Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Dynamical behaviors of a two-competitive metapopulation system with impulsive control

Shasha Tian, Yepeng Xing* and Tao Ma

*Correspondence:
13818008538@163.com
Department of Mathematics,
Shanghai Normal University,
Shanghai, 200234, P.R. China

Abstract

In this paper, we study the dynamical behaviors of a two-competitive metapopulation system with impulsive control and focus on the stable coexistence of the superior and inferior species. A Poincaré map is introduced to prove the existence of a periodic solution and its stability. It is also shown that a stably positive periodic solution bifurcates from the semi-trivial periodic solution through a transcritical bifurcation.

MSC: 34D09; 34D99

Keywords: metapopulation system; impulsive state feedback; Poincaré map; bifurcation

1 Introduction and preliminaries

Metapopulation is a population in which individuals are spatially distributed in a habitat in two or more subpopulations. Populations of butterflies and coral-reef fishes are good examples of metapopulation. Human activities and natural disasters are the main causes of metapopulation as they increase the population that occurs as metapopulatons. Such factors cause the fragmentation of a large habitat into patches. This may be an important reason whereby models of metapopulation dynamics become important methods in the field of conservation biology. Readers can refer to the references [1–15] for details.

The name metapopulation was first used in 1969 [8] by Levins to describe a population dynamics model for insect pests inhabiting crop growing areas; however, the idea has since been most broadly applied to species in fragmented habitats. Let p be the fraction of sites occupied by a species which will be called its abundance. Levins posed a simple, general model for the dynamics of the site occupancy in such a system

$$\frac{dp}{dt} = cp(1-p) - mp, \tag{1.1}$$

where c is the colonization rate and m is the morality (local extinction) rate. Propagules dispersed randomly among all sites. The rate of propagules production by the occupied sites cp is multiplied by the proportion of sites that are not occupied $1-p$ to give the rate of production of newly colonized sites. The morality rate m is multiplied by the proportion of occupied sites p to give the density-independent rate at which occupied sites become vacant. A site becomes vacant when the individual occupying that site dies.

It is impossible that only one species exists in a region. Nevertheless, the emergence of other species in the plaque will cause competition, symbiosis, predation and other relations, and these are the potential effects of the two species' extinction and survival. So the Levins model was developed to the metapopulation model of two interacting species or multi interacting species. That is a study of metacommunity, which is the improvement and perfection of the theory of metapopulation.

Tilman considered in [15] the co-existence of two competitors. In his paper, the superior competitor (the first species p_1) has the same equation as would a species living by itself and thus is totally unaffected by the inferior competitor (the second species p_2). The inferior competitor can colonize only sites in which both it and the superior species are absent (the term $1 - p_1 - p_2$). However, superior species can invade into and displace inferior species (the term $-c_1p_1p_2$). Thus, this leads to the following system:

$$\begin{cases} \frac{dp_1}{dt} = c_1p_1(1 - p_1) - e_1p_1, \\ \frac{dp_2}{dt} = c_2p_1(1 - p_1 - p_2) - e_2p_2 - c_1p_1p_2. \end{cases} \tag{1.2}$$

It is easy to see that the superior species grows logistically and approaches its equilibrial abundance. Once the first species is at or very near to equilibrium, the second species grows logistically to its equilibrium. In this paper, the system is written as

$$\begin{cases} \dot{x} = ax(1 - x) - bx, \\ \dot{y} = cy(1 - x - y) - dy - axy, \end{cases} \tag{1.3}$$

for all $t > 0$. Obviously, the linear part of system (1.3) at equilibrium is determined by the matrix

$$D(x, y) = \begin{pmatrix} a - b - 2ax & 0 \\ -cy - ay & c - d - (a + c)x - 2cy \end{pmatrix}.$$

The dynamics of system (1.3) in the neighborhood of an equilibrium (x, y) directly depends on the property of eigenvalues of the matrix $D(x, y)$.

By calculation, we know that equilibriums can be grouped into the following three cases:

(1) If $c > d$, $a > b$, $-a^2 + bc + ab - ad > 0$, then system (1.3) has one trivial equilibrium $E_0 = (0, 0)$, two semi-trivial equilibria $E_1 = (0, \frac{c-d}{c})$ and $E_2 = (\frac{a-b}{a}, 0)$, one positive equilibrium $E_* = (\frac{a-b}{a}, -a^2 + bc + ab - ad)$.

E_0 is an unstable node, E_1, E_2 are saddles and E_* is a stable node (see Figure 1(a)).

(2) If $c > d$, $a > b$, $-a^2 + bc + ab - ad < 0$, then E_* disappears and it leaves only three equilibriums E_0, E_1 and E_2. E_0 is an unstable node, E_1 is a saddle and E_2 is a stable node (see Figure 1(b)).

(3) If $c > d$, $a > b$, $-a^2 + bc + ab - ad = 0$, then system (1.3) also has three equilibria, E_0 is an unstable node, E_1 is a saddle and E_2 is a saddle-node (see Figure 1(c)).

Consequently, $c > d$, $a > b$, $-a^2 + bc + ab - ad > 0$ are necessary and sufficient conditions for the stable coexistence of a superior and an inferior competitor in a subdivided habitat. However, in the case of (2) and (3), the inferior competitor goes extinct finally. The purpose of this paper is to find control strategy to ensure the stable coexistence of two competitors.

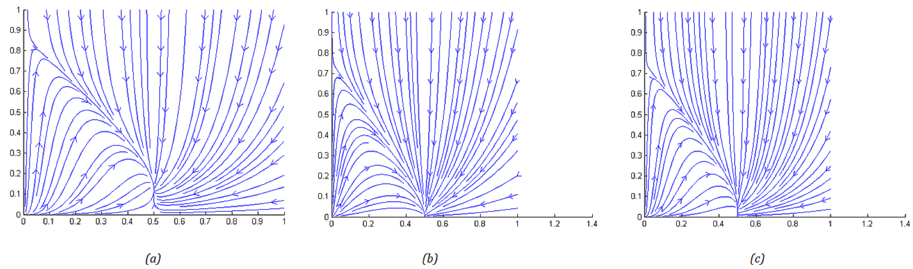


Figure 1 The phase portrait of system (1.3).

