. This together with Lemma 3.1 implies that $T:\overline{K}_r\to\overline{K}_r$ is a completely continuous operator. In the same way, if $(u,v)\in\overline{K}_c$, then assumption (H_5) yields $\|T(u,v)\|<c$. Hence, condition (ii) of Lemma 2.7 is satisfied.

To check condition (i) of Lemma 2.7, we let $u(t)+v(t)=\frac{d}{\Lambda}$ for $t\in[a,b]$. It is easy to verify that $u(t)+v(t)=\frac{d}{\Lambda}\in K(\psi,d,\frac{d}{\Lambda})$ and $\psi(u,v)=\frac{d}{\Lambda}>d$, and so $\{(u,v)\in K(\psi,d,\frac{d}{\Lambda}):\psi(u,v)>d\}\neq\emptyset$. Thus, for all $(u,v)\in K(\psi,d,\frac{d}{\Lambda})$, we have that $d\le u(t)+v(t)\le\frac{d}{\Lambda}$ for $t\in I$ and $T(u,v)\in K$. From Lemma 2.4, Lemma 2.5 and (H_4), one has

$$\begin{aligned}\psi\bigl(T(u,v)(t)\bigr) &= \min_{t\in I}\bigl|T(u,v)(t)\bigr|\\ &= \min_{t\in I}\biggl(\int_a^b G_1(t,s)\varphi_q\biggl(\int_a^b H_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\\ &\quad+\int_a^b G_2(t,s)\varphi_q\biggl(\int_a^b H_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\biggr)\\ &\ge \biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \Lambda G_1(b,s)\varphi_q\biggl(\int_a^b H_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\\ &\quad+\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \Lambda G_2(b,s)\varphi_q\biggl(\int_a^b H_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\biggr)\\ &\ge \biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \Lambda G_1(b,s)\varphi_q\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} h(\tau)H_{11}(\tau,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\\ &\quad+\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \Lambda G_2(b,s)\varphi_q\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} h(\tau)H_{21}(\tau,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\biggr)\end{aligned}$$

$$
\begin{aligned}
&> \frac{dN}{2\Lambda}\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \Lambda G_1(b,s)\varphi_q\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} h(\tau)H_{11}(\tau,\tau)\,d\tau\biggr)\,ds \\
&\quad + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \Lambda G_2(b,s)\varphi_q\biggl(\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} h(\tau)H_{21}(\tau,\tau)\,d\tau\biggr)\,ds\biggr) \\
&\geq \frac{d}{2}+\frac{d}{2}=d.
\end{aligned}
$$

This shows that condition (i) of Lemma 2.7 holds.

Secondly, we verify that (iii) of Lemma 2.7 is satisfied. By Lemma 3.1, we have

$$\min_{t\in I}\bigl|T(u,v)(t)\bigr| \geq \Lambda\bigl\|T(u,v)\bigr\| > d \quad \text{for } (u,v)\in K(\psi,d,r) \text{ with } \bigl\|T(u,v)\bigr\| > \frac{d}{\Lambda},$$

which shows that condition (iii) of Lemma 2.7 holds.

To sum up, all the conditions of Lemma 2.7 are satisfied, it follows from Lemma 2.7 that there exist three positive solutions (u_1,v_1), (u_2,v_2) and (u_3,v_3) satisfying

$$
\begin{aligned}
&\max_{a\leq t\leq b}\bigl|u_1(t)+v_1(t)\bigr| < c, \qquad d < \min_{t\in I}\bigl|u_2(t)+v_2(t)\bigr| < \max_{a\leq t\leq b}\bigl|u_2(t)+v_2(t)\bigr| < r, \\
&c < \max_{a\leq t\leq b}\bigl|u_3(t)+v_3(t)\bigr| < r, \qquad \min_{t\in I}\bigl|u_3(t)+v_3(t)\bigr| < d.
\end{aligned}
$$

