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Existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations

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Abstract

In the article, the existence and uniqueness of positive solutions for a class of fractional differential equation with nonlinear boundary conditions are discussed. By applying some fixed point theorems on cone, we gain a unique positive solution and construct two iterative sequences to approximate the solution. Moreover, as applications of our main results, some examples are also presented.

Keywords: fractional differential equations; fixed point theorem; mixed monotone operator; positive solution

1 Introduction

As is well known, fractional differential equations has been widely applied to various fields of physical, engineering, biology, chemistry, etc.; see [1–6]. Recently, more and more authors have paid attention to the research on the solutions for fractional differential equations. There are many great results as regards positive solutions of boundary value problems (see [7–19]). For example, Zhai, Yan and Yang [7] studied the existence and uniqueness of positive solutions to fractional differential equation boundary value problems. In [10], Yuan was concerned with the $(n - 1, 1)$ -type semipositone conjugate boundary value problems. Yang [14] and the authors [17] considered nonlinear fractional differential equations with integral boundary conditions. In [15], the authors were interested in the nonlinear fractional differential equation with a derivative term.

In this paper, we study the nonlinear fractional differential equation given by

$$\begin{cases} D_{0+}^{\nu}x(t) + f(t, x(t), D_{0+}^{\gamma}x(t)) + g(t, x(t)) = 0, & t \in (0, 1), n - 1 < \nu < n; \\ x^{(i)}(0) = 0, & i = 0, 1, 2, 3, \dots, n - 2; \\ [D_{0+}^{\beta}x(t)]_{t=1} = k(x(1)), \end{cases} \quad (1.1)$$

where $n > 3$, $1 \leq \gamma \leq \beta \leq n - 2$, $f : [0, 1] \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$, $g : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$, and $k : [0, \infty) \rightarrow [0, \infty)$ are continuous functions, D_{0+}^{ν} stands for the Riemann-Liouville fractional derivative of order ν , and $x^{(i)}$ represents the i th (ordinary) derivative of x .

When $k(x) = 0$ and $f(t, x, D_{0^+}^\gamma x(t)) = 0$ in (1.1), the problem (1.1) is changed into the form of the following fractional differential equation considered in [8]:

$$\begin{cases} D_{0^+}^\nu x(t) + g(t, x(t)) = 0, & t \in (0, 1), n - 1 < \nu \leq n; \\ x^{(i)}(0) = 0, & i = 0, 1, 2, 3, \dots, n - 2; \\ [D_{0^+}^\beta x(t)]_{t=1} = 0, & 1 \leq \beta \leq n - 2. \end{cases} \tag{1.2}$$

By means of the Krasnosel'skii fixed point theorem in cones, Goodrich only came to the conclusion that problem (1.2) has at least one positive solution. Meanwhile, he did not mention the uniqueness.

When $k(x) = 0$ and $\gamma = 0$ in (1.1), Mohamed Jleli and Bessem Samet [9] considered the following problem:

$$\begin{cases} D_{0^+}^\nu x(t) + f(t, x(t), x(t)) + g(t, x(t)) = 0, & t \in (0, 1), n - 1 < \nu \leq n; \\ x^{(i)}(0) = 0, & i = 0, 1, 2, 3, \dots, n - 2; \\ [D_{0^+}^\beta x(t)]_{t=1} = 0, & 2 \leq \beta \leq n - 2. \end{cases} \tag{1.3}$$

They investigated a unique positive solution of problem (1.3) by using a fixed point theorem of a mixed monotone operator.

Compared with [8] and [9], we not only generalize the boundary conditions but also consider the derivative term. As far as we know, few papers can be found in the literature on the existence and uniqueness of positive solutions for the (1.1), especially the analysis relying on fixed point theorems of a sum operator. Our main contribution in this paper is to give some distinct methods to obtain the main results of [8, 9]. By some fixed theorems of a sum operator on cone, we establish a unique solution and set up two iterative schemes to approximate the solution.

The construction of this paper is displayed as follows. In Section 2, we recall some basic definitions and notations and several useful lemmas. Section 3 contains the existence and uniqueness of solution for nonlinear fractional differential equations (1.1), based on some fixed point theorems. In Section 4, we give some examples to illustrate our main results.

2 Preliminaries

For convenience, we present some basic definitions, lemmas and preliminary results that will be used in this manuscript.

Definition 2.1 ([3]) The Riemann-Liouville fractional derivative of order ($p > 0$) of a function $h \in C[0, 1]$ is given by

$$D_{0^+}^p h(t) = \frac{1}{\Gamma(n - p)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{h(s)}{(t - s)^{p - n + 1}} ds.$$

The Riemann-Liouville fractional integral of order ($q > 0$) of h is given by

$$I_{0^+}^p h(t) = \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p - 1} h(s) ds,$$

where $n = [p] + 1$, $[p]$ denotes the integer part of number p , provided that the right side is pointwise defined on $(0, 1)$.

Lemma 2.1 ([3]) *Assume $h \in C[0, 1]$, $q \geq p \geq 0$, then*

$$D_{0+}^p I_{0+}^q h(t) = I_{0+}^{q-p} h(t).$$

Lemma 2.2 ([3]) *Let $h \in C[0, 1] \cap L^1[0, 1]$, $p > 0$, then*

$$I_{0+}^p D_{0+}^p h(t) = c_1 t^{p-1} + c_2 t^{p-2} + \dots + c_n t^{p-n},$$

where $c_i \in R$, $i = 1, 2, 3, \dots, n$ ($n = [p] + 1$).

Lemma 2.3 *Let $h \in C[0, 1]$, then the unique solution of the linear problem*

$$D_{0+}^\nu x(t) + h(t) = 0, \quad t \in (0, 1), n - 1 < \nu \leq n; \tag{2.1}$$

$$x^i(0) = 0, \quad i = 0, 1, 2, 3, \dots, n - 2; \tag{2.2}$$

$$[D_{0+}^\beta x(t)]_{t=1} = k(x(1)), \quad 1 \leq \beta \leq n - 2, \tag{2.3}$$

is given by

$$x(t) = \int_0^1 G(t, s)h(s) ds + \frac{\Gamma(\nu - \beta)}{\Gamma(\nu)} k(x(1))t^{\nu-1},$$

where

$$G(t, s) = \begin{cases} \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}-(t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \leq s \leq t \leq 1; \\ \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu)}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.4}$$

is the Green's function.

Proof In view of Lemma 2.2 and (2.1), we have

$$x(t) = -I_{0+}^\nu h(t) + c_1 t^{\nu-1} + c_2 t^{\nu-2} + \dots + c_n t^{\nu-n}.$$

The boundary condition (2.2) implies that $c_n = c_{n-1} = c_{n-2} = \dots = c_2 = 0$. Thus, the solution of (2.1) is equivalent to the following one:

$$x(t) = -I_{0+}^\nu h(t) + c_1 t^{\nu-1}. \tag{2.5}$$

Using the equality $D_{0+}^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$ and Lemma 2.1, (2.5) reduces to

$$\begin{aligned} D_{0+}^\beta x(t) &= -I_{0+}^{\nu-\beta} h(t) + \frac{\Gamma(\nu)}{\Gamma(\nu - \beta)} t^{\nu-\beta-1} c_1 \\ &= - \int_0^t \frac{(t-s)^{\nu-\beta-1}}{\Gamma(\nu - \beta)} h(s) ds + \frac{\Gamma(\nu)}{\Gamma(\nu - \beta)} t^{\nu-\beta-1} c_1. \end{aligned} \tag{2.6}$$

From (2.3) and (2.6), then we conclude

$$- \int_0^1 \frac{(1-s)^{\nu-\beta-1}}{\Gamma(\nu - \beta)} h(s) ds + \frac{\Gamma(\nu)}{\Gamma(\nu - \beta)} c_1 = k(x(1)). \tag{2.7}$$

Equation (2.7) may be simplified to get

$$c_1 = \frac{\Gamma(v - \beta)}{\Gamma(v)}k(x(1)) + \int_0^1 \frac{(1 - s)^{v-\beta-1}}{\Gamma(v)}h(s) ds. \tag{2.8}$$