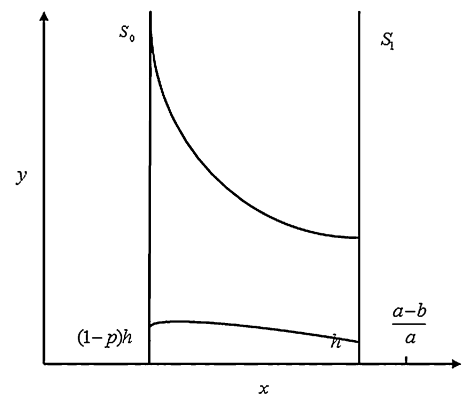
In ecological practice, controlling the amount of strong competition is our first consideration. Integrated strategy is a complete control strategy that uses a combination of biological, cultural and chemical tactics that hold the strong competition population on suitable levels with less cost and minimal effect on the environment. It has very important practical significance on how to protect the rare and endangered species (which is always the inferior competitor), maintain biodiversity and maintain ecological balance, and it also has great theoretical significance on the mathematical model of population ecology, genetic drift and biological evolution of the genetics. When the amount of the strong competition population reaches a threshold, integrated strategy may be taken. Consequently, the superior competitor population decreases, while the amount of the inferior competitor increases abruptly. Reader can refer to the references [16–25] for details. Now we consider the competition system (1.3) by introducing a state feedback control strategy rather than the usual fixed-time control strategy. This design is often feasible in practice. We can control the population size by catching or poisoning the superior competitor and releasing the inferior competitor when the amount of the strong competition reaches the threshold value. The controlled system is modeled by the following equations:

$$\begin{cases} \left.\begin{aligned} &\dot{x} = ax(1-x) - bx, \\ &\dot{y} = cy(1-x-y) - dy - axy \end{aligned}\right\} x \neq h, \\ \left.\begin{aligned} &\Delta x = -px, \\ &\Delta y = qy + \tau \end{aligned}\right\} x = h, \end{cases} \tag{1.4}$$

where the parameters $p \in (0,1)$, $h > 0$, $q > 0$, $\tau \geq 0$, and $\Delta x(t) = x(t^+) - x(t)$, $\Delta y(t) = y(t^+) - y(t)$. When the amount of the strong competition reaches the threshold value h at the time t_{k-1}, the controlling measures are taken and the amount of strong competition and weak competition abruptly turn to $(1-p)h$ and $(1+q)y(t_{k-1}) + \tau$, respectively. In this paper, $X(t) = (x(t), 0)$ is called a semi-trivial solution for $y \equiv 0$, $X(t)$ is called a periodic solution if $X(nT) = X((n+1)T)$ for all $n > 0$. According to the actual significance, we can only consider the solutions with nonnegative components, continuously differentiable in $R^+ - \{t_{k-1}\}$.

See that the superior species grows logistically and approaches its equilibrial abundance $\frac{a-b}{a}$. Nevertheless, our control strategy will not work if the amount of superior competitor exceeds $\frac{a-b}{a}$. It is reasonable that for protecting the inferior species, we firstly decrease the density of superior species down under its equilibrium $\frac{a-b}{a}$ by catching or poisoning, etc. Also we assume that the threshold value satisfies $h < \frac{a-b}{a}$ for possible stable coexistence of the two species.

Figure 2 The simple phase portrait of system (1.3).



There may be two types of trajectories when the value of x is on the left of h (see Figure 2). With the parameters p, h given, the purpose of this paper is to obtain the control measure of how to release the strong competition to control the weak competition by studying the dynamics of system (1.4). Therefore, it is easy to see that q and τ are control parameters.

The following lemma is used extensively to prove the stability of periodic solutions for impulsive differential equations.

Lemma 1 ([26]) *The T-periodic solution $(x,y) = (\xi(t), \eta(t))$ of the system*

$$\begin{cases} \frac{dx}{dt} = P(x,y), \frac{dy}{dt} = Q(x,y), & \textit{if } \varphi(x,y) \neq 0, \\ \Delta x = \alpha(x,y), \Delta y = \beta(x,y), & \textit{if } \varphi(x,y) \neq 0, \end{cases}$$

is orbitally asymptotically stable if the Floquet multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

$$\mu_2 = \prod_{k=1}^{q} \Delta_k \exp\left(\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))\right) dt\right)$$

with

$$\Delta_k = \frac{P_+(\frac{\partial\beta}{\partial y}\frac{\partial\varphi}{\partial x} - \frac{\partial\beta}{\partial x}\frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial x}) + Q_+(\frac{\partial\alpha}{\partial x}\frac{\partial\varphi}{\partial y} - \frac{\partial\alpha}{\partial y}\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y})}{P\frac{\partial\varphi}{\partial x} + Q\frac{\partial\varphi}{\partial y}}$$

and P, Q, $\frac{\partial\alpha}{\partial x}$, $\frac{\partial\alpha}{\partial y}$, $\frac{\partial\beta}{\partial x}$, $\frac{\partial\beta}{\partial y}$, $\frac{\partial\varphi}{\partial x}$ and $\frac{\partial\varphi}{\partial y}$ are calculated at the point $(\xi(t_k), \eta(t_k))$, $P_+ = P(\xi(t_k^+), \eta(t_k^+))$ and $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$. Here $\varphi(x,y)$ is a sufficiently smooth function such that grad $\varphi(x,y) = 0$, and t_k $(k \in N)$ is the time of the kth jump.

Next lemma is employed to prove the existence of bifurcation of a mapping.