□

4 Example

Example 1 Consider the system of fractional differential equations BVP

$$D_{0^+}^{\frac{3}{2}}\bigl(\varphi_p\bigl(D_{0^+}^{\frac{5}{2}}u(t)\bigr)\bigr) = f_1\bigl(t,u(t),v(t)\bigr), \quad t\in(0,1), \tag{4.1}$$

$$D_{0^+}^{\frac{3}{2}}\bigl(\varphi_p\bigl(D_{0^+}^{\frac{5}{2}}v(t)\bigr)\bigr) = f_2\bigl(t,u(t),v(t)\bigr), \quad t\in(0,1), \tag{4.2}$$

with the boundary conditions

$$\begin{cases} u(0)=u'(0)=0, & u^{(\frac{5}{4})}(1)=\frac{1}{4}u^{(\frac{5}{4})}(\frac{1}{4}), \\ \varphi_p(D_{0^+}^{\frac{5}{2}}u(0))=0, & D_{0^+}^{\frac{1}{4}}(\varphi_p(D_{0^+}^{\frac{5}{2}}u(1)))=\frac{1}{3}D_{0^+}^{\frac{1}{4}}(\varphi_p(D_{0^+}^{\frac{5}{2}}u(\frac{1}{4}))), \end{cases} \tag{4.3}$$

$$\begin{cases} v(0)=v'(0)=0, & v^{(\frac{5}{4})}(1)=\frac{1}{4}v^{(\frac{5}{4})}(\frac{1}{4}), \\ \varphi_p(D_{0^+}^{\frac{5}{2}}v(0))=0, & D_{0^+}^{\frac{1}{4}}(\varphi_p(D_{0^+}^{\frac{5}{2}}v(1)))=\frac{1}{3}D_{0^+}^{\frac{1}{4}}(\varphi_p(D_{0^+}^{\frac{5}{2}}v(\frac{1}{4}))), \end{cases} \tag{4.4}$$

where

$$f_1(t,u,v)=\begin{cases} \frac{4(u+v)^2}{5(1+t^2)}, & 0\leq t\leq 1, 0\leq u+v\leq\frac{1}{2}, \\ \frac{18{,}559.6(u+v)-9{,}279.6}{1+t^2}, & 0\leq t\leq 1, \frac{1}{2}<u+v\leq 1, \\ \frac{145}{(1+t^2)2^{u+v-7}}, & 0\leq t\leq 1, u+v>1, \end{cases}$$

$$f_2(t,u,v)=\begin{cases} \frac{1}{2}(1-t)(u+v), & 0\leq t\leq 1, 0\leq u+v\leq\frac{1}{2}, \\ (1-t)(11{,}839.5(u+v)-5{,}919.5), & 0\leq t\leq 1, \frac{1}{2}<u+v\leq 1, \\ 92.5(1-t)2^{-(u+v)+7}, & 0\leq t\leq 1, u+v>1. \end{cases}$$

We note that $a=0$, $b=1$, $n=m=3$, $\alpha_1=\alpha_2=\frac{5}{2}$, $\beta_1=\beta_2=\frac{3}{2}$, $\mu_1=\mu_3=\frac{5}{4}$, $\gamma_1=\eta_1=\xi_1=\mu_2=\mu_4=\frac{1}{4}$, $\delta_1=\frac{1}{3}$, $\gamma_i=\delta_i=0$ $(i=2,3,4,\ldots)$. Let $p=2$, an easy computation shows that

$\Phi_1 = \Phi_2 \approx 0.8232$, $\Delta_1 = \Delta_2 \approx 0.7643$, $\Lambda = 0.125$, $N \approx 23.1092$, $M \approx 2.6109$. Then, if we choose $c = \frac{1}{4}$, $d = \frac{1}{2}$, $r = 7{,}109$, then $f_i(t,u,v)$ for $i = 1,2$ satisfies the following conditions:

(H_3) $f_i(t,u,v) < \varphi_p(\frac{rM}{2}) \approx 9{,}280.44$ for $i = 1,2$, for all $t \in [0,1]$, $(u,v) \in [0,7{,}109] \times [0,7{,}109]$;

(H_4) $f_i(t,u,v) > \varphi_p(\frac{dN}{2\Lambda}) = 46.2184$ for $i = 1,2$, for all $t \in [\frac{1}{4},\frac{3}{4}]$, $(u,v) \in [\frac{1}{2},4] \times [\frac{1}{2},4]$;

(H_5) $f_i(t,u,v) < \varphi_p(\frac{cM}{2}) \approx 0.3264$ for $i = 1,2$, for all $t \in [0,1]$, $(u,v) \in [0,\frac{1}{4}] \times [0,\frac{1}{4}]$.