Finally, plugging (2.8) into (2.5), the unique solution of problem (2.1), (2.2), (2.3) is given by

$$\begin{aligned} x(t) &= -I_{0^+}^v h(t) + t^{v-1} \left\{ \frac{\Gamma(v - \beta)}{\Gamma(v)}k(x(1)) + \int_0^1 \frac{(1 - s)^{v-\beta-1}}{\Gamma(v)}h(s) ds \right\} \\ &= - \int_0^t \frac{(t - s)^{v-1}}{\Gamma(v)}h(s) ds + \frac{\Gamma(v - \beta)}{\Gamma(v)}k(x(1))t^{v-1} + \int_0^1 \frac{(1 - s)^{v-\beta-1}t^{v-1}}{\Gamma(v)}h(s) ds \\ &= \int_0^1 G(t, s)h(s) ds + \frac{\Gamma(v - \beta)}{\Gamma(v)}k(x(1))t^{v-1}, \end{aligned}$$

where $G(t, s)$ defined as (2.4). □

Lemma 2.4 *The Green's function (2.4) has the following properties:*

$$0 \leq t^{v-1}(1 - s)^{v-\beta-1}[1 - (1 - s)^\beta] \leq \Gamma(v)G(t, s) \leq t^{v-1}(1 - s)^{v-\beta-1}, \quad t, s \in [0, 1], \tag{2.9}$$

$$\begin{aligned} 0 &\leq t^{v-\gamma-1}(1 - s)^{v-\beta-1}[1 - (1 - s)^{\beta-\gamma}] \leq \Gamma(v - \gamma)D_{0^+}^\gamma G(t, s) \\ &\leq t^{v-\gamma-1}(1 - s)^{v-\beta-1}, \quad t, s \in [0, 1]. \end{aligned} \tag{2.10}$$

Proof The first inequality (2.9) has been proved by [9]. Now we prove the inequality (2.10). From (2.4), we get

$$D_{0^+}^\gamma G(t, s) = \begin{cases} \frac{t^{v-\gamma-1}(1-s)^{v-\beta-1}-(t-s)^{v-\gamma-1}}{\Gamma(v-\gamma)}, & 0 \leq s \leq t \leq 1; \\ \frac{t^{v-\gamma-1}(1-s)^{v-\beta-1}}{\Gamma(v-\gamma)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.11}$$

We can find that the right term in (2.10) is clear. So we only prove the right term. If $0 \leq t \leq s \leq 1$, since $\beta \geq \gamma$, then

$$\begin{aligned} \Gamma(v - \gamma)G(t, s) &= t^{v-\gamma-1}(1 - s)^{v-\beta-1} \\ &\geq t^{v-\gamma-1}(1 - s)^{v-\beta-1}[1 - (1 - s)^{\beta-\gamma}]. \end{aligned}$$

If $0 \leq s \leq t \leq 1$, since $n - 1 < v \leq n$ with $n > 3$ and $1 \leq \gamma \leq n - 2$, then we get $t - s \leq t - ts = (1 - s)t$, which implies that

$$(t - s)^{v-\gamma-1} \leq (1 - s)^{v-\gamma-1}t^{v-\gamma-1}.$$

Then

$$\begin{aligned} \Gamma(v - \gamma)G(t, s) &= t^{v-\gamma-1}(1 - s)^{v-\beta-1} - (t - s)^{v-\gamma-1} \\ &\geq t^{v-\gamma-1}(1 - s)^{v-\beta-1} - (1 - s)^{v-\gamma-1}t^{v-\gamma-1} \\ &= t^{v-\gamma-1}(1 - s)^{v-\beta-1}[1 - (1 - s)^{\beta-\gamma}]. \end{aligned}$$

Therefore equation (2.10) is demonstrated. □

In the sequel, we state here some basic concepts in ordered Banach spaces and some fixed point theorems, which will be used later.

Suppose that $(E, \|\cdot\|)$ is a real Banach space, which is partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E . Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies: (1) $x \in P, r > 0 \Rightarrow rx \in P$; (2) $x \in P, -x \in P \Rightarrow x = \theta$.

P is called normal if $\exists N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. In this case, the smallest constant satisfying this inequality is called the normality constant of P . For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0, \mu > 0$ such that $\lambda y \leq x \leq \mu y$. Clearly, \sim is an equivalence relation. Given $h > \theta$, we denote by P_h the set $P_h = \{x \in E | x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.2 An operator $A : P \rightarrow P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tAx, \quad \forall t \in (0, 1), x \in P. \tag{2.12}$$

Definition 2.3 Let $D = P$ or $D = \dot{P}$ and α be a real number with $0 \leq \alpha < 1$. An operator $A : D \rightarrow D$ is said to be α -concave if it satisfies

$$A(tx) \geq t^\alpha Ax, \quad \forall t \in (0, 1), x \in D. \tag{2.13}$$

Definition 2.4 ([8]) An operator $A : P \times P \rightarrow P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in first component and decreasing in second component, i.e., $u_i, v_i (i = 1, 2) \in P, u_1 \leq u_2, v_1 \geq v_2$, implies $A(u_1, v_1) \leq A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

In [20], the operator equation $Ax + Bx + C(x, x) = x$ is considered. Here, the operator $A : P \rightarrow P$ is an increasing, $B : P \rightarrow P$ is a decreasing operator, $C : P \times P \rightarrow P$ is a mixed monotone operator. Then Wang and Zhang get the following main results.

Lemma 2.5 Let $\alpha \in (0, 1)$ and P is a normal cone of E . Assume that A is sub-homogeneous and B, C satisfy the following conditions:

$$B(t^{-1}y) \geq tB(y), \quad C(tx, t^{-1}y) \geq t^\alpha C(x, y), \quad \forall t \in (0, 1), x, y \in P. \tag{2.14}$$

Suppose that

- (A₁) there exists $h_0 \in P_h$ such that $A(h_0) \in P_h, B(h_0) \in P_h, C(h_0, h_0) \in P_h$;
- (A₂) there exists a constant $\delta > 0$ such that $C(x, y) \geq \delta(Ax + By), \forall x, y \in P$.

Then

- (1) $A : P_h \rightarrow P_h, B : P_h \rightarrow P_h, C : P_h \times P_h \rightarrow P_h$;
- (2) there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that

$$rv_0 \leq u_0 < v_0, \quad u_0 \leq Au_0 + Bv_0 + C(u_0, v_0) \leq Av_0 + Bu_0 + C(v_0, u_0) \leq v_0;$$

- (3) the operator equation $Ax + Bx + C(x, x) = x$ has a unique solution x^* in P_h ;

(4) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{aligned} x_n &= Ax_{n-1} + By_{n-1} + C(x_{n-1}, y_{n-1}), \quad n = 1, 2, \dots, \\ y_n &= Ay_{n-1} + Bx_{n-1} + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots, \end{aligned}$$

we have $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$.

Lemma 2.6 Let $\alpha \in (0, 1)$ and P is a normal cone of E . Assume that A sub-homogeneous operator, and the following inequalities hold for B, C :

$$B(t^{-1}y) \geq t^\alpha B(y), \quad C(tx, t^{-1}y) \geq tC(x, y), \quad \forall t \in (0, 1), x, y \in P.$$

Suppose that

- (A'_1) there exists $h_0 \in P_h$ such that $A(h_0) \in P_h, B(h_0) \in P_h, C(h_0, h_0) \in P_h$;
- (A'_2) there exists a constant $\delta > 0$ such that $Ax + C(x, y) \leq \delta By, \forall x, y \in P$.

Then the conclusions (1)-(4) in Lemma 2.5 hold.

Lemma 2.7 Let $\alpha \in (0, 1)$ and P is a normal cone of E . Assume that A is a α -concave operator, the operators B, C satisfy

$$B(t^{-1}y) \geq tB(y), \quad C(tx, t^{-1}y) \geq tC(x, y), \quad \forall t \in (0, 1), x, y \in P.$$

Suppose that

- (A''_1) there exists $h_0 \in P_h$ such that $A(h_0) \in P_h, B(h_0) \in P_h, C(h_0, h_0) \in P_h$;
- (A''_2) there exists a constant $\delta > 0$ such that $By + C(x, y) \leq \delta Ax, \forall x, y \in P$.

