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Existence and multiplicity of weak solutions for a nonlinear impulsive (q,p)-Laplacian dynamical system

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Abstract

In this paper, we investigate the existence and multiplicity of nontrivial weak solutions for a class of nonlinear impulsive (q,p)-Laplacian dynamical systems. The key contributions of this paper lie in (i) Exploiting the least action principle, we deduce that the system we are interested in has at least one weak solution if the potential function has sub-(q,p) growth or (q,p) growth; (ii) Employing a critical point theorem due to Ding (Nonlinear Anal. 25(11):1095-1113, 1995), we derive that the system involved has infinitely many weak solutions provided that the potential function is even.

MSC: 34C25; 58E50

Keywords: (q,p)-Laplacian; existence; multiplicity; nontrivial solution; variational methods

1 Introduction and main results

For $N \in \mathbb{N}$, let $(\mathbb{R}^N, \langle\cdot,\cdot\rangle, |\cdot|)$ be the N-dimensional Euclidean space. For fixed $l,k \in \mathbb{N}$, set $B := \{1,2,\ldots,l\}$ and $C := \{1,2,\ldots,k\}$. If $f:\mathbb{R}^n \to \mathbb{R}$ is a smooth function, let ∇f stand for the gradient operator. For a smooth function $f:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, denote by $\nabla_{x_1} f$ and $\nabla_{x_2} f$ the gradient operator with respect to the first component and the second component, respectively. For a mapping $f:\mathbb{R}_+ \to \mathbb{R}$, $f(t^+)$ and $f(t^-)$ mean the right-hand side limit and the left-land side limit at t, respectively. For functions $f:\mathbb{R}^n \to \mathbb{R}^n$ and $g:\mathbb{R}_+ \to \mathbb{R}^n$, let $\triangle(f(g(t))) = f(g(t^+)) - f(g(t^-))$.

In this paper, we consider a nonlinear system with impulsive effects on $\mathbb{H}^N := \mathbb{R}^N \times \mathbb{R}^N$ for any $p,q > 1$, $\lambda > 0$, $j \in B$, and $m \in C$,

$$\begin{cases} -\frac{d(\Phi_q(\dot{u}_1(t)))}{dt} + \Phi_q(u_1(t)) = \lambda \nabla_{u_1} F(t,u_1(t),u_2(t)), & \text{a.e. } t \in [0,T], \\ -\frac{d(\Phi_p(\dot{u}_2(t)))}{dt} + \Phi_p(u_2(t)) = \lambda \nabla_{u_2} F(t,u_1(t),u_2(t)), & \text{a.e. } t \in [0,T], \\ \Delta(\Phi_q(\dot{u}_1(t_j))) = \nabla I_j(u_1(t_j)), \\ \Delta(\Phi_p(\dot{u}_2(s_m))) = \nabla K_m(u_2(s_m)), \end{cases} \tag{1.1}$$

with the initial condition $(\dot{u}_1(0),\dot{u}_2(0)) = (u_1(0),u_2(0)) \in \mathbb{H}^N$ and the terminal condition $(\dot{u}_1(T),\dot{u}_2(T)) = (0,0) \in \mathbb{H}^N$, where $\Phi_\mu(z) := |z|^{\mu-2}z$ for any $\mu > 1$ and $z \in \mathbb{R}^N$; $F:\mathbb{R}_+ \times \mathbb{H}^N \to \mathbb{R}$; $(t_j)_{j\in B}$ and $(s_m)_{m\in C}$ are impulsive times with $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1} = T$,

$0 = s_0 < s_1 < s_2 < \cdots < s_k < s_{k+1} = T$, and for $j \in B$ and $m \in C$, $I_j : \mathbb{R}^N \to \mathbb{R}$ and $K_m : \mathbb{R}^N \to \mathbb{R}$ are continuously differentiable.

For the nonlinear term $F : [0, T] \times \mathbb{H}^N \to \mathbb{R}$, we assume that

(A1) For fixed $t \in [0, T]$ and $x \in \mathbb{H}^N$, $F(\cdot, x)$ is measurable and $F(t, \cdot)$ is continuously differentiable;

(A2) There exist $a_1, a_2 \in C(\mathbb{R}_+; \mathbb{R}_+)$ and $b \in L^1([0, T]; \mathbb{R}_+)$ such that

$$\begin{aligned} &\left|F(t,x_1,x_2)\right| \le \left[a_1\left(|x_1|\right) + a_2\left(|x_2|\right)\right]b(t),\\ &\left|\nabla F(t,x_1,x_2)\right| \le \left[a_1\left(|x_1|\right) + a_2\left(|x_2|\right)\right]b(t),\\ &\left|I_j(x_1)\right| \le a_1\left(|x_1|\right), \qquad \left|\nabla I_j(x_1)\right| \le a_1\left(|x_1|\right), \quad j \in B,\\ &\left|K_m(x_2)\right| \le a_2\left(|x_2|\right), \qquad \left|\nabla K_m(x_2)\right| \le a_2\left(|x_2|\right), \quad m \in C \end{aligned}$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$.