Lemma 2 ([27]) *Let $F: R \times R \to R$ be a one-parameter family of C^2 maps satisfying*

(i) $F(0,\mu) = 0$,
(ii) $\frac{\partial F}{\partial x}(0,0) = 1$,
(iii) $\frac{\partial^2 F}{\partial x \partial \mu}(0,0) > 0$,
(iv) $\frac{\partial^2 F}{\partial x^2}(0,0) < 0$.

Then F has two branches of fixed points for μ near zero. The first branch is $x_1(\mu) = 0$ for all μ. The second bifurcation branch $x_2(\mu)$ changes its value from negative to positive as μ increases through $\mu = 0$. The fixed points of the first branch is stable if $\mu < 0$ and unstable if $\mu > 0$, while those of the bifurcating branches have the opposite stability.

To understand this lemma, according to the conditions, we may assume that the mapping $F(x,\mu)$ has the normal form

$$F(x,\mu) = x + \alpha\mu x - \beta x^2, \quad \text{where } \alpha > 0, \beta > 0,$$

and

$$F'_x(x,\mu) = 1 + \alpha\mu - 2\beta x.$$

If we let $F(x,\mu) = x$, then we get $x_1(\mu) = 0$ and $x_2(\mu) = \frac{\alpha}{\beta}$. Obviously, $F'_x(0,\mu) = 1 + \alpha\mu$ and $F'_x(\frac{\alpha}{\beta},\mu) = 1 - \alpha\mu$ and so that the two fixed points have opposite stability.

The rest of this paper is organized as follows. In Section 2, the sufficient conditions for the existence and stability of a semi-trivial periodic solution are given. The Poincaré maps are constructed and theoretical results of dynamical behaviors are presented, including the transcritical and flip bifurcations.

2 Dynamical properties

2.1 Poincaré map

In order to discuss the dynamical behavior of system (1.4), two types of Poincaré maps are constructed. Firstly, we set Poincaré section $S_0 = \{(x,y) | x = (1-p)h, y \geq 0\}$ and Poincaré section $S_1 = \{(x,y) | x = h, y \geq 0\}$. Suppose that system (1.4) has a periodic solution $(\varphi(t), \phi(t))$ (see Figure 3 and Figure 4).

Now we establish a type of Poincaré mapping. Suppose that the periodic trajectory with the initial point $E^+ = ((1-p)h, y_0)$ intersects the Poincaré section S_1 at the point $E = (h, y_1)$, then jumps to the point E^+ on line S_0 due to the impulsive effects $\Delta x = -px$ and $\Delta y = qy + \tau$. Therefore

$$\varphi(0) = (1-p)h, \qquad \phi(0) = y_0, \qquad \varphi(T) = h, \qquad \phi(T) = y_1 = \frac{y_0 - \tau}{1+q}. \tag{2.1}$$

Figure 3 The phase portrait of system (1.4).

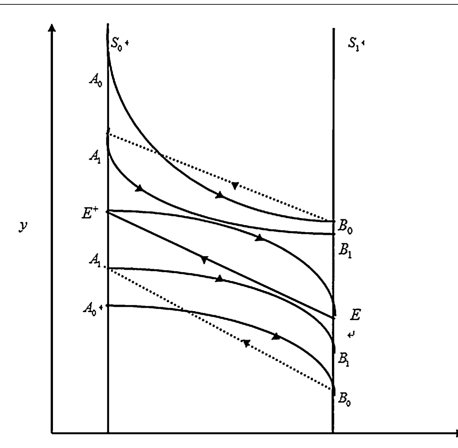
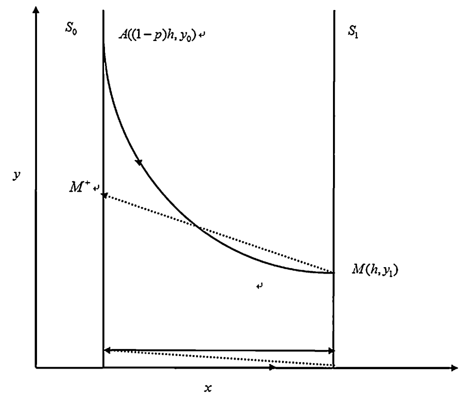


Figure 4 The simple phase portrait of system (1.4) with $\tau > 0$.



Consider another solution $(\overline{\varphi}(t), \overline{\phi}(t))$ with initial point $A_0 = ((1-p)h, y_0 + \delta y_0)$, where δy_0 is small enough. This disturbed trajectory starting from the point A_0 first intersects the Poincaré section S_1 at the point $B_0 = (h, \overline{y_1})$ at moment $t = T + \delta t$ and then jumps to the point $A_1 = ((1-p)h, \overline{y})$ on the line S_0. Hence,

$$\overline{\varphi}(0) = (1-p)h, \qquad \overline{\phi}(0) = y_0 + \delta y_0, \qquad \overline{\varphi}(T + \delta t) = h, \qquad \overline{\phi}(T + \delta t) = \overline{y_1}. \tag{2.2}$$

Let $\delta x = \overline{\varphi}(t) - \varphi(t)$ and $\delta y = \overline{\phi}(t) - \phi(t)$, then $\delta x_0 = \overline{\varphi}(0) - \varphi(0) = (1-p)h - (1-p)h = 0$ and $\delta y_0 = \overline{\phi}(0) - \phi(0) = |\overline{A_0 E^+}|$. Set $\delta y_1 = |\overline{A_1 E^+}|$ and $\delta y_0^* = |\overline{B_0 E}|$, and then the relation between δy_0 and δy_1 determines one type of Poincaré map P:

$$\delta y_1 = f(q, \tau, \delta y_0). \tag{2.3}$$

We next establish another type of Poincaré map. According to the above discussions, any trajectory through the initial point $((1-p)h, u)$ will always intersect with the line S_1 at the point $(h, g(u))$. Thus, for any point (x, y) which satisfies $\dot{x} > 0$, system (1.3) can be transformed as follows:

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \tag{2.4}$$

where $Q(x,y) = cy(1-x-y) - dy - axy$ and $P(x,y) = ax(1-x) - bx$.