Then the conclusions (1)-(4) in Lemma 2.5 hold.

3 Main results

In the section, we use Lemmas 2.5, 2.6, 2.7 to obtain the existence and uniqueness of positive solutions for the problem (1.1).

Set $E = \{x|x \in C[0, 1], D_{0^+}^\nu x \in C[0, 1]\}$ is a Banach space with the norm

$$\|x(t)\| = \max \left\{ \max_{t \in [0, 1]} |x(t)|, \max_{t \in [0, 1]} D_{0^+}^\nu |x(t)| \right\}.$$

E is endowed with an order relation $u \leq v$ if $u(t) \leq v(t), D_{0^+}^\nu u(t) \leq D_{0^+}^\nu v(t)$. Moreover, let $P \subset E$ be defined by

$$P = \{x \in E : x(t) \geq 0, D_{0^+}^\nu x(t) \geq 0, \forall t \in [0, 1]\}.$$

Clearly, P is a normal cone and $P_h \subset E$.

By Lemma 2.3, we know that the unique solution for the problem (1.1) has an integral formulation:

$$x(t) = \int_0^1 G(t, s)f(s, x(s), D_{0^+}^\nu x(s)) ds + \int_0^1 G(t, s)g(s, x(s)) ds + \frac{\Gamma(\nu - \beta)}{\Gamma(\nu)} k(x(1))t^{\nu-1},$$

where $G(t, s)$ is given by (2.4).

We define the operators A, B, C :

$$A(u)(t) = \int_0^1 G(t,s)g(s,u(s)) \, ds, \quad t \in [0,1]; \tag{3.1}$$

$$B(v)(t) = \frac{\Gamma(v-\beta)}{\Gamma(v)}k(v(1))t^{v-1}, \quad t \in [0,1]; \tag{3.2}$$

$$C(u,v)(t) = \int_0^1 G(t,s)f(s,u(s),D_{0+}^\gamma v(s)) \, ds, \quad t \in [0,1]. \tag{3.3}$$

Evidently u is the solution of problem (1.1) if and only if $u = A(u) + B(u) + C(u, u)$.

Theorem 3.1 *Assume that*

- (H₁) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \rightarrow [0,+\infty)$, $g : [0,1] \times [0,+\infty) \rightarrow [0,+\infty)$, and $k : [0,+\infty) \rightarrow [0,+\infty)$ are continuous;
- (H₂) $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, $k(y)$ is decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ with $k(y(1)) \neq 0$;
- (H₃) there exists a constant $\alpha \in (0,1)$ such that $f(t,\eta x,\eta^{-1}y) \geq t^\alpha f(t,x,y)$, $\forall \eta \in (0,1)$, $t \in [0,1]$, $x,y \in [0,+\infty)$; $g(t,x)$, $k(y)$ satisfy $g(t,\eta x) \geq \eta g(t,x)$, $k(\eta^{-1}y) \geq \eta k(y)$, $\forall \eta \in (0,1)$, $x,y \in [0,+\infty)$;
- (H₄) $g(t,0) \neq 0$ for $t \in [0,1]$ and there exist two constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $f(t,x,y) \geq \delta_1 g(t,x)$, $f(t,x,y) \geq \delta_2 \geq k(y)$, $t \in [0,1]$, $x,y \in [0,+\infty)$.

Then we have the following conclusions:

- (1) there exist $u_0, v_0 \in P_h \subset E$ and $r \in (0,1)$ such that

$$\begin{aligned}
 &rv_0 \leq u_0 < v_0, \quad \text{that is,} \quad rv_0 \leq u_0 < v_0, \quad rD_{0+}^\gamma v_0 \leq D_{0+}^\gamma u_0 < D_{0+}^\gamma v_0; \\
 &u_0(t) \leq \int_0^1 G(t,s)f(s,u_0(s),D_{0+}^\gamma v_0(s)) \, ds + \int_0^1 G(t,s)g(s,u_0(s)) \, ds \\
 &\quad + \frac{\Gamma(v-\beta)}{\Gamma(v)}k(v_0(1))t^{v-1}, \quad t \in [0,1]; \\
 &D_{0+}^\gamma u_0(t) \leq \int_0^1 D_{0+}^\gamma G(t,s)f(s,u_0(s),D_{0+}^\gamma v_0(s)) \, ds + \int_0^1 D_{0+}^\gamma G(t,s)g(s,u_0(s)) \, ds \\
 &\quad + \frac{\Gamma(v-\beta)}{\Gamma(v-\gamma)}k(v_0(1))t^{v-\gamma-1}, \quad t \in [0,1]; \\
 &v_0(t) \geq \int_0^1 G(t,s)f(s,v_0(s),D_{0+}^\gamma u_0(s)) \, ds + \int_0^1 G(t,s)g(s,v_0(s)) \, ds \\
 &\quad + \frac{\Gamma(v-\beta)}{\Gamma(v)}k(u_0(1))t^{v-1}, \quad t \in [0,1]; \\
 &D_{0+}^\gamma v_0(t) \geq \int_0^1 D_{0+}^\gamma G(t,s)f(s,v_0(s),D_{0+}^\gamma u_0(s)) \, ds + \int_0^1 D_{0+}^\gamma G(t,s)g(s,v_0(s)) \, ds \\
 &\quad + \frac{\Gamma(v-\beta)}{\Gamma(v-\gamma)}k(u_0(1))t^{v-\gamma-1}, \quad t \in [0,1];
 \end{aligned}$$

where $h(t) = t^{v-1}$, $t \in [0,1]$ and $G(t,s)$ is defined by (2.4);

- (2) the problem (1.1) has a unique positive solution x^* in P_h ;
- (3) for any initial value $x_0, y_0 \in P_h$, there are two iterative sequences $\{x_n\}, \{y_n\}$ for approximating x^* , that is, $x_n \rightarrow x^*, y_n \rightarrow x^*$, as $n \rightarrow \infty$, where

$$\begin{aligned}
 x_n(t) &= \int_0^1 G(t,s)f(s, x_{n-1}(s), D_{0+}^\gamma y_{n-1}(s)) \, ds + \int_0^1 G(t,s)g(s, x_{n-1}(s)) \, ds \\
 &\quad + \frac{\Gamma(v-\beta)}{\Gamma(v)}k(y_{n-1}(1))t^{v-1}, \quad n = 1, 2, \dots; \\
 y_n(t) &= \int_0^1 G(t,s)f(s, y_{n-1}(s), D_{0+}^\gamma x_{n-1}(s)) \, ds + \int_0^1 G(t,s)g(s, y_{n-1}(s)) \, ds \\
 &\quad + \frac{\Gamma(v-\beta)}{\Gamma(v)}k(x_{n-1}(1))t^{v-1}, \quad n = 1, 2, \dots
 \end{aligned}$$

Proof By (3.1)-(3.3), we can calculate that

$$D_{0+}^\gamma A(u)(t) = \int_0^1 D_{0+}^\gamma G(t,s)g(s, u(s)) \, ds, \quad t \in [0, 1]; \tag{3.4}$$

$$D_{0+}^\gamma B(v)(t) = \frac{\Gamma(v-\beta)}{\Gamma(v-\gamma)}k(v(1))t^{v-\gamma-1}, \quad t \in [0, 1]; \tag{3.5}$$

$$D_{0+}^\gamma C(u, v)(t) = \int_0^1 D_{0+}^\gamma G(t,s)f(s, u(s), D_{0+}^\gamma v(s)) \, ds, \quad t \in [0, 1]. \tag{3.6}$$

To get the conclusions, we need to prove that the operators A, B, C satisfies all conditions in Lemma 2.5.