For $N = 1$, $p = q = 2$, $F(t,x_1,x_2) = F(t,x_1)$, and $I_j \equiv 0$ $(j \in B)$, system (1.1) reduces to the following second order impulsive differential equation:

$$\begin{cases} u_1''(t) + u_1(t) = \lambda_1 \nabla_{u_1} F(t, u_1(t)), \quad \text{a.e. } t \in [0, +\infty),\\ u_1'(0) = u_1(0),\\ u_1'(T) = 0,\\ \Delta(u_1'(t_j)) = I_j(u_1(t_j)), \quad j \in B. \end{cases} \tag{1.2}$$

Recently, Chen and Sun [2] investigated the following second order impulsive differential equation:

$$\begin{cases} u_1''(t) + u_1(t) = \lambda f(t, u_1(t)), \quad \text{a.e. } t \in [0, +\infty),\\ u_1'(0^+) = g(u_1(0)),\\ u_1'(+\infty) = 0,\\ \Delta(u_1'(t_j)) = I_j(u_1(t_j)), \quad j \in B, \end{cases} \tag{1.3}$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$, $g, I_j \in C(\mathbb{R}; \mathbb{R})$. In [2], the authors not only established the variational structure of equation (1.3) but also obtained that (1.3) enjoys three solutions by using an abstract critical point theorem taken from [3]. More precisely, they obtained the following theorem.

Theoremm A ([2], Theorem 3.1) *Suppose that*

(H1) *$g(u), I_j(u)$ are nondecreasing, and $g(u)u \ge 0$, $I_j(u)u \ge 0$ for any $u \in \mathbb{R}$;*

(H2) *There exist $a > 0, l \in (0, 2)$, $b \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, and $c \in L^2(\mathbb{R}_+; \mathbb{R}_+)$ such that*

$$F(t,u) \le b(t)\left(a + |u|^l\right), \qquad f(t,u) \le c(t)|u|^{l-1},$$

for a.e. $t \ge 0$ and $u \in \mathbb{R}$, where $F(t,u) := \int_0^u f(t,s)\,ds$;

(H3) *There exist $d, m, M > 0$ such that*

$$\frac{d^2}{M^2} < m^2 + 2\sum_{j=1}^{l} \int_0^{me^{-t_j}} I_j(s)\,ds + 2\int_0^m g(s)\,ds;$$

(H4)

$$\frac{M^2 \int_0^{+\infty} \max_{|\xi|\le d} F(t,\xi)\,dt}{d^2} < \frac{\int_0^{+\infty} F(t, me^{-t})\,dt}{m^2 + 2\sum_{j=1}^{l} \int_0^{me^{-t_j}} I_j(s)\,ds + 2\int_0^m g(s)\,ds}.$$

Then, for each

$$\lambda \in \left[\frac{\frac{m^2}{2} + \sum_{j=1}^{l} \int_0^{me^{-t_j}} I_j(s)\,ds + \int_0^m g(s)\,ds}{\int_0^{+\infty} F(t, me^{-t})\,dt}, \frac{d^2}{2M^2 \int_0^{+\infty} \max_{|\xi|\le d} F(t,\xi)\,dt}\right],$$

(1.3) *has at least three classical solutions.*

Also, Dai and Zhang [4] showed by using the least action principle that (1.3) has at least one solution if the potential function has subquadratic growth and, by taking advantage of the fountain theorem due to [5], that (1.3) has infinitely many solutions if the potential function is even.

To be precise, they obtained the following theorems.

Theoremm B ([4], Theorem 3.1) *Suppose that*

(S1) $(I_j)_{j\in B}$ *and* g *satisfy* $\int_0^u I_j(s)\,ds \ge 0$ *and* $\int_0^u g(s)\,ds \ge 0$, $u \in \mathbb{R}$, *respectively*;

(S2) *There exist* $a > 0$, $\alpha \in (1,2)$, *and* $b \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ *such that*

$$F(t,u) \le b(t)\big(a + |u|^\alpha\big)$$

for a.e. $t \ge 0$ *and all* $u \in \mathbb{R}$.

Then, for $\lambda > 0$, (1.3) *has at least one classical solution.*

Theoremm C ([4], Theorem 3.2) *Besides* (S1) *above, for a.e.* $t \ge 0$ *and all* $u \in \mathbb{R}$, *assume that*

(S3) *There exist* $\alpha \in (1,2)$ *and* $d \in L^{\frac{2}{2-\alpha}}(\mathbb{R}_+;\mathbb{R}_+)$ *such that*

$$F(t,u) \ge d(t)|u|^\alpha;$$

(S4) *There exist* $\gamma \in (0,1)$ *and* $h_1, h_2 \in L^1(\mathbb{R}_+;\mathbb{R}_+)$ *such that*

$$f(t,u) \le h_1(t)|u|^\gamma + h_2(t);$$

(S5) *There exist* $\gamma_j > \alpha - 1$, $\theta > \alpha - 1$, *and* $q_j, q > 0, j \in B$, *such that*

$$I_j(u) \le q_j|u|^{\gamma_j}, \qquad g(u) \le q|u|^\theta;$$

(S6) $f(t,u)$, $I_j(u)$, *and* $g(u)$ *are odd about* u.