Let $(x, y(x; x_0, y_0))$ be an orbit of system (2.5), and set $x_0 = (1-p)h$, $y_0 = u$, then we have

$$y\bigl(x; (1-p)h, u\bigr) \equiv y(x,u), \qquad (1-p)h \le x \le h$$

and

$$\frac{\partial y(x,u)}{\partial u} = \exp\left\{ \int_{(1-p)h}^{x} \frac{\partial}{\partial y}\left(\frac{Q(s, y(s,u))}{P(s, y(s,u))} \right) ds \right\} > 0. \tag{2.5}$$

Then

$$\frac{\partial^2 y(x,u)}{\partial u^2} = \frac{\partial y(x,u)}{\partial u} \int_{(1-p)h}^{x} \frac{\partial^2}{\partial y^2}\left(\frac{Q(s, y(s,u))}{P(s, y(s,u))} \right) \frac{\partial y(s,u)}{\partial u}\, ds. \tag{2.6}$$

Taking into account that

$$\frac{\partial^2}{\partial y^2}\left(\frac{Q(s,y(s,u))}{P(s,y(s,u))}\right)=\frac{-2c}{(a-b)x-ax^2},$$

and $(1-p)h \leq x \leq h < (a-b)/a$, we have that

$$\frac{\partial^2 y(x,u)}{\partial u^2}<0. \tag{2.7}$$

Suppose that the point $A_k((1-h)h, y_k)$ is on the Poincaré section S_0, $B_k(h, g(y_k)) \in S_1$. Then $B_k^+((1-p)h, (1+q)g(y_k)+\tau) := A_{k+1}$ is on the line S_0 again due to the impulsive effects. Hence we get the following Poincaré map $\hat{P}$: $R \to R$:

$$\hat{P}(u,q,\tau)=(1+q)g(u)+\tau. \tag{2.8}$$

Noting that $\frac{Q(x,y)}{P(x,y)}$ is continuous on the stripe region $\{(x,y)|(1-p)h \leq x \leq h, 0 \leq y < \infty\}$, we have the following proposition.

Lemma 3 *The Poincaré map* $\hat{P}: R \to R$, $\hat{P}(u,q,\tau)=(1+q)g(u)+\tau$ *is continuous in* u. (1.4) *has a periodic solution on the stripe region* $\{(x,y)|(1-p)h \leq x \leq h, 0 \leq y < \infty\}$ *if and only if* $\hat{P}$ *has a fixed point*.

From the definition of Poincaré map and (2.5), we have that

$$P'(u)=(1+q)g'(u)=(1+q)\exp\left\{\int_{(1-p)h}^{h}\frac{\partial}{\partial y}\left(\frac{Q(s,y(s,u))}{P(s,y(s,u))}\right)ds\right\}>0.$$

Lemma 4 *Assume that the Poincaré map* $\hat{P}$ *has a fixed point* y^*. *Then the periodic solution is stable* $\hat{P}'(y^*)<1$ *and unstable* $\hat{P}'(y^*)>1$.

Proof Suppose that $\hat{P}'(y^*)<1$. Then there exist positive constants ρ and δ such that for $|u-y^*|<\delta$

$$\begin{aligned}\left|\hat{P}(u)-y^*\right| &= \left|\hat{P}\left(y^*\right)+\hat{P}'\left(y^*\right)\left(u-y^*\right)+o\left(\left|u-y^*\right|^2\right)-y^*\right| \\ &= \left|\hat{P}\left(y^*\right)\left(u-y^*\right)+o\left(\left|u-y^*\right|^2\right)\right|<\rho\left|u-y^*\right|.\end{aligned}$$

Let $\hat{P}^2(u)=\hat{P}(\hat{P}(u)), \hat{P}^3(u)=\hat{P}(\hat{P}^2(u)),\ldots,\hat{P}^n(u)=\hat{P}(\hat{P}^{n-1}(u)),\ldots$. Then we have for $|u-y^*|<\delta$ that

$$\left|\hat{P}^2(u)-y^*\right|<\rho\left|\hat{P}(u)-y^*\right|<\rho^2\left|u-y^*\right|,\ldots,\left|\hat{P}^n(u)-y^*\right|<\rho\left|\hat{P}^{n-1}(u)-y^*\right|<\rho^n\left|u-y^*\right|.$$

Thus,

$$\hat{P}^n(u)\to y^*, \quad n\to\infty.$$

By the continuity of solution with respect to the initial values of equation (2.4), we have that $y(x,h,\hat{P}^n(u)) \rightrightarrows y(x,h,y^*)$ uniformly for $x \in [(1-p)h, h]$, and therefore the periodic solution $y(x,h,y^*)$ is stable.

Similarly, we can prove that the periodic solution is unstable $\hat{P}'(y^*) > 1$. □

2.2 The case of $\tau = 0$

It is easy to see that the semi-trivial periodic solution with $y = 0$ of system (1.4) exists if and only if $\tau = 0$. If $\tau = 0$, system (1.4) has the following form:

$$\begin{cases} \left.\begin{aligned} &\dot{x} = ax(1-x) - bx, \\ &\dot{y} = cy(1-x-y) - dy - axy \end{aligned}\right\} x \neq h, \\ \left.\begin{aligned} &\Delta x = -px, \\ &\Delta y = qy \end{aligned}\right\} x = h. \end{cases} \tag{2.9}$$

In what follows, the dynamical properties of periodic solutions of system (2.9) are discussed in the case where we assume that q is a control parameter.