Firstly, the proof of $A : P \rightarrow P, B : P \rightarrow P$ and $C : P \times P \rightarrow P$ is presented. $\forall u, v \in P$, from (H_1) and Lemma 2.4, it is easy to obtain $A(u)(t) \geq 0, D_{0+}^\gamma A(u)(t) \geq 0, B(v)(t) \geq 0, D_{0+}^\gamma B(v)(t) \geq 0, C(u, v)(t) \geq 0, D_{0+}^\gamma C(u, v)(t) \geq 0, \forall t \in [0, 1]$. That is, $A(u) \in P, B(v) \in P, C(u, v) \in P$. Therefore, $A : P \rightarrow P, B : P \rightarrow P$ and $C : P \times P \rightarrow P$. Next, we show A is increasing and B is decreasing. $\forall u, v \in P$ and $u \leq v$, we know that $u(t) \leq v(t), D_{0+}^\gamma u(t) \leq D_{0+}^\gamma v(t), \forall t \in [0, 1]$. From (H_2) and $G(t, s) \geq 0$, we have

$$\begin{aligned}
 A(u) &= \int_0^1 G(t,s)g(s, u(s)) \, ds \leq \int_0^1 G(t,s)g(s, v(s)) \, ds = A(v); \\
 D_{0+}^\gamma A(u) &= \int_0^1 D_{0+}^\gamma G(t,s)g(s, u(s)) \, ds \leq \int_0^1 D_{0+}^\gamma G(t,s)g(s, v(s)) \, ds = D_{0+}^\gamma A(v); \\
 B(u) &= \frac{\Gamma(v-\beta)}{\Gamma(v)}k(u(1))t^{v-1} \geq \frac{\Gamma(v-\beta)}{\Gamma(v)}k(v(1))t^{v-1} = B(v); \\
 D_{0+}^\gamma B(u) &= \frac{\Gamma(v-\beta)}{\Gamma(v-\gamma)}k(u(1))t^{v-\gamma-1} \geq \frac{\Gamma(v-\beta)}{\Gamma(v-\gamma)}k(v(1))t^{v-\gamma-1} = D_{0+}^\gamma B(v).
 \end{aligned}$$

Hence, $A(u) \leq A(v)$ and $B(u) \geq B(v)$.

Further, we prove that C is a mixed monotone operator. $\forall u_1, u_2, v_1, v_2 \in P$ with $u_1 \leq u_2, v_1 \geq v_2$, that is, $u_1(t) \leq u_2(t), D_{0+}^\gamma u_1(t) \leq D_{0+}^\gamma u_2(t), v_1(t) \geq v_2(t), D_{0+}^\gamma v_1(t) \geq D_{0+}^\gamma v_2(t), t \in$

[0, 1]. By (H_2) and $G(t, s) \geq 0$, we get

$$\begin{aligned} C(u_1, v_1) &= \int_0^1 G(t, s)f(s, u_1(s), D_{0+}^\gamma v_1(s)) \, ds \\ &\leq \int_0^1 G(t, s)f(s, u_2(s), D_{0+}^\gamma v_2(s)) \, ds = C(u_2, v_2), \\ D_{0+}^\gamma C(u_1, v_1) &= \int_0^1 D_{0+}^\gamma G(t, s)f(s, u_1(s), D_{0+}^\gamma v_1(s)) \, ds \\ &\leq \int_0^1 D_{0+}^\gamma G(t, s)f(s, u_2(s), D_{0+}^\gamma v_2(s)) \, ds = D_{0+}^\gamma C(u_2, v_2). \end{aligned}$$

Thus, $C(u_1, v_1) \leq C(u_2, v_2)$.

Now we prove A satisfies (2.13) and B and C satisfy (2.14). From (H_3) , $\eta \in (0, 1)$, we obtain

$$\begin{aligned} A(\eta x) &= \int_0^1 G(t, s)g(s, \eta x(s)) \, ds \geq \eta \int_0^1 G(t, s)g(s, x(s)) \, ds = \eta A(x), \\ D_{0+}^\gamma A(\eta x) &= \int_0^1 D_{0+}^\gamma G(t, s)g(s, \eta x(s)) \, ds \geq \eta \int_0^1 D_{0+}^\gamma G(t, s)g(s, x(s)) \, ds = \eta D_{0+}^\gamma A(x), \end{aligned}$$

that is, $A(tx) \geq tAx$, $x \in P$. Then A is a sub-homogeneous operator. Clearly, the following inequalities are also tenable:

$$\begin{aligned} B(\eta^{-1}y) &= \frac{\Gamma(v - \beta)}{\Gamma(v)} k(\eta^{-1}y(1))t^{v-1} \geq \eta \frac{\Gamma(v - \beta)}{\Gamma(v)} k(y(1))t^{v-1} = \eta B(y), \\ D_{0+}^\gamma B(\eta^{-1}y) &= \frac{\Gamma(v - \beta)}{\Gamma(v - \gamma)} k(\eta^{-1}y(1))t^{v-\gamma-1} \geq \eta \frac{\Gamma(v - \beta)}{\Gamma(v - \gamma)} k(y(1))t^{v-\gamma-1} = \eta D_{0+}^\gamma B(y), \\ C(\eta x, \eta^{-1}y) &= \int_0^1 G(t, s)f(s, \eta x(s), \eta^{-1}D_{0+}^\gamma y(s)) \, ds \\ &\geq \eta^\alpha \int_0^1 G(t, s)f(s, x(s), D_{0+}^\gamma y(s)) \, ds = \eta^\alpha C(x, y), \\ D_{0+}^\gamma C(\eta x, \eta^{-1}y) &= \int_0^1 D_{0+}^\gamma G(t, s)f(s, \eta x(s), \eta^{-1}D_{0+}^\gamma y(s)) \, ds \\ &\geq \eta^\alpha \int_0^1 D_{0+}^\gamma G(t, s)f(s, x(s), D_{0+}^\gamma y(s)) \, ds = \eta^\alpha D_{0+}^\gamma C(x, y). \end{aligned}$$

Hence, $B(t^{-1}y) \geq tBy$ and $C(x, y) \geq t^\alpha C(x, y)$.

In the sequel, we demonstrate that the assumptions (A_1) and (A_2) are satisfied. In view of (H_1) , (H_2) and (2.9) in Lemma 2.4, we have

$$\begin{aligned} A(h) &= \int_0^1 G(t, s)g(s, s^{v-1}) \, ds \\ &\leq \int_0^1 \frac{t^{v-1}(1-s)^{v-\beta-1}}{\Gamma(v)} g(s, s^{v-1}) \, ds \\ &\leq t^{v-1} \int_0^1 \frac{(1-s)^{v-\beta-1}g(s, 1)}{\Gamma(v)} \, ds, \end{aligned}$$

$$\begin{aligned}
 A(h) &\geq \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}[1-(1-s)^\beta]}{\Gamma(\nu)} g(s, s^{\nu-1}) \, ds \\
 &\geq t^{\nu-1} \int_0^1 \frac{(1-s)^{\nu-\beta-1}[1-(1-s)^\beta]g(s, 0)}{\Gamma(\nu)} \, ds, \\
 C(h, h) &= \int_0^1 G(t, s)f(s, s^{\nu-1}, D_{0+}^\gamma s^{\nu-1}) \, ds \\
 &\leq \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu)} f\left(s, s^{\nu-1}, \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1}\right) \, ds \\
 &\leq t^{\nu-1} \int_0^1 \frac{(1-s)^{\nu-\beta-1}}{\Gamma(\nu)} f(s, 1, 0) \, ds, \\
 C(h, h) &\geq \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}[1-(1-s)^\beta]}{\Gamma(\nu)} f\left(s, s^{\nu-1}, \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1}\right) \, ds \\
 &\geq t^{\nu-1} \int_0^1 \frac{(1-s)^{\nu-\beta-1}[1-(1-s)^\beta]}{\Gamma(\nu)} f\left(s, 0, \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)}\right) \, ds.
 \end{aligned}$$