Then, for any $\lambda > 0$, (1.3) *has infinitely many solutions*.

Recently, by applying the least action principle and saddle point theorem, [6–8] investigated the existence of periodic solutions for the following dynamical systems:

$$\begin{cases} \frac{d}{dt}\Phi_q(\dot{u}_1(t)) = \nabla_{u_1}F(t,u_1(t),u_2(t)), & \text{a.e. } t \in [0,T], \\ \frac{d}{dt}\Phi_p(\dot{u}_2(t)) = \nabla_{u_2}F(t,u_1(t),u_2(t)), & \text{a.e. } t \in [0,T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0 \end{cases} \tag{1.4}$$

and

$$\begin{cases} \frac{d}{dt}\Phi_q(\dot{u}_1(t)) + \nabla_{u_1}F(t,u_1(t),u_2(t)) = 0, & \text{a.e. } t \in [0,T], \\ \frac{d}{dt}\Phi_p(\dot{u}_2(t)) + \nabla_{u_2}F(t,u_1(t),u_2(t)) = 0, & \text{a.e. } t \in [0,T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \end{cases} \tag{1.5}$$

respectively. Subsequently, by variational approach, Yang and Chen [9, 10] discussed the existence and multiplicity of periodic solutions for the following two classes of nonlinear (q,p)-Laplacian dynamical systems with impulsive effects:

$$\begin{cases} \frac{d(\rho_1(t)\Phi_q(\dot{u}_1(t)))}{dt} - \rho_2(t)\Phi_\lambda(u_1(t)) + \nabla_{u_1}F(t,u_1(t),u_2(t)) = 0, & \text{a.e. } t \in [0,T], \\ \frac{d(\gamma_1(t)\Phi_p(\dot{u}_2(t)))}{dt} - \gamma_2(t)\Phi_\eta(u_2(t)) + \nabla_{u_2}F(t,u_1(t),u_2(t)) = 0, & \text{a.e. } t \in [0,T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \\ \Delta(\rho_1(t_j)\Phi_q(\dot{u}_1(t_j))) = \nabla I_j(u_1(t_j)), & j \in B, \\ \Delta(\gamma_1(s_m)\Phi_p(\dot{u}_2(s_m))) = \nabla K_m(u_2(s_m)), & m \in C, \end{cases} \tag{1.6}$$

and

$$\begin{cases} \frac{d(\Phi_q(\dot{u}_1(t)))}{dt} = \nabla_{u_1}F(t,u_1(t),u_2(t)), & \text{a.e. } t \in [0,T], \\ \frac{d(\Phi_p(\dot{u}_2(t)))}{dt} = \nabla_{u_2}F(t,u_1(t),u_2(t)), & \text{a.e. } t \in [0,T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \\ \Delta(\Phi_q(\dot{u}_1(t_j))) = \nabla I_j(u_1(t_j)), & j \in B, \\ \Delta(\Phi_p(\dot{u}_2(s_m))) = \nabla K_m(u_2(s_m)), & m \in C, \end{cases} \tag{1.7}$$

respectively, where $p,q,\lambda,\eta > 1$ and $\rho_1,\rho_2,\gamma_1,\gamma_2 \in C([0,T];\mathbb{R}_+)$.

Motivated by [2, 4, 6–10], in this paper, we are interested in the existence and multiplicity of a nontrivial weak solution for system (1.1) by using the least action principle and a critical point theorem due to Ding [1]. To be precise, we obtain the following results.

Theorem 1.1 *Suppose that*

(HIK1) *For* $x_1, x_2 \in \mathbb{R}^N$,

$$\sum_{j=1}^{l} I_j(x_1) \geq 0, \qquad \sum_{m=1}^{k} K_m(x_2) \geq 0;$$

(HF1) *There exist* $\alpha_1 \in [0,q)$, $\alpha_2 \in [0,p)$, $a_1 > 0$, *and* $d_1 \in L^1([0,T];\mathbb{R}_+)$ *such that*

$$F(t,x_1,x_2) \le d_1(t)\big(a_1 + |x_1|^{\alpha_1} + |x_2|^{\alpha_2}\big), \quad \forall (x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Then, for each $\lambda > 0$, *system* (1.1) *has at least one weak solution in* $X_q \times X_p$, *where, for* $s > 1$,

$$X_s = \big\{u : [0,T] \to \mathbb{R}^N | u \text{ is absolutely continuous and } \dot{u} \in L^s[0,T]\big\}.$$