Let $y(t) \equiv 0$ for $t \in (0, \infty)$, then we have

$$\begin{cases} \dot{x} = ax(1-x) - bx, & x \neq h, \\ \Delta x = -px, & x = h. \end{cases} \tag{2.10}$$

The solution of equation $\dot{x} = ax(1-x) - bx$ can be calculated as

$$x(t) = \frac{(a-b)(1-p)h}{a(1-p)h + [(a-b) - a(1-p)h]\exp(-(a-b)(t-t_k))}.$$

Set $T = \frac{1}{a-b}\ln\frac{a-b-a(1-p)h}{(1-p)(a-b-ah)}$, $x(T) = h$, $x(T^+) = (1-p)h$. Then we have the semi-trivial periodic solution of system (2.9) for $(k-1)T \le t < kT$, $k = 1, 2, 3, \ldots$:

$$\begin{cases} \xi(t) = \frac{(a-b)(1-p)h}{a(1-p)h + [(a-b) - a(1-p)h]\exp(-(a-b)(t-(k-1)T))}, \\ \eta(t) = 0. \end{cases} \tag{2.11}$$

Then we discuss the stability of the semi-trivial periodic solution in two ways by using Lemma 1 and Lemma 4, respectively.

Case I: $c > d$, $a > b$, $-a^2 + bc + ab - ad < 0$.

Theorem 1 *Suppose that $h < \frac{a-b}{a}$, $c > d$, $a > b$ and $-a^2 + bc + ab - ad < 0$. Then the semi-trivial periodic solution $(\xi(t), 0)$ of system* (2.9) *is stable if the following condition holds*:

$$0 < q < q^*, \tag{2.12}$$

where $q^* = -1 + (1-p)^{\frac{c-d}{a-b}}\left(\frac{a-b-ah}{a-b-a(1-p)h}\right)^{\frac{-a^2+ab+bc-ad}{a(a-b)}}$.

Proof Two different ways are introduced to prove the theorem. One is to use Lemma 1 given by Simeonov and Bainov. Another one is to employ the Poincaré map shown in Section 2.1.

(Method I) See that

$$P(x, y) = ax(1-x) - bx, \qquad Q(x, y) = cy(1-x-y) - dy - axy,$$

$$\alpha(x,y) = -px, \qquad \beta(x,y) = qy, \qquad \varphi(x,y) = x - h,$$
$$\big(\xi(T), \eta(T)\big) = (h,0), \qquad \big(\xi\big(T^+\big), \eta\big(T^+\big)\big) = \big((1-p)h, 0\big).$$

Then, by virtue of Lemma 1, we can get

$$\begin{aligned}
&\frac{\partial P}{\partial x} = a - b - 2ax, \qquad \frac{\partial Q}{\partial y} = c - d - (a+c)x - 2cy,\\
&\frac{\partial \alpha}{\partial x} = -P, \qquad \frac{\partial \alpha}{\partial y} = 0, \qquad \frac{\partial \beta}{\partial x} = 0,\\
&\frac{\partial \beta}{\partial y} = q, \qquad \frac{\partial \varphi}{\partial x} = 1, \qquad \frac{\partial \varphi}{\partial y} = 0,\\
&\Delta_1 = \frac{P_+(\frac{\partial \beta}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial \beta}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial x}) + Q_+(\frac{\partial \alpha}{\partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y})}{P\frac{\partial \varphi}{\partial x} + Q\frac{\partial \varphi}{\partial y}}\\
&\quad\; = \frac{P_+(\xi(T^+), \eta(T^+))(1+q)}{P(\xi(T), \eta(T))} = \left(\frac{a - b - ah + aph}{a - b - ah}\right)(1-p)(1+q).
\end{aligned}$$

Note that $h < \frac{a-b}{a}$ implies $a - b - ax$ stays positive on the interval $[(1-p)h, h]$. Then we compute directly to get

$$\begin{aligned}
\theta &= \exp\left(\int_0^T \left(\frac{\partial P}{\partial x}\big(\xi(t), \eta(t)\big) + \frac{\partial Q}{\partial y}\big(\xi(t), \eta(t)\big)\right) dt\right)\\
&= \exp\left(\int_0^T (a - b - 2ax + c - d - ax - cx - 2cy)\, dt\right)\\
&= \exp\left(\int_0^T \big((a - b + c - d) - (3a + c)x\big)\, dt\right)\\
&= \exp\big((a - b + c - d)T\big) \exp\left(-(3a + c)\int_{(1-p)h}^h \frac{x\, dx}{(a-b)x - ax^2}\right)\\
&= e^{(a-b+c-d)T} \exp\left(\frac{3a + c}{a}\int_{(1-p)h}^h \frac{-a\, dx}{-ax + a - b}\right)\\
&= \left(\frac{1}{1-p}\right)^{1+\frac{c-d}{a-b}} \left(\frac{a - b - a(1-p)h}{a - b - ah}\right)^{\frac{-2a^2+2ab-ad+bc}{a(a-b)}}.
\end{aligned}$$

Thus the Floquet multiplier μ_2 can be calculated directly as follows:

$$\mu_2 = \Delta_1\theta = \left(\frac{1}{1-p}\right)^{\frac{c-d}{a-b}} \left(\frac{a - b - a(1-p)h}{a - b - ah}\right)^{\frac{-a^2+ab-ad+bc}{a(a-b)}} (1+q).$$

It is easy to see that $|\mu_2| < 1$ if and only if (2.12) holds. That is the proof of Theorem 1.
(Method II) See that $\hat{P}'(0) = (1+q)g'(0)$ and

$$\begin{aligned}
g'(0) &= \exp\left\{\int_{(1-p)h}^h \frac{\partial}{\partial y}\left(\frac{Q(s, y(s,u))}{P(s, y(s,u))}\right) ds\right\} = \exp\left\{\int_{(1-p)h}^h \frac{c - d - (a+c)s}{(a-b)s - as^2}\, ds\right\}\\
&= \exp\left\{\int_{(1-p)h}^h \frac{c - d}{(a-b)s - as^2} - \frac{a + c}{a - b - as}\, ds\right\}
\end{aligned}$$

$$= \exp\left\{\frac{a^2-ab-bc+ad}{a(a-b)}\ln\left(\frac{a-b-ah}{a-b-a(1-p)h}\right)+\frac{c-d}{a-b}\ln\left(\frac{1}{1-p}\right)\right\}$$
$$= \left(\frac{1}{1-p}\right)^{\frac{c-d}{a-b}}\left(\frac{a-b-a(1-p)h}{a-b-ah}\right)^{\frac{-a^2+ab-ad+bc}{a(a-b)}}.$$