On the other hand, (H_1) , (H_2) and (2.10) imply

$$\begin{aligned}
 D_{0+}^\gamma A(h) &= \int_0^1 D_{0+}^\gamma G(t, s)g(s, s^{\nu-1}) \, ds \\
 &\leq \int_0^1 \frac{t^{\nu-\gamma-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu-\gamma)} g(s, s^{\nu-1}) \, ds \\
 &\leq \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1} \int_0^1 \frac{(1-s)^{\nu-\beta-1}g(s, 1)}{\Gamma(\nu)} \, ds, \\
 D_{0+}^\gamma A(h) &\geq \int_0^1 \frac{t^{\nu-\gamma-1}(1-s)^{\nu-\beta-1}[1-(1-s)^{\beta-\gamma}]}{\Gamma(\nu-\gamma)} g(s, s^{\nu-1}) \, ds \\
 &\geq \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1} \int_0^1 \frac{(1-s)^{\nu-\beta-1}[1-(1-s)^{\beta-\gamma}]g(s, 0)}{\Gamma(\nu)} \, ds, \\
 D_{0+}^\gamma C(h, h) &= \int_0^1 D_{0+}^\gamma G(t, s)f(s, s^{\nu-1}, D_{0+}^\gamma s^{\nu-1}) \, ds \\
 &\leq \int_0^1 \frac{t^{\nu-\gamma-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu-\gamma)} f\left(s, s^{\nu-1}, \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1}\right) \, ds \\
 &\leq \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1} \int_0^1 \frac{(1-s)^{\nu-\beta-1}}{\Gamma(\nu)} f(s, 1, 0) \, ds, \\
 D_{0+}^\gamma C(h, h) &\geq \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}[1-(1-s)^{\beta-\gamma}]}{\Gamma(\nu)} f\left(s, s^{\nu-1}, \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1}\right) \, ds \\
 &\geq \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1} \int_0^1 \frac{(1-s)^{\nu-\beta-1}[1-(1-s)^{\beta-\gamma}]}{\Gamma(\nu)} f\left(s, 0, \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)}\right) \, ds.
 \end{aligned}$$

Set

$$\begin{aligned}
 c_1 &= \int_0^1 \frac{(1-s)^{\nu-\beta-1}[1-(1-s)^{\beta-\gamma}]g(s, 0)}{\Gamma(\nu)} \, ds, \\
 c_2 &= \int_0^1 \frac{(1-s)^{\nu-\beta-1}g(s, 1)}{\Gamma(\nu)} \, ds,
 \end{aligned} \tag{3.7}$$

$$\begin{aligned}
 c_3 &= \int_0^1 \frac{(1-s)^{\nu-\beta-1} [1-(1-s)^{\beta-\gamma}] f(s, 0, \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)})}{\Gamma(\nu)} ds, \\
 c_4 &= \int_0^1 \frac{(1-s)^{\nu-\beta-1}}{\Gamma(\nu)} f(s, 1, 0) ds.
 \end{aligned}
 \tag{3.8}$$

From (H_2) and (H_4) , it is clear that

$$c_2 \geq c_1 > 0, \quad c_4 \geq c_3 \geq \delta_1 c_1 > 0.$$

Consequently,

$$c_1 h \leq A(h) \leq c_2 h, \quad c_3 h \leq C(h, h) \leq c_4 h.$$

Further,

$$B(h) = t^{\nu-1} \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(1) = \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(1) h(t), \tag{3.9}$$

$$D_{0+}^\gamma B(h) = \frac{\Gamma(\nu)}{\Gamma(\nu-\gamma)} t^{\nu-\gamma-1} \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(1) = \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(1) D_{0+}^\gamma h(t). \tag{3.10}$$

From $k(y(1)) \neq 0$, we have $B(h) \in P_h$. Therefore, (A_1) in Lemma 2.5 is proved.

Next, we show the proof of the condition (A_2) of Lemma 2.5. By (H_4) ,

$$\begin{aligned}
 C(x, y) &= \int_0^1 G(t, s) f(s, x(s), D_{0+}^\gamma y(s)) ds \\
 &\geq \delta_1 \int_0^1 G(t, s) g(s, x(s)) ds \\
 &= \delta_1 Ax, \\
 D_{0+}^\gamma C(x, y) &= \int_0^1 D_{0+}^\gamma G(t, s) f(s, x(s), D_{0+}^\gamma y(s)) ds \\
 &\geq \delta_1 \int_0^1 D_{0+}^\gamma G(t, s) g(s, x(s)) ds \\
 &= \delta_1 D_{0+}^\gamma Ax.
 \end{aligned}$$

Then

$$C(x, y) \geq \delta_1 Ax.$$

From (H_4) and Lemma 2.4, we have

$$\begin{aligned}
 C(x, y) &= \int_0^1 G(t, s) f(s, x(s), D_{0+}^\gamma y(s)) ds \\
 &\geq \int_0^1 \frac{t^{\nu-1} (1-s)^{\nu-\beta-1} [1-(1-s)^\beta]}{\Gamma(\nu)} f(s, x(s), D_{0+}^\gamma y(s)) ds \\
 &\geq \frac{t^{\nu-1}}{\Gamma(\nu)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu} \right) \delta_2
 \end{aligned}$$

$$\begin{aligned} &\geq \frac{t^{\nu-1}}{\Gamma(\nu)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) k(y(1)) \\ &= \frac{1}{\Gamma(\nu-\beta)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) By, \\ D_{0+}^{\gamma} C(x, y) &= \int_0^1 D_{0+}^{\gamma} G(t, s) f(s, x(s), D_{0+}^{\gamma} y(s)) \, ds \\ &\geq \frac{t^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)} \int_0^1 (1-s)^{\nu-\beta-1} (1-(1-s)^{\beta-\gamma}) f(s, x(s), D_{0+}^{\gamma} y(s)) \, ds \\ &\geq \frac{t^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) k(y(1)) \\ &= \frac{1}{\Gamma(\nu-\beta)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) D_{0+}^{\gamma} By. \end{aligned}$$

That means $C(x, y) \geq \frac{1}{\Gamma(\nu-\beta)} (\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma}) By$. Let $\delta = \min\{\delta_1, \frac{1}{\Gamma(\nu-\beta)} (\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma})\}$. Then we have

$$C(x, y) \geq \delta(Ax + By).$$

Finally, as an application of Lemma 2.5, we see that the conclusion (2) in Lemma 2.5 means that there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that

$$\begin{aligned} rv_0 \leq u_0 < v_0, \quad \text{that is,} \quad rv_0 \leq u_0 < v_0, \quad rD_{0+}^{\gamma} v_0 \leq D_{0+}^{\gamma} u_0 < D_{0+}^{\gamma} v_0; \\ u_0(t) &\leq \int_0^1 G(t, s) f(s, u_0(s), D_{0+}^{\gamma} v_0(s)) \, ds + \int_0^1 G(t, s) g(s, u_0(s)) \, ds \\ &\quad + \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(v_0(1)) t^{\nu-1}, \quad t \in [0, 1]; \\ D_{0+}^{\gamma} u_0(t) &\leq \int_0^1 D_{0+}^{\gamma} G(t, s) f(s, u_0(s), D_{0+}^{\gamma} v_0(s)) \, ds + \int_0^1 D_{0+}^{\gamma} G(t, s) g(s, u_0(s)) \, ds \\ &\quad + \frac{\Gamma(\nu-\beta)}{\Gamma(\nu-\gamma)} k(v_0(1)) t^{\nu-\gamma-1}, \quad t \in [0, 1]; \\ v_0(t) &\geq \int_0^1 G(t, s) f(s, v_0(s), D_{0+}^{\gamma} u_0(s)) \, ds + \int_0^1 G(t, s) g(s, v_0(s)) \, ds \\ &\quad + \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(u_0(1)) t^{\nu-1}, \quad t \in [0, 1]; \\ D_{0+}^{\gamma} v_0(t) &\leq \int_0^1 D_{0+}^{\gamma} G(t, s) f(s, v_0(s), D_{0+}^{\gamma} u_0(s)) \, ds + \int_0^1 D_{0+}^{\gamma} G(t, s) g(s, v_0(s)) \, ds \\ &\quad + \frac{\Gamma(\nu-\beta)}{\Gamma(\nu-\gamma)} k(u_0(1)) t^{\nu-\gamma-1}, \quad t \in [0, 1]. \end{aligned}$$