Remark 1.1 There exist examples satisfying Theorem 1.1. For example, let $q = 4$, $p = 3$,

$$I_j(x_1) = \big(|x_1| + c_1\big)^{\xi_1} \quad (j \in B), \qquad K_m(x_2) = \ln\big(|x_2| + c_2\big)^{\xi_2} \quad (m \in C), \tag{1.8}$$

where $c_1, c_2, \xi_1, \xi_2 > 0$, and for all $t \in [0,T]$,

$$F(t,x_1,x_2) = \sin t|x_1|^3 + \cos t|x_2|^2,$$

or

$$F(t,x_1,x_2) = t^2 \ln\big(1 + |x_1|^2\big) + \frac{e^t|x_2|^4}{1 + |x_2|^2}.$$

Theorem 1.2 *In addition to* (HIK1), *we assume that*

(HF2) *There exist* $a_2 > 0$ *and* $d_2 \in L^1([0,T];\mathbb{R}_+)$ *such that*

$$F(t,x_1,x_2) \le d_2(t)\big(a_2 + |x_1|^q + |x_2|^p\big), \quad \forall (x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Then, for each $0 < \lambda < \min\{\frac{1}{q(D_0(q))^q}, \frac{1}{p(D_0(p))^p}\}$, (1.1) *has at least one weak solution in* $X_q \times X_p$, *where*

$$D_0(s) := \left(T^{-\frac{1}{s-1}} + \frac{(s-1)T}{2^{\frac{s}{s-1}} \cdot (2s-1)}\right)^{\frac{s-1}{s}}, \quad s = p, q.$$

Remark 1.2 There exist examples satisfying Theorem 1.2. For example, let $q = 4$, $p = 3$, and I_j, K_m defined by (1.8). For all $t \in [0,T]$, let

$$F(t,x_1,x_2) = e^t|x_1|^4 + \sin t|x_2|^3,$$

or

$$F(t,x_1,x_2) = t^2|x_1|^3 \ln\big(1 + |x_1|^2\big) + \big(t^3 + 1\big)|x_2|^3.$$

Theorem 1.3 *Along with* (HIK1) *and* (HF2), *for* $x_1, x_2 \in \mathbb{R}^N$, $j \in B, m \in C$, *and* $t \in [0,T]$, *we suppose that*

(HIK2) *There exist* $\nu_1 \ge q$, $\nu_2 \ge q$, *and* $\delta_0 > 0$ *such that*

$$I_j(x_1) \le d_3|x_1|^{\nu_1}, \qquad K_m(x_2) \le d_4|x_2|^{\nu_2}, \quad |x| \le \delta_0;$$

(HIK3) $I_j(x_1) = I_j(-x_1)$, $K_m(x_2) = K_m(-x_2)$, $I_j(0) = 0$, $K_m(0) = 0$;

(HF3) *There exist* $\mu_1 \in (1,q)$, $\mu_2 \in (1,p)$, $d_5 > 0$, *and* $\delta_1 > 0$ *such that*

$$F(t,x_1,x_2) \geq d_5\big(|x_1|^{\mu_1} + |x_2|^{\mu_2}\big), \quad |x_1| \leq \delta_1, |x_2| \leq \delta_1;$$

(HF5) $F(t,x_1,x_2) = F(t,-x_1,-x_2), F(t,0,0) \equiv 0$.

Then, for each $0 < \lambda < \min\{\frac{1}{q(D_0(q))^q}, \frac{1}{p(D_0(p))^p}\}$, (1.1) *has infinitely many weak solutions in* $X_q \times X_p$.

Remark 1.3 There exist examples satisfying Theorem 1.3. For example, let $q = 4$, $p = 3$, and

$$I_j(x_1) = c_3|x_1|^5 \quad (j \in B), \qquad K_m(x_2) = c_4|x_2|^4 \quad (m \in C),$$

where $c_3, c_4 > 0$. For all $t \in [0,T]$, let

$$F(t,x_1,x_2) = \big(e^t + 1\big)|x_1|^3 + \big(t^2 + 1\big)|x_2|^2.$$

If we take $\nu_1 = 4.5$, $\nu_2 = 3.5$, $\mu_1 = 3.5$, and $\mu_2 = 2.5$, it is easy to see that the example satisfies Theorem 1.3.

2 Variational structure and some preliminaries

For $u \in X_s$ with $s = q,p$, define

$$\|u\|_{X_s} = \left(\int_0^T \big|u(t)\big|^s\,dt + \int_0^T \big|\dot{u}(t)\big|^s\,dt\right)^{1/s}.$$

Set

$$\|u\|_s := \left(\int_0^T \big|u(t)\big|^s\,dt\right)^{1/s} \quad \text{and} \quad \|u\|_\infty := \max_{t\in[0,T]} \big|u(t)\big|.$$

Set $X := X_q \times X_p$ and define the norm $\|(u_1,u_2)\|_X = \|u_1\|_{X_q} + \|u_2\|_{X_p}$. Obviously, X is a reflexive Banach space. Let

$$\mathcal{C} = \big\{u : [0,T] \to \mathbb{R}^N | u \text{ is continuous}\big\}.$$

X_s embeds into $\mathcal{C}$ continuously and, according to [11], Lemma 2.4,

$$\|u\|_\infty \leq D_0(s)\|u\|_{X_s} \quad \text{for any } u \in X_s. \tag{2.1}$$

Lemma 2.1 ([12], Proposition 1.2) *If* u_k *converges to* u *weakly, then* u_k *uniformly converges to* u *on* $[0,T]$.