Thus, $\hat{P}'(0) < 1$ if and only if

$$\left(\frac{1}{1-p}\right)^{\frac{c-d}{a-b}}\left(\frac{a-b-a(1-p)h}{a-b-ah}\right)^{\frac{-a^2+ab-ad+bc}{a(a-b)}}(1+q) < 1.$$

Then the conclusion follows by Lemma 4. The proof is therefore completed. □

Remarks (1) It follows from the proof that our construction of the Poincaré map is proper. (2) It is clear that a bifurcation may occur at $q = q^*$ for $|\mu_2| = 1$. As a result, a positive periodic solution may appear when $q > q^*$.

We study the problem of the bifurcation of a nontrivial periodic solution of system (2.9) near the semi-trivial one $(\xi(t), 0)$. Consider the Poincaré map in (2.5) with $\tau = 0$. We have

$$\hat{P}(y,q,0) \equiv (1+q)g(y) := F(y,q). \tag{2.13}$$

In order to discuss the bifurcation of map (2.13), we will use Lemma 2. To apply this lemma, for $F(u,q) = (1+q)g(u)$, we need to compute $\frac{\partial F}{\partial x}(0,q^*)$, $\frac{\partial^2 F}{\partial x\partial\mu}(0,q^*)$ and $\frac{\partial^2 F}{\partial x^2}(0,q^*)$.

Clearly, we can compute $g'(0)$ and $g''(0)$ as follows:

$$g'(0) = \left(\frac{1}{1-p}\right)^{\frac{c-d}{a-b}}\left(\frac{a-b-a(1-p)h}{a-b-ah}\right)^{\frac{-a^2+ab-ad+bc}{a(a-b)}}.$$

Taking into account the range of p and h, we know $g'(0) > 0$. Furthermore, we have

$$g''(0) = g'(0)\int_{(1-p)h}^{x}\frac{\partial}{\partial y}\left(\frac{-2cy+c-d-(a+c)s}{(a-b)s-as^2}\right)\frac{\partial y(s,0)}{\partial u}\,ds$$
$$= g'(0)\int_{(1-p)h}^{x}\frac{-2c}{(a-b)s-as^2}\frac{\partial y(s,0)}{\partial u}\,ds.$$

It follows by (2.5) that $g''(0) < 0$. By Lemma 2, we know a transcritical bifurcation occurs whenever $q = q^*$. Therefore, a stable positive fixed point appears when the parameter q changes through q^* from left to right.

Theorem 2 *Suppose that $h < \frac{a-b}{a}$, $c > d$, $a > b$ and $-a^2+bc+ab-ad < 0$. System (2.9) has a stable positive periodic solution if $q \in (q^*, q^*+\varepsilon)$ with $\varepsilon > 0$, where $q^* = -1 + (1-p)^{\frac{c-d}{a-b}}(\frac{a-b-ah}{a-b-a(1-p)h})^{\frac{-a^2+ab+bc-ad}{a(a-b)}}$.*

Proof It is easy to verify that

$$F(0,q) = g(0) = 0, \quad q \in (0,+\infty),$$
$$\frac{\partial F(0,q)}{\partial u} = (1+q)g'(0) = (1+q)\left(\frac{1}{1-p}\right)^{\frac{c-d}{a-b}}\left(\frac{a-b-a(1-p)h}{a-b-ah}\right)^{\frac{-a^2+ab-ad+bc}{a(a-b)}},$$

which yields $\frac{\partial F(0,q^*)}{\partial u} = 1$. This means that $(0,q^*)$ is a fixed point with the eigenvalue 1 of map (2.13).

Then

$$\frac{\partial^2 F(0,q^*)}{\partial u\,\partial q} = g'(0) > 0.$$

Finally,

$$\frac{\partial^2 F(0,q^*)}{\partial u^2} = \left(1+q^*\right)g''(0) < 0.$$

Then all the conditions in Lemma 2 are satisfied and so the proof of Theorem 2 is completed. □

Case II: $c > d$, $a > b$, $-a^2 + bc + ab - ad = 0$.

In this case, for any $h > 0$, $0 < p < 1$, we find that

$$\left(\frac{1}{1-p}\right)^{\frac{c-d}{a-b}}\left(\frac{a-b-a(1-p)h}{a-b-ah}\right)^{\frac{-a^2+ab-ad+bc}{a(a-b)}}(1+q) = \left(\frac{1}{1-p}\right)^{\frac{c-d}{a-b}}(1+q) > 1$$

is always true. It follows from Lemma 4 that the semi-trivial solution is unstable. This means there exists $\bar{y}$, $0 < \bar{y} \le \delta \ll 1$, such that the Poincaré map

$$F(\bar{y},q) \equiv (1+q)g(\bar{y}) > \bar{y}. \tag{2.14}$$

On the other hand, we notice that $Q(x,y) = -cy^2 + [c-d-(a+c)x]y \to -\infty$ as $y \to +\infty$ and $P(x,y) = ax(1-x)$ is bounded by $\lambda < P(x,y) < \frac{(a-b)^2}{4a}$, where $\lambda = \min\{(a-b)(1-p)h - a(1-p)^2h^2, (a-b)h - ah^2\}$. It follows that there must be $\bar{y}$ such that

$$\frac{Q(x,y)}{P(x,y)} + \frac{q\bar{y}}{ph} < 0, \quad \forall y \ge \bar{y}, (1-p)h \le x \le h.$$