Equation (3) in Lemma 2.5 implies that the problem (1.1) has a unique positive solution x^* in P_h ; Lemma 2.5(4) means that, for any initial value $x_0, y_0 \in P_h$, there are two iterative sequences $\{x_n\}, \{y_n\}$ for approximating x^* , that is, $x_n \rightarrow x^*, y_n \rightarrow x^*$, as $n \rightarrow \infty$, where

$$\begin{aligned} x_n(t) &= \int_0^1 G(t, s) f(s, x_{n-1}(s), D_{0+}^{\gamma} y_{n-1}(s)) \, ds + \int_0^1 G(t, s) g(s, x_{n-1}(s)) \, ds \\ &\quad + \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(y_{n-1}(1)) t^{\nu-1}, \quad n = 1, 2, \dots; \end{aligned}$$

$$\begin{aligned}
 y_n(t) &= \int_0^1 G(t,s)f(s,y_{n-1}(s),D_{0+}^\gamma x_{n-1}(s)) \, ds + \int_0^1 G(t,s)g(s,y_{n-1}(s)) \, ds \\
 &\quad + \frac{\Gamma(v-\beta)}{\Gamma(v)}k(x_{n-1}(1))t^{v-1}, \quad n = 1, 2, \dots
 \end{aligned}
 \tag*{\square}$$

Theorem 3.2 Assume (H_1) , (H_2) and suppose that

- (H_3) there exists a constant $\alpha \in (0, 1)$ such that $k(\eta^{-1}y) \geq \eta^\alpha k(y)$, $\forall \eta \in (0, 1)$, $t \in [0, 1]$, $y \in [0, +\infty)$ And $g(t, x), f(t, x, y)$ satisfy $g(t, \eta x) \geq \eta g(t, x)$, $f(t, \eta x, \eta^{-1}y) \geq \eta f(t, x, y)$, $\forall \eta \in (0, 1)$, $x, y \in [0, +\infty)$.
- (H_4) $g(t, 0) \neq 0$, $f(t, 0, \frac{\Gamma(v)}{\Gamma(v-\beta)}) \neq 0$ for $t \in [0, 1]$ and there exists a constant δ_0 such that $g(t, x) + f(t, x, y) \leq \delta_0 \leq k(y)$, $t \in [0, 1]$, $x, y \in [0, +\infty)$.

Then the conclusions (1)-(3) of Theorem 3.1 hold.

Proof Similar to the proof of Theorem 3.1, from (H_1) and (H_2) , we see that $A, B : P \rightarrow P$, $C : P \times P \rightarrow P$ and A is increasing, B is decreasing, C is a mixed monotone operator. From (H_3) , we have

$$\begin{aligned}
 A(\eta x) &= \int_0^1 G(t,s)g(s,\eta x(s)) \, ds \geq \eta \int_0^1 G(t,s)g(s,x(s)) \, ds = \eta A(x), \\
 D_{0+}^\gamma A(\eta x) &= \int_0^1 D_{0+}^\gamma G(t,s)g(s,\eta x(s)) \, ds \geq \eta \int_0^1 D_{0+}^\gamma G(t,s)g(s,x(s)) \, ds = \eta D_{0+}^\gamma A(x), \\
 B(\eta^{-1}y) &= \frac{\Gamma(v-\beta)}{\Gamma(v)}k(\eta^{-1}y(1))t^{v-1} \geq \eta^\alpha \frac{\Gamma(v-\beta)}{\Gamma(v)}k(y(1))t^{v-1} = \eta^\alpha B(y), \\
 D_{0+}^\gamma B(\eta^{-1}y) &= \frac{\Gamma(v-\beta)}{\Gamma(v-\gamma)}k(\eta^{-1}y(1))t^{v-\gamma-1} \geq \eta^\alpha \frac{\Gamma(v-\beta)}{\Gamma(v-\gamma)}k(y(1))t^{v-\gamma-1} = \eta^\alpha D_{0+}^\gamma B(y), \\
 C(\eta x, \eta^{-1}y) &= \int_0^1 G(t,s)f(s,\eta x(s),\eta^{-1}D_{0+}^\gamma y(s)) \, ds \\
 &\geq \eta \int_0^1 G(t,s)f(s,x(s),D_{0+}^\gamma y(s)) \, ds = \eta C(x, y), \\
 D_{0+}^\gamma C(\eta x, \eta^{-1}y) &= \int_0^1 D_{0+}^\gamma G(t,s)f(s,\eta x(s),\eta^{-1}D_{0+}^\gamma y(s)) \, ds \\
 &\geq \eta \int_0^1 D_{0+}^\gamma G(t,s)f(s,x(s),D_{0+}^\gamma y(s)) \, ds = \eta D_{0+}^\gamma C(x, y).
 \end{aligned}$$

Hence, $A(tx) \geq tAx$, $B(t^{-1}y) \geq t^\alpha By$ and $C(x, y) \geq tC(x, y)$. Then A is a sub-homogeneous operator. Since $f(t, 0, \frac{\Gamma(v)}{\Gamma(v-\beta)}) \neq 0$ and by (H_2) , we obtain $c_4 \geq c_3 > 0$. Obviously, $C(h, h) \in P_h$. From $g(t, 0) \neq 0$ and (H_2) , we have $c_2 \geq c_1 > 0$, which implies $A(h) \in P_h$. Here, c_1, c_2, c_3, c_4 is given by (3.7), (3.8). From (3.9) and (3.10) in Theorem 3.1 and $k(y(1)) \neq 0$, we get $B(h) \in P_h$.

Now, we show the assumption (A'_2) is satisfied. From (H_4) and Lemma 2.4, for $t \in (0, 1)$, $x, y \in [0, +\infty)$, we obtain

$$\begin{aligned}
 Ax + C(x, y) &= \int_0^1 G(t,s)(g(s,x(s)) + f(s,x(s),D_{0+}^\gamma y(s))) \, ds \\
 &\leq \int_0^1 \frac{t^{v-1}(1-s)^{v-\beta-1}}{\Gamma(v)}(g(s,x(s)) + f(s,x(s),D_{0+}^\gamma y(s))) \, ds
 \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu)} \delta_0 \, ds \\
 &\leq \frac{1}{(\nu-\beta)\Gamma(\nu-\beta)} \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(y(1))t^{\nu-1} \\
 &= \frac{1}{\Gamma(\nu-\beta+1)} By, \\
 D_{0+}^\gamma (Ax + C(x, y)) &= \int_0^1 D_{0+}^\gamma G(t, s)(g(s, x(s)) + f(s, x(s), D_{0+}^\gamma y(s))) \, ds \\
 &\leq \int_0^1 \frac{t^{\nu-\gamma-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu-\gamma)} (g(s, x(s)) + f(s, x(s), D_{0+}^\gamma y(s))) \, ds \\
 &\leq \int_0^1 \frac{t^{\nu-\gamma-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu-\gamma)} \delta_0 \, ds \\
 &\leq \frac{1}{(\nu-\beta)\Gamma(\nu-\beta)} \frac{\Gamma(\nu-\beta)}{\Gamma(\nu-\gamma)} k(y(1))t^{\nu-\gamma-1} \\
 &= \frac{1}{\Gamma(\nu-\beta+1)} D_{0+}^\gamma By.
 \end{aligned}$$

Let $\delta = \frac{1}{\Gamma(\nu-\beta+1)}$. Then

$$Ax + C(x, y) \leq \delta By.$$

Therefore, by Lemma 2.6, the conclusions (1)-(3) in Theorem 3.1 hold. □

Theorem 3.3 *Assume (H_1) , (H_2) and suppose that*

- (H_3'') *there exists a constant $\alpha \in (0, 1)$ such that $g(t, \eta x) \geq \eta^\alpha g(t, x)$, $\forall \eta \in (0, 1)$, $t \in [0, 1]$, $x \in [0, +\infty)$. And $k(y), f(t, x, y)$ satisfy $k(\eta^{-1}y) \geq \eta k(y)$, $f(t, \eta x, \eta^{-1}y) \geq \eta f(t, x, y)$, $\forall \eta \in (0, 1)$, $x, y \in [0, +\infty)$.*
- (H_4'') *$f(t, 0, \frac{\Gamma(\nu)}{\Gamma(\nu-\beta)}) \neq 0$ for $t \in [0, 1]$, $k(y(1)) \neq 0$ and there exist two constants δ_1, δ_2 such that $k(y) \leq \delta_1 \leq g(t, x)$ and $f(t, x, y) \leq \delta_2 g(t, x)$, $t \in [0, 1]$, $x, y \in [0, +\infty)$.*

Then the conclusions (1)-(3) of Theorem 3.1 hold.