If $u \in X_s$, then u is absolutely continuous, whereas $\dot{u}$ need not be continuous. Hence, it is possible that $\Delta\Phi_s(\dot{u}(t)) = \Phi_s(\dot{u}(t^+)) - \Phi_s(\dot{u}(t^-)) \neq 0$, which leads to impulse effects.

Following the idea [13], multiplying by $v_1 \in X_q$ on both sides of the first equation in (1.1) and integrating from 0 to T yields that

$$\int_0^T \left[-\frac{d}{dt}\big(\big|\dot{u}_1(t)\big|^{q-2}\dot{u}_1(t)\big) + \big|u_1(t)\big|^{q-2}u_1(t) - \lambda\nabla_{x_1}F\big(t,u_1(t),u_2(t)\big)\right]v_1(t)\,dt = 0. \tag{2.2}$$

Since v_1 is continuous, $v_1(t_j^-) = v_1(t_j^+) = v_1(t_j)$. Combining $\dot{u}_1(T) = 0$ with $\dot{u}_1(0) = u_1(0)$ implies that

$$\begin{aligned}
&\int_0^T \left(\frac{d(\Phi_q(\dot{u}_1(t))}{dt}, v_1(t) \right) dt \\
&= \sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} \left(\frac{d(\Phi_q(\dot{u}_1(t)))}{dt}, v_1(t) \right) dt \\
&= \sum_{j=0}^{l} \big[\big(\Phi_q\big(\dot{u}_1\big(t_{j+1}^-\big)\big), v_1\big(t_{j+1}^-\big)\big) - \big(\Phi_q\big(\dot{u}_1\big(t_j^+\big)\big), v_1\big(t_j^+\big)\big) \big] dt \\
&\quad - \sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} \big(\Phi_q\big(\dot{u}_1(t)\big), \dot{v}_1(t)\big) dt \\
&= \big(\Phi_q\big(\dot{u}_1(T)\big), v_1(T)\big) - \big(\Phi_q\big(\dot{u}_1(0)\big), v_1(0)\big) \\
&\quad - \sum_{j=1}^{l} \big(\Delta\Phi_q\big(\dot{u}_1(t_j)\big), v_1(t_j)\big) - \int_0^T \big(\Phi_q\big(\dot{u}_1(t)\big), \dot{v}_1(t)\big) dt \\
&= -\big(\Phi_q\big(u_1(0)\big), v_1(0)\big) - \sum_{j=1}^{l} \big(\Delta\Phi_q\big(\dot{u}_1(t_j)\big), v_1(t_j)\big) - \int_0^T \big(\Phi_q\big(\dot{u}_1(t)\big), \dot{v}_1(t)\big) dt \\
&= -\big(\Phi_q\big(u_1(0)\big), v_1(0)\big) - \sum_{j=1}^{l} \big(\nabla I_j\big(u_1(t_j)\big), v_1(t_j)\big) - \int_0^T \big(\Phi_q\big(\dot{u}_1(t)\big), \dot{v}_1(t)\big) dt,
\end{aligned}$$

which, together with (2.2), further leads to

$$\begin{aligned}
&\big(\Phi_q\big(u_1(0)\big), v_1(0)\big) + \sum_{j=1}^{l} \big(\nabla I_j\big(u_1(t_j)\big), v_1(t_j)\big) + \int_0^T \big(\Phi_q\big(\dot{u}_1(t)\big), \dot{v}_1(t)\big) dt \\
&\quad + \int_0^T \big|u_1(t)\big|^{q-2} \big(u_1(t), v_1(t)\big) dt - \lambda \int_0^T \big(\nabla_{x_1} F\big(t, u_1(t), u_2(t)\big), v_1(t)\big) dt = 0.
\end{aligned}$$

Analogously, for any $v_2 \in X_p$,

$$\begin{aligned}
&\big(\Phi_p\big(u_2(0)\big), v_2(0)\big) + \sum_{m=1}^{k} \big(\nabla K_m\big(u_2(t_m)\big), v_2(t_m)\big) + \int_0^T \big(\Phi_p\big(\dot{u}_2(t)\big), \dot{v}_2(t)\big) dt \\
&\quad + \int_0^T \big|u_2(t)\big|^{p-2} \big(u_2(t), v_2(t)\big) dt - \lambda \int_0^T \big(\nabla_{x_2} F\big(t, u_1(t), u_2(t)\big), v_2(t)\big) dt = 0.
\end{aligned}$$

With the two equalities above in hand, we present the notion of weak solutions for (1.1).