We show that the y-coordinate of the solution starting from point $((1-p)h, (1+q)\bar{y})$ cannot stay above $\hat{y}$ for $x \in [(1-p)h, h]$. Otherwise

$$\begin{aligned} y(h) - y\big((1-p)h\big) = y(h) - (1+q)\bar{y} &= \int_{(1-p)h}^{h} \frac{Q(x,y)}{P(x,y)}\,dx \\ &< -\int_{(1-p)h}^{h} \frac{q\bar{y}}{ph}\,dx < -q\bar{y}. \end{aligned}$$

This implies that

$$y(h) < \bar{y},$$

which is a contradiction. Therefore there exists $\bar{x}$ such that $y(\bar{x}) = \bar{y}$ and $y(x) < \bar{y}$, $\bar{x} < x \le h$. Thus, $g((1+q)\bar{y}) < \bar{y}$, and so $((1+q)g((1+q)\bar{y})) < (1+q)\bar{y}$. This is equivalent to saying that the Poincaré map satisfies

$$F\big((1+q)\bar{y},q\big) \equiv (1+q)g\big((1+q)\bar{y}\big) < (1+q)\bar{y}.$$

To sum up, we have for $F(u,q)$ that

(A) $F(\delta,q) > 0$, $F((1+q)\bar{y},q) < (1+q)\bar{y}$, $\bar{y} > 0$, $0 < \delta \ll 1$;
(B) $F''(u,q) < 0$, $\forall 0 < u < +\infty$;
(C) $F'(0,q) > 1$.

If follows from (A) that $F(u,q)$ must have a fixed point. Denote $y^* = \inf\{y | y > \delta, F(y,q) = y\}$. We firstly show that $F'(y^*,q) < 1$. Otherwise, by (B) and (C) we have that

$$F'(u,q) \geq 1, \quad 0 \leq u \leq y^*.$$

This implies by the Lagrange mean value theorem that

$$F\big(y^*,q\big) = F(0,q) + F'(\theta)\big(y^* - 0\big) > y^*,$$

where $\theta \in (0,y^*)$. This is a contradiction and so $F'(y^*) < 1$. We next prove that y^* is the unique fixed point. Otherwise, we have $y^{**} > y^*$, $F(y^{**},q) = y^{**}$, and also by (B) there holds

$$F'(u,q) < 1, y^* \leq u \leq y^{**}. \tag{2.15}$$

Then, by using the Lagrange mean value theorem again, we have that there exists ϑ, $y^* < \vartheta < y^{**}$ such that

$$F'(\vartheta,q) = \frac{F(y^{**},q) - F(y^*,q)}{y^{**} - y^*} = 1.$$

This contradicts (2.15). To conclude, F has a unique positive fixed point y^* and $F'(y^*) < 1$. Thus, we obtain the following theorem.

Theorem 3 *Suppose that $h < \frac{a-b}{a}$, $c > d$, $a > b$ and $-a^2 + bc + ab - ad = 0$. Then the semi-trivial solution of system* (2.9) *is unstable. Moreover, system* (2.9) *has a unique positive periodic solution and it is stable.*

2.3 The case of $\tau > 0$

In this section, we discuss the existence of a positive periodic solution with $\tau > 0$ by using Poincaré map (2.5). In the case of $\tau > 0$, it is obvious that system (1.4) has no semi-trivial solution. Moreover, we see the fact that $\tilde{P}(0,q,\tau) = \tau > 0$. Next we try to find $\tilde{y} > 0$ such that $\tilde{P}(\tilde{y},q,\tau) < \tilde{y}$.

We notice that $Q(x,y) = -cy^2 + [c-d-(a+c)x]y \to -\infty$ as $y \to +\infty$ and $P(x,y) = ax(1-x)$ is bounded by $\lambda < P(x,y) < \frac{(a-b)^2}{4a}$, where $\lambda = \min\{(a-b)(1-p)h - a(1-p)^2h^2, (a-b)h - ah^2\}$. It follows that there must be $\hat{y}$ such that

$$\frac{Q(x,y)}{P(x,y)} + \frac{q\hat{y} + \tau}{ph} < 0, \quad \forall y \geq \hat{y}, (1-p)h \leq x \leq h.$$

We next show that the y-coordinate of the solution starting from point $((1-p)h,(1+q)\hat{y}+\tau)$ cannot stay above $\hat{y}$ for $x \in [(1-p)h, h]$. Otherwise

$$y(h) - y\big((1-p)h\big) = y(h) - (1+q)\hat{y} - \tau = \int_{(1-p)h}^{h} \frac{Q(x,y)}{P(x,y)}\,dx - \tau$$
$$< -\int_{(1-p)h}^{h} \frac{q\hat{y}+\tau}{ph}\,dx - \tau < q\hat{y}.$$

This implies that

$$y(h) < \hat{y},$$

which is a contradiction. Consequently,

$$\tilde{P}\big((1+q)\hat{y}, q, \tau\big) \equiv (1+q)g\big((1+q)\hat{y}\big) < (1+q)\hat{y}.$$

To sum up, we have for $\tilde{P}(u,q,\tau)$ that

($\tilde{\mathrm{A}}$) $\tilde{P}(0,q,\tau) > 0$, $\tilde{P}((1+q)\hat{y},q,\tau) < (1+q)\hat{y}$, $\hat{y} > 0$;
($\tilde{\mathrm{B}}$) $\tilde{P}(u,q,\tau)'' < 0$, $\forall 0 < u < +\infty$;
($\tilde{\mathrm{C}}$) $\tilde{P}(u,q,\tau)' > 1$.

Deducing similarly as in the proof of Theorem 3, we have the following theorem.

Theorem 4 *Suppose that* $h < \frac{a-b}{a}$, $c > d$, $a > b$ *and* $-a^2 + bc + ab - ad \leq 0$. *Then the semi-trivial solution of system* (1.4) *has a unique positive periodic solution and this solution is stable*.