Proof From the proof of Theorem 3.1, we know that $A : P \rightarrow P$ is an increasing operator, $B : P \rightarrow P$ is a decreasing operator, $C : P \times P \rightarrow P$ is a mixed monotone operator.

In the following, we demonstrate that A, B, C satisfy all assumptions in Lemma 2.7. From (H_3'') , we know that

$$\begin{aligned}
 A(\eta x) &= \int_0^1 G(t, s)g(s, \eta x(s)) \, ds \geq \eta^\alpha \int_0^1 G(t, s)g(s, x(s)) \, ds = \eta^\alpha A(x), \\
 D_{0+}^\gamma A(\eta x) &= \int_0^1 D_{0+}^\gamma G(t, s)g(s, \eta x(s)) \, ds \geq \eta^\alpha \int_0^1 D_{0+}^\gamma G(t, s)g(s, x(s)) \, ds = \eta D_{0+}^\gamma A(x), \\
 B(\eta^{-1}y) &= \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(\eta^{-1}y(1))t^{\nu-1} \geq \eta \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(y(1))t^{\nu-1} = \eta B(y), \\
 D_{0+}^\gamma B(\eta^{-1}y) &= \frac{\Gamma(\nu-\beta)}{\Gamma(\nu-\gamma)} k(\eta^{-1}y(1))t^{\nu-\gamma-1} \geq \eta \frac{\Gamma(\nu-\beta)}{\Gamma(\nu-\gamma)} k(y(1))t^{\nu-\gamma-1} = \eta D_{0+}^\gamma B(y),
 \end{aligned}$$

$$\begin{aligned}
 C(\eta x, \eta^{-1}y) &= \int_0^1 G(t,s)f(s, \eta x(s), \eta^{-1}D_{0+}^\gamma y(s)) \, ds \\
 &\geq \eta \int_0^1 G(t,s)f(s, x(s), D_{0+}^\gamma y(s)) \, ds = \eta C(x, y), \\
 D_{0+}^\gamma C(\eta x, \eta^{-1}y) &= \int_0^1 D_{0+}^\gamma G(t,s)f(s, \eta(s), \eta^{-1}D_{0+}^\gamma y(s)) \, ds \\
 &\geq \eta \int_0^1 D_{0+}^\gamma G(t,s)f(s, x(s), D_{0+}^\gamma y(s)) \, ds = \eta D_{0+}^\gamma C(x, y).
 \end{aligned}$$

That is, $A(tx) \geq t^\alpha Ax$, $B(t^{-1}y) \geq tBy$ and $C(x, y) \geq tC(x, y)$. Then A is a α -concave operator. From (H_2) , (H_4'') , we obtain $c_4 \geq c_3 > 0$ and $c_1 \geq c_3 > 0$, where c_1, c_2, c_3, c_4 are defined by (3.7), (3.8). It is evident that $C(h, h) \in P_h$ and $A(h) \in P_h$. From the proof of Theorem 3.1, we get $B(h) \in P_h$. Therefore, the condition (A_1'') in Lemma 2.7 is proved.

Next, we present the assumption (A_2'') . From (H_4'') and Lemma 2.4, for $t \in (0, 1)$, $x, y \in [0, +\infty)$, we obtain

$$\begin{aligned}
 C(x, y) &= \int_0^1 G(t,s)f(s, x(s), D_{0+}^\gamma y(s)) \, ds \\
 &\leq \delta_2 \int_0^1 G(t,s)g(s, x(s)) \, ds \\
 &= \delta_2 Ax, \\
 D_{0+}^\gamma C(x, y) &= \int_0^1 D_{0+}^\gamma G(t,s)f(s, x(s), D_{0+}^\gamma y(s)) \, ds \\
 &\leq \delta_2 \int_0^1 D_{0+}^\gamma G(t,s)g(s, x(s)) \, ds \\
 &= \delta_2 D_{0+}^\gamma Ax.
 \end{aligned}$$

On the other side, from (H_4'') and Lemma 2.4,

$$\begin{aligned}
 A(x) &= \int_0^1 G(t,s)g(s, x(s)) \, ds \\
 &\geq \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}[1-(1-s)^\beta]}{\Gamma(\nu)} g(s, x(s)) \, ds \\
 &\geq \delta_1 \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-\beta-1}[1-(1-s)^\beta]}{\Gamma(\nu)} \, ds \\
 &\geq \frac{1}{\Gamma(\nu-\beta)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu} \right) \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(y(1)) t^{\nu-1} \\
 &\geq \frac{1}{\Gamma(\nu-\beta)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) \frac{\Gamma(\nu-\beta)}{\Gamma(\nu)} k(y(1)) t^{\nu-1} \\
 &= \frac{1}{\Gamma(\nu-\beta)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) By, \\
 D_{0+}^\gamma A(x) &= D_{0+}^\gamma \int_0^1 G(t,s)g(s, x(s)) \, ds \\
 &\geq \int_0^1 \frac{t^{\nu-\gamma-1}(1-s)^{\nu-\beta-1}[1-(1-s)^{\beta-\gamma}]}{\Gamma(\nu-\gamma)} g(s, x(s)) \, ds
 \end{aligned}$$

$$\begin{aligned} &\geq \delta_1 \int_0^1 \frac{t^{\nu-\gamma-1}(1-s)^{\nu-\beta-1}[1-(1-s)^{\beta-\gamma}]}{\Gamma(\nu-\gamma)} ds \\ &\geq \frac{1}{\Gamma(\nu-\beta)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) \frac{\Gamma(\nu-\beta)}{\Gamma(\nu-\gamma)} k(y(1)) t^{\nu-\gamma-1} \\ &= \frac{1}{\Gamma(\nu-\beta)} \left(\frac{1}{\nu-\beta} - \frac{1}{\nu-\gamma} \right) D_{0^+}^\gamma B y. \end{aligned}$$

That is, $C(x, y) \leq \delta_2 Ax$ and $By \leq \frac{(\nu-\gamma)\Gamma(\nu-\beta+1)}{\beta-\gamma} Ax$. Let $\delta = 2 \max\{\delta_2, \frac{(\nu-\gamma)\Gamma(\nu-\beta+1)}{\beta-\gamma}\}$. Then

$$By + C(x, y) \leq \delta Ax.$$

Therefore, applying Lemma 2.7, the results (1)-(3) in Theorem 3.1 are tenable. □

Remark 3.1 Comparing Theorem 3.1, Theorem 3.2 and Theorem 3.3 with the main results in [9], we provide some alternative approaches to the study of the same type of problems (when the parameter $k(x) = 0$ and $\gamma = 0$) under different conditions. Our results cannot only guarantee the existence of a unique positive solution but also we obtain two sequences approximating the solution.

4 Examples

As applications, three examples are presented to illustrate our main results.

4.1 Example 1

Consider the following fractional differential equation:

$$\begin{cases} D_{0^+}^{\frac{5}{2}} x(t) + 2(x(t))^{\frac{1}{4}} + t^2 + (x'(t) + 1)^{-\frac{1}{2}} + 1 = 0, & t \in (0, 1), \\ x(0) = x'(0) = x''(0) = x'''(0) = 0, \\ D_{0^+}^{\frac{5}{2}} x(t) = \frac{1}{1+x(1)}. \end{cases} \tag{4.1}$$