Thus, all the hypotheses of Theorem 3.2 are satisfied. Hence, the system of fractional differential equations BVP (4.1)-(4.4) has at least three positive solutions.

Acknowledgements
The authors thank the editor and referees for their careful reading of the manuscript and a number of excellent suggestions. The work is sponsored by the Natural Science Foundation of China (11571136 and 11271364), the University Natural Science Key Research Program of Anhui Province under Grant (KJ2015A196) and the Anhui Provincial Natural Science Foundation of key projects (KJ2014A200).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Hefei Normal University, Hefei, Anhui 230061, P.R. China. [2]School of Mathematical Science, Huaiyin Normal University, Huaian, Jiangsu 223300, P.R. China.

Publisher's Note

Received: 12 January 2017 Accepted: 26 May 2017 Published online: 15 June 2017

References

1. Al-Hossain, AY: Existence of positive solutions for system of p-Laplacian fractional order boundary value problems. Differ. Equ. Dyn. Syst. (2016). doi:10.1007/s12591-016-0275-0
2. Yuan, C, Jiang, D, O'Regan, D, Agarwal, RP: Multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2012**, Article ID 13 (2012)
3. Bai, C: Existence of positive solutions for boundary value problems of fractional differential equations. Electron. J. Qual. Theory Differ. Equ. **2010**, Article ID 30 (2010)
4. Ma, D: Positive solutions of multi-point boundary value problem of fractional differential equation. Arab J. Math. Sci. **21**, 225-236 (2015)
5. Henderson, J, Luca, R: Nonexistence of positive solutions for a system of coupled fractional boundary value problems. Bound. Value Probl. **2015**, Article ID 138 (2015)
6. Henderson, J, Luca, R: Existence and multiplicity of positive solutions for a system of fractional boundary value problems. Bound. Value Probl. **2014**, Article ID 60 (2014)
7. Prasad, KR, Krushna, BMB, Sreedhar, N: Even number of positive solutions for the system of (p, q)-Laplacian fractional order two-point boundary value problems. Differ. Equ. Dyn. Syst. (2016). doi:10.1007/s12591-016-0281-2
8. Zhao, K, Liu, J: Multiple monotone positive solutions of integral BVPs for a higher-order fractional differential equation with monotone homomorphism. Adv. Differ. Equ. **2016**, Article ID 20 (2016)
9. Yang, L, Shen, C, Xie, D: Multiple positive solutions for nonlinear boundary value problem of fractional order differential equation with the Riemann-Liouville derivative. Adv. Differ. Equ. **2014**, Article ID 284 (2014)
10. Luca, R, Tudorache, A: Positive solutions to a system of semipositone fractional boundary value problems. Adv. Differ. Equ. **2014**, Article ID 179 (2014)
11. Leggett, RW, Williams, LR: Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math. J. **28**, 673-688 (1979)
12. Feng, W: Topological methods on solvability, multiplicity and eigenvalues of a nonlinear fractional boundary value problem. Electron. J. Qual. Theory Differ. Equ. **2015**, Article ID 70 (2015)
13. Liu, X, Jia, M, Ge, W: The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator. Appl. Math. Lett. **65**, 56-62 (2017)
14. Han, Z, Lu, H, Zhang, C: Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian. Appl. Math. Comput. **257**, 526-536 (2015)
15. Kilbas, AA, Srivastava, HH, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
16. Jiang, W, Qiu, J, Yang, C: The existence of positive solutions for p-Laplacian boundary value problems at resonance. Bound. Value Probl. **2016**, Article ID 175 (2016)