Definition 2.1 For any $v = (v_1, v_2) \in X_q \times X_p$, if

$$\begin{aligned}
&\big(\Phi_q\big(u_1(0)\big), v_1(0)\big) + \sum_{j=1}^{l} \big(\nabla I_j\big(u_1(t_j)\big), v_1(t_j)\big) + \int_0^T \big(\Phi_q\big(\dot{u}_1(t)\big), \dot{v}_1(t)\big) dt \\
&\quad + \int_0^T \big|u_1(t)\big|^{q-2} \big(u_1(t), v_1(t)\big) dt - \lambda \int_0^T \big(\nabla_{x_1} F\big(t, u_1(t), u_2(t)\big), v_1(t)\big) dt = 0
\end{aligned}$$

and

$$\begin{aligned}&\big(\Phi_p\big(u_2(0)\big),v_2(0)\big)+\sum_{m=1}^{k}\big(\nabla K_m\big(u_2(t_m)\big),v_2(t_m)\big)+\int_0^T\big(\Phi_p\big(\dot{u}_2(t)\big),\dot{v}_2(t)\big)\,dt\\&\quad+\int_0^T\big|u_2(t)\big|^{p-2}\big(u_2(t),v_2(t)\big)\,dt-\lambda\int_0^T\big(\nabla_{x_2}F\big(t,u_1(t),u_2(t)\big),v_2(t)\big)\,dt=0,\end{aligned}$$

then $u=(u_1,u_2)\in X_q\times X_p$ is called a weak solution of (1.1).

For $u=(u_1,u_2)\in X_q\times X_p$, define the functional $\varphi:X\to\mathbb{R}$ by

$$\begin{aligned}\varphi(u)&=\varphi(u_1,u_2)\\&=\frac{1}{q}\int_0^T\big|\dot{u}_1(t)\big|^q\,dt+\frac{1}{p}\int_0^T\big|\dot{u}_2(t)\big|^p\,dt-\lambda\int_0^T F\big(t,u_1(t),u_2(t)\big)\,dt\\&\quad+\frac{1}{q}\int_0^T\big|u_1(t)\big|^q\,dt+\frac{1}{p}\int_0^T\big|u_2(t)\big|^p\,dt\\&\quad+\sum_{j=1}^{l}I_j\big(u_1(t_j)\big)+\sum_{m=1}^{k}K_m\big(u_2(s_m)\big)\\&\quad+\frac{1}{q}\big|u_1(0)\big|^q+\frac{1}{p}\big|u_2(0)\big|^p\\&=\phi(u_1,u_2)+\psi(u_1,u_2),\end{aligned}$$

where

$$\begin{aligned}\phi(u_1,u_2)&:=\frac{1}{q}\int_0^T\big|\dot{u}_1(t)\big|^q\,dt+\frac{1}{p}\int_0^T\big|\dot{u}_2(t)\big|^p\,dt-\lambda\int_0^T F\big(t,u_1(t),u_2(t)\big)\,dt\\&\quad+\frac{1}{q}\int_0^T\big|u_1(t)\big|^q\,dt+\frac{1}{p}\int_0^T\big|u_2(t)\big|^p\,dt\\&\quad+\frac{1}{q}\big|u_1(0)\big|^q+\frac{1}{p}\big|u_2(0)\big|^p,\\\psi(u_1,u_2)&:=\sum_{j=1}^{l}I_j\big(u_1(t_j)\big)+\sum_{m=1}^{k}K_m\big(u_2(s_m)\big).\end{aligned}$$

By virtue of (A1) and (A2), by following the argument of [12], Theorem 1.4, one has $\phi\in C^1(X_q\times X_p,\mathbb{R})$. Thanks to continuous differentiability of $(I_j)_{j\in B}$ and $(K_m)_{m\in C}$, we have $\psi\in C^1(X_q\times X_p,\mathbb{R})$. As a consequence, $\varphi\in C^1(X,\mathbb{R})$ and, for all $(v_1,v_2)\in X_q\times X_p$,

$$\begin{aligned}&\big\langle\varphi'(u_1,u_2),(v_1,v_2)\big\rangle\\&\quad=\int_0^T\big(\Phi_q\big(\dot{u}_1(t)\big),\dot{v}_1(t)\big)\,dt+\int_0^T\big(\Phi_p\big(\dot{u}_2(t)\big),\dot{v}_2(t)\big)\,dt\\&\qquad+\int_0^T\big(\Phi_q\big(u_1(t)\big),v_1(t)\big)\,dt+\int_0^T\big(\Phi_p\big(u_2(t)\big),v_2(t)\big)\,dt\\&\qquad+\big(\Phi_q\big(u_1(0)\big),v_1(0)\big)+\big(\Phi_p\big(u_2(0)\big),v_2(0)\big)\end{aligned}$$

$$-\lambda \int_0^T \big(\nabla_{x_1} F\big(t, u_1(t), u_2(t)\big), v_1(t)\big)\, dt - \lambda \int_0^T \big(\nabla_{x_2} F\big(t, u_1(t), u_2(t)\big), v_2(t)\big)\, dt$$
$$+ \sum_{j=1}^{l} \big(\nabla I_j\big(u_1(t_j)\big), v_1(t_j)\big) + \sum_{m=1}^{k} \big(\nabla K_m\big(u_2(s_m)\big), v_2(s_m)\big).$$

Definition 2.1 shows that the critical point of φ is the weak solution of system (1.1).