3 Conclusion

In this paper, we study the dynamical behaviors of a two-competitive metapopulation system with impulsive control. To show the stability of the semi-trivial periodic solution, we use two different methods: one is the theorem developed by Simeonov and Bainov, and the other one is the method of Poincaré map. The first method can be used only when the explicit expression of the solution is given. Undoubtedly, this restriction narrows its application. So when it comes to the stability of a positive periodic solution, we have to use the Poincaré map together with its particular geometric properties with respect to our specific system. Also, in the case of $h < \frac{a-b}{a}$, $c > d$, $a > b$ and $-a^2 + bc + ab - ad < 0$, we find the positive periodic solution and obtain its stability through a transcritical bifurcation in which q is taken as a bifurcation parameter.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ST carried out the main results of this paper and drafted the manuscript. YX directed the study and helped to inspect the manuscript. TM helped to revise the manuscript. All authors read and approved the final manuscript.

Acknowledgements
This work was supported by NNSF of China No. 11431008 and NNSF of China No.11271261.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 June 2016 Accepted: 31 March 2017 Published online: 02 June 2017

References

1. Barbour, AD, Luczak, MJ: Individual and patch behaviour in structured metapopulation models. J. Math. Biol. **71**(3), 713-733 (2015)
2. Chen, SL: Mathematical Models and Methods in Ecology. Science Press, Beijing (1988)
3. Edwards, PJ, Webb, NR, May, RM: Large Scale Ecology and Conservation Biology. Blackwell Sci., Oxford (1994)
4. Gui, JZ: Biological Model and Computer Dynamic Simulation. Science Press, Beijing (2005)
5. Hanski, I: Metapopulation Ecology. Oxford University Press, Oxford (1999)
6. Hui, C, Li, Z, Yue, D: Metapopulation dynamics and distribution, and environmental heterogeneity induced by niche construction. Ecol. Model. **177**(1-2), 107-118 (2004)
7. Iggidr, A, Sallet, G, Tsanou, B: Global stability analysis of a metapopulation SIS epidemic model. Math. Popul. Stud. **19**(3), 115-129 (2012)
8. Levins, R: Some demographic and genetic consequences of environmental heterogeneity for biological control. Bull. Entomol. Soc. Am. **15**, 237-240 (1969)
9. Lumi, N, Laas, K, Mankin, R: Rising relative fluctuation as a warning indicator of discontinuous transitions in symbiotic metapopulations. Physica, A **437**, 109-118 (2015)
10. Ngoc Doanh, N, de la Parra, RB, Zavala, MA, Auger, P: Competition and species coexistence in a metapopulation model: can fast asymmetric migration reverse the outcome of competition in a homogeneous environment? J. Theor. Biol. **266**(2), 256-263 (2010)
11. Salinas, RA, Lenhart, S, Gross, LJ: Control of a metapopulation harvesting model for black bears. Nat. Resour. Model. **18**(3), 307-321 (2005)
12. Sanchirico, JN, Wilen, JE: Dynamics of spatial exploitation: a metapopulation approach. Nat. Resour. Model. **14**(3), 391-418 (2001)
13. Sardanyés, J, Fontich, E: On the metapopulation dynamics of autocatalysis: extinction transients related to ghosts. Int. J. Bifurc. Chaos Appl. Sci. Eng. **20**(4), 1261-1268 (2010)
14. Terry, AJ: Pulse vaccination strategies in a metapopulation SIR model. Math. Biosci. Eng. **7**(2), 455-477 (2010)
15. Tilman, D: Competition and biodiversity in spatially structured habitats. Ecology **75**(1), 2-16 (1994)
16. Guo, H, Chen, L, Song, X: Qualitative analysis of impulsive state feedback control to an algae-fish system with bistable property. Appl. Math. Comput. **271**, 905-922 (2015)
17. Guo, H, Chen, L, Song, X: Geometric properties of solution of a cylindrical dynamic system with impulsive state feedback control. Nonlinear Anal. Hybrid Syst. **15**, 98-111 (2015)
18. Jiang, G, Lu, Q: Impulsive state feedback control of a predator-prey model. J. Comput. Appl. Math. **200**(1), 193-207 (2007)
19. Li, Z, Zhao, Z, Chen, L: Bifurcation of a three molecular saturated reaction with impulsive input. Nonlinear Anal., Real World Appl. **12**(4), 2016-2030 (2011)
20. Pang, G, Chen, L, Xu, W, Fu, G: A stage structure pest management model with impulsive state feedback control. Commun. Nonlinear Sci. Numer. Simul. **23**(1-3), 78-88 (2015)
21. Qian, L, Lu, Q, Meng, Q, Feng, Z: Dynamical behaviors of a prey-predator system with impulsive control. J. Math. Anal. Appl. **363**(1), 345-356 (2010)
22. Sun, M, Liu, Y, Liu, S, Hu, Z, Chen, L: A novel method for analyzing the stability of periodic solution of impulsive state feedback model. Appl. Math. Comput. **273**, 425-434 (2016)
23. Wang, T, Chen, C, Chen, L: Periodic solution of a microbial pesticide model with the monod growth rate and impulsive state feedback control. J. Biomath. **28**(4), 577-585 (2013)
24. Wei, C, Chen, L: Homoclinic bifurcation of prey-predator model with impulsive state feedback control. Appl. Math. Comput. **237**, 282-292 (2014)
25. Xu, W, Chen, L, Chen, S, Pang, G: An impulsive state feedback control model for releasing white-headed langurs in captive to the wild. Commun. Nonlinear Sci. Numer. Simul. **34**, 199-209 (2016)
26. Simeonov, PE, Bainov, DD: Orbital stability of periodic solutions of autonomous systems with impulse effect. Int. J. Syst. Sci. **19**, 2562-2585 (1988)
27. Rasband, SN: Chaotic Dynamics of Nonlinear Systems. Wiley, New York (1990)