Let $g(t, x) = (x(t))^{\frac{1}{4}} + t^2$ and $g(t, 0) = t^2 \neq 0, f(t, x, y) = (x(t))^{\frac{1}{4}} + (y(t) + 1)^{-\frac{1}{2}} + 1$ and $k(y) = \frac{1}{1+y}$. Clearly, $g : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty), f : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ and $k : [0, +\infty) \rightarrow [0, +\infty)$ are continuous. It is easy to see that $g(t, x)$ is increasing in $x \in [0, +\infty)$ for fixed $t \in (0, 1), k(y)$ is decreasing in $y \in [0, +\infty)$ and $f(t, x, y)$ is increasing in $x \in [0, +\infty)$ for fixed $t \in (0, 1)$ and $y \in [0, +\infty),$ decreasing in $y \in [0, +\infty)$ for fixed $t \in (0, 1)$ and $x \in [0, +\infty)$. In addition, $\forall \eta \in (0, 1)$ we get

$$\begin{aligned} g(t, \eta x) &= (\eta x(t))^{\frac{1}{4}} + t^2 \geq \eta^{\frac{1}{4}} (x(t))^{\frac{1}{4}} + \eta t^2 \geq \eta \left((x(t))^{\frac{1}{4}} + t^2 \right) g(t, x) = \eta g(t, x), \\ f(t, \eta x, \eta^{-1} y) &\geq \eta^{\frac{1}{4}} (x(t))^{\frac{1}{4}} + \eta^{\frac{1}{2}} (y(t) + 1)^{-\frac{1}{2}} + 1 \geq \eta^{\frac{1}{2}} \left((x(t))^{\frac{1}{4}} + \eta^{\frac{1}{2}} (y(t) + 1)^{-\frac{1}{2}} + 1 \right) \\ &= \eta^{\frac{1}{2}} f(t, x, y), \\ k(\eta^{-1} y) &= \frac{1}{1 + \eta^{-1} y} \geq \eta k(y). \end{aligned}$$

Moreover, set $\delta_1 = \delta_2 = 1,$

$$\begin{aligned} f(t, x, y) &= (x(t))^{\frac{1}{4}} + (y(t) + 1)^{-\frac{1}{2}} + 1 \geq \delta_2 \geq \frac{1}{1 + y} = k(y), \\ f(t, x, y) &= (x(t))^{\frac{1}{4}} + (y(t) + 1)^{-\frac{1}{2}} + 1 \geq (x(t))^{\frac{1}{4}} + t^2 = \delta_1 g(t, x). \end{aligned}$$

Hence, all the conditions of Theorem 3.1 are satisfied. Then the problem (4.1) has a unique positive solution x^* in P_{h_1} , where $h_1 = t^{\frac{7}{2}}$.

4.2 Example 2

$$\begin{cases} D_{0^+}^{\frac{11}{3}}x(t) + t + t^3 + \frac{2x}{1+x} + \frac{1}{D_{0^+}^{\frac{5}{3}}x(t)+1} = 0, & t \in (0, 1), \\ x(0) = x'(0) = x''(0) = 0, \\ x''(1) = (x(1))^{-\frac{1}{3}} + 5. \end{cases} \tag{4.2}$$

Let $g(t, x) = t + \frac{x}{1+x}$, $f(t, x, y) = t^3 + \frac{x}{1+x} + \frac{1}{y+1}$ and $k(y) = y^{-\frac{1}{3}} + 5$. Obviously, $g : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$, $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ and $k : [0, +\infty) \rightarrow [0, +\infty)$ are continuous. It is clear that $g(t, x)$ is increasing in $x \in [0, +\infty)$ for fixed $t \in (0, 1)$, $k(y)$ is decreasing in $y \in [0, +\infty)$ for fixed $t \in (0, 1)$ and $f(t, x, y)$ is increasing in $x \in [0, +\infty)$ for fixed $t \in (0, 1)$ and $y \in [0, +\infty)$, decreasing in $y \in [0, +\infty)$ for fixed $t \in (0, 1)$ and $x \in [0, +\infty)$. In addition, $\forall \eta \in (0, 1)$ we get

$$\begin{aligned} g(t, \eta x) &= t + \frac{\eta x}{1 + \eta x} \geq \eta t + \eta \frac{x}{1 + x} = \eta g(t, x), \\ f(t, \eta x, \eta^{-1}y) &= t^3 + \frac{\eta x}{1 + \eta x} + \frac{1}{\eta^{-1}y + 1} \geq t^3 + \frac{\eta x}{1 + x} + \frac{\eta}{y + 1} \geq \eta f(t, x, y), \\ k(\eta^{-1}y) &= (\eta^{-1}y)^{-\frac{1}{3}} + 5 \geq \eta^{\frac{1}{3}}k(y). \end{aligned}$$

Besides, $g(t, 0) = t \neq 0$ and $f(t, 0, \frac{\Gamma(\nu)}{\Gamma(\nu-\beta)}) = t^3 + \frac{1}{\frac{\Gamma(\nu)}{\Gamma(\nu-\beta)}+1} \neq 0$, set $\delta_0 = 5$,

$$f(t, x, y) + g(t, x) = t + t^3 + \frac{2x}{1+x} + \frac{1}{y(t)+1} \leq \delta_0 \leq (y(t))^{-\frac{1}{3}} + 5 = k(y).$$

Therefore, we proved all hypotheses of Theorem 3.2. Then we deduce that (4.1) has a unique positive solution x^* in P_{h_2} , where $h_2 = t^{\frac{8}{3}}$.

4.3 Example 3

Consider the following fractional differential equation:

$$\begin{cases} D_{0^+}^{\frac{9}{4}}x(t) + 2 + 2(x(t))^{\frac{1}{4}} + \cos^2 t + \frac{1}{D_{0^+}^{\frac{5}{4}}x(t)+1} = 0, & t \in (0, 1), \\ x(0) = x'(0) = 0, \\ [D_{0^+}^{\frac{3}{2}}x(t)]_{t=1} = \frac{1}{1+x(1)^{\frac{1}{2}}}. \end{cases} \tag{4.3}$$

Let $g(t, x) = 2 + (x(t))^{\frac{1}{4}}$, $f(t, x, y) = \cos^2 t + (x(t))^{\frac{1}{4}} + \frac{1}{y+1}$ and $k(y) = \frac{1}{1+y^{\frac{1}{2}}}$. Evidently, $g : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$, $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ and $k : [0, +\infty) \rightarrow [0, +\infty)$ are continuous. It is not difficult to verify that $g(t, x)$ is increasing in $x \in [0, +\infty)$ for fixed $t \in (0, 1)$, $k(y)$ is decreasing in $y \in [0, +\infty)$ and $f(t, x, y)$ is increasing in $x \in [0, +\infty)$ for fixed $t \in (0, 1)$ and $y \in [0, +\infty)$, decreasing in $y \in [0, +\infty)$ for fixed $t \in (0, 1)$ and $x \in [0, +\infty)$. Moreover, we get

$$g(t, \eta x) = 2 + (\eta x(t))^{\frac{1}{4}} \geq \eta^{\frac{1}{4}}2 + \eta^{\frac{1}{4}}(x(t))^{\frac{1}{4}} = \eta^{\frac{1}{4}}g(t, x),$$

$$f(t, \eta x, \eta^{-1}y) = \cos^2 t + (x(t))^{\frac{1}{4}} + \frac{1}{\eta^{-1}y + 1} \geq \eta \cos^2 t + \eta (x(t))^{\frac{1}{4}} + \frac{\eta}{y + 1} \geq \eta f(t, x, y),$$

$$k(\eta^{-1}y) = \frac{1}{1 + (\eta^{-1}y)^{\frac{1}{2}}} \geq \frac{\eta^{\frac{1}{2}}}{1 + y^{\frac{1}{2}}} \geq \frac{\eta}{1 + y^{\frac{1}{2}}} = \eta k(y).$$

Besides, $f(t, 0, \frac{\Gamma(v)}{\Gamma(v-\beta)}) = \cos^2 t + \frac{1}{\frac{\Gamma(v)}{\Gamma(v-\beta)} + 1} \neq 0$, set $\delta_1 = 2$, $\delta_2 = 1$,

$$k(y) = \frac{1}{1 + y^{\frac{1}{2}}} \leq \delta_1 \leq 2 + (x(t))^{\frac{1}{4}} = g(t, x),$$

$$f(t, x, y) = \cos^2 t + (x(t))^{\frac{1}{4}} + \frac{1}{y + 1} \leq 2 + (x(t))^{\frac{1}{4}} = \delta_2 g(t, x).$$

In consequence, an application of Theorem 3.3 means that (4.1) has a unique positive solution x^* in P_{h_3} , where $h_3 = t^{\frac{5}{4}}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

LZ participated in the design of the study and drafted the main part of manuscript. HT carried out the theoretical studies and helped to draft the manuscript. All authors read and approved the final manuscript.

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