The following lemma plays a crucial role in achieving the critical point of φ.

Lemma 2.2 ([14]) *Assume that $\varphi \in C^1(E, \mathbb{R})$ is bounded from below (above) and satisfies the (PS) condition. Then*

$$c = \inf_{u \in E} \varphi(u) \quad \Big(c = \sup_{u \in E} \varphi(u)\Big)$$

is a critical value of φ.

Lemma 2.3 ([1]) *Let E be an infinite dimensional Banach space, and let $\varphi \in C^1(E, \mathbb{R})$ with $\varphi(0) = 0$ be even and satisfy (PS). If $E = E_1 \oplus E_2$, where E_1 is finite dimensional, and φ satisfies that*

(φ_1) *φ is bounded from above on E_2;*

(φ_2) *for each finite dimensional subspace $\tilde{E} \subset E$, there are positive constants $\rho = \rho(\tilde{E})$ and $\sigma = \sigma(\tilde{E})$ such that $\varphi \geq 0$ on $B_\rho \cap \tilde{E}$ and $\varphi|_{\partial B_\rho \cap \tilde{E}} \geq \sigma$, where $B_\rho = \{x \in E; \|x\| \leq \rho\}$,*

then φ possesses infinitely many nontrivial critical points.

3 Proofs of theorems

Proof of Theorem 1.1 It follows from (HIK1), (HF1), and (2.1) that

$$\begin{aligned}
\varphi(u) &= \varphi(u_1, u_2) \\
&= \frac{1}{q} \int_0^T \big|\dot{u}_1(t)\big|^q\, dt + \frac{1}{p} \int_0^T \big|\dot{u}_2(t)\big|^p\, dt - \lambda \int_0^T F\big(t, u_1(t), u_2(t)\big)\, dt \\
&\quad + \frac{1}{q} \int_0^T \big|u_1(t)\big|^q\, dt + \frac{1}{p} \int_0^T \big|u_2(t)\big|^p\, dt \\
&\quad + \sum_{j=1}^{l} I_j\big(u_1(t_j)\big) + \sum_{m=1}^{k} K_m\big(u_2(s_m)\big) + \frac{1}{q}\big|u_1(0)\big|^q + \frac{1}{p}\big|u_2(0)\big|^p \\
&\geq \frac{1}{q}\|u_1\|_{X_q}^q + \frac{1}{p}\|u_2\|_{X_p}^p - \lambda \int_0^T d_1(t)\big(a_1 + \big|u_1(t)\big|^{\alpha_1} + \big|u_2(t)\big|^{\alpha_2}\big)\, dt \\
&\geq \frac{1}{q}\|u_1\|_{X_q}^q + \frac{1}{p}\|u_2\|_{X_p}^p - \lambda \|u_1\|_\infty^{\alpha_1} \int_0^T d_1(t)\, dt \\
&\quad -\lambda \|u_2\|_\infty^{\alpha_2} \int_0^T d_1(t)\, dt - \lambda a_1 \int_0^T d_1(t)\, dt \\
&\geq \frac{1}{q}\|u_1\|_{X_q}^q + \frac{1}{p}\|u_2\|_{X_p}^p - \lambda\big(D_0(q)\big)^{\alpha_1} \|u_1\|_{X_q}^{\alpha_1} \int_0^T d_1(t)\, dt \\
&\quad - \lambda\big(D_0(p)\big)^{\alpha_2} \|u_2\|_{X_p}^{\alpha_2} \int_0^T d_1(t)\, dt - \lambda a_1 \int_0^T d_1(t)\, dt.
\end{aligned}$$

Owing to $\alpha_1 \in [0,q)$ and $\alpha_2 \in [0,p)$, we readily obtain that $\varphi(u) \to +\infty$ as $\|u\|_X \to \infty$, i.e., φ satisfies the coercive condition on X. So φ is bounded below on X.

Hereinafter, we claim that φ satisfies the (PS) condition. If $\{\varphi(u_{1n}, u_{2n})\}$ is bounded and $\|\varphi'(u_{1n}, u_{2n})\| \to 0$ as $n \to \infty$, then there exists a positive constant D_1 such that

$$\bigl|\varphi(u_{1n}, u_{2n})\bigr| \le D_1, \qquad \bigl\|\varphi'(u_{1n}, u_{2n})\bigr\| \le D_1, \quad \forall n \in \mathbb{N}.$$

Since φ satisfies a coercive condition on X, we infer that $\|u_{1n}\|_{X_q}$ and $\|u_{2n}\|_{X_p}$ is bounded. Next, in light of the reflexive property of X_s, there exists a subsequence, still denoted by $\{u_n = (u_{1n}, u_{2n})\}$, such that

$$u_{1n} \rightharpoonup u_1 \quad \text{on } X_q, \qquad u_{2n} \rightharpoonup u_2 \quad \text{on } X_p.$$

Thus, Lemma 2.1 gives that

$$u_{1n} \to u_1 \quad \text{in } C\bigl(0,T;\mathbb{R}^N\bigr) \quad \text{and} \quad u_{2n} \to u_2 \quad \text{in } C\bigl(0,T;\mathbb{R}^N\bigr).$$

Following the argument in [15–17], we can derive that $\|u_n - u\|_X \to 0$, where $u = (u_1, u_2)$. Consequently, φ satisfies the (PS) condition. Thus, with the help of Lemma 2.2, we deduce that φ has at least one critical point on X. Hence system (1.1) has at least one solution on X. □

Proof of Theorem 1.2 By (HIK1), (HF2), and (2.1), it follows that

$$\begin{aligned}
\varphi(u) &= \varphi(u_1, u_2) \\
&= \frac{1}{q}\int_0^T \bigl|\dot{u}_1(t)\bigr|^q\,dt + \frac{1}{p}\int_0^T \bigl|\dot{u}_2(t)\bigr|^p\,dt - \lambda\int_0^T F\bigl(t, u_1(t), u_2(t)\bigr)\,dt \\
&\quad + \frac{1}{q}\int_0^T \bigl|u_1(t)\bigr|^q\,dt + \frac{1}{p}\int_0^T \bigl|u_2(t)\bigr|^p\,dt \\
&\quad + \sum_{j=1}^{l} I_j\bigl(u_1(t_j)\bigr) + \sum_{m=1}^{k} K_m\bigl(u_2(s_m)\bigr) \\
&\quad + \frac{1}{q}\bigl|u_1(0)\bigr|^q + \frac{1}{p}\bigl|u_2(0)\bigr|^p \\
&\ge \frac{1}{q}\|u_1\|_{X_q}^q + \frac{1}{p}\|u_2\|_{X_p}^p - \lambda\int_0^T \bigl[d_2(t)\bigl(a_2 + \bigl|u_1(t)\bigr|^q + \bigl|u_2(t)\bigr|^p\bigr)\bigr]\,dt \\
&\ge \frac{1}{q}\|u_1\|_{X_q}^q + \frac{1}{p}\|u_2\|_{X_p}^p - \lambda\|u_1\|_\infty^q\int_0^T d_2(t)\,dt \\
&\quad - \lambda\|u_2\|_\infty^p\int_0^T d_2(t)\,dt - \lambda a_2\int_0^T d_2(t)\,dt \\
&\ge \frac{1}{q}\|u_1\|_{X_q}^q + \frac{1}{p}\|u_2\|_{X_p}^p - \lambda\bigl(D_0(q)\bigr)^q\|u_1\|_{X_q}^q\int_0^T d_2(t)\,dt \\
&\quad - \lambda\bigl(D_0(p)\bigr)^p\|u_2\|_{X_p}^p\int_0^T d_2(t)\,dt - \lambda a_2\int_0^T d_2(t)\,dt.
\end{aligned} \tag{3.1}$$

In view of $\lambda < \min\{\frac{1}{q(D_0(q))^q}, \frac{1}{p(D_0(p))^p}\}$, one has $\varphi(u) \to +\infty$ as $\|u\|_X \to \infty$, that is, φ satisfies the coercive condition on X. Hence φ is bounded below on X. By carrying out a similar argument to derive Theorem 1.1, we get that system (1.1) has at least one solution in X. □

Proof of Theorem 1.3 Keep in mind that φ and $-\varphi$ have the same critical points. Let $\Theta = -\varphi$. In the sequel, we aim at verifying that all conditions in Lemma 2.3 are fulfilled by Θ. In fact, from (HIK3) and (HF5), we find that Θ is even and $\Theta(0) = 0$. Taking (HIK1), (HF2), and (3.1) into account, we obtain that $\Theta(u) \to -\infty$ as $\|u\|_X \to \infty$. Hence Θ is bounded above on X so that Θ satisfies (φ_1) in Lemma 2.3.

Assume that $\tilde{X} \subset X$ is finite-dimensional. For any $u = (u_1, u_2) \in \tilde{X} = \tilde{X}_q \times \tilde{X}_p$, where $\tilde{X}_q \subset X_q$ and $\tilde{X}_p \subset X_p$, we deduce that $\|u_1\|_{\mu_1}$ is equivalent to $\|u_1\|_{X_q}$, and $\|u_2\|_{\mu_2}$ is equivalent to $\|u_2\|_{X_p}$. Hence there exist constants $d_6, d_7 > 0$ such that

$$\|u_1\|_{\mu_1} \geq d_6\|u_1\|_{X_q}, \qquad \|u_2\|_{\mu_2} \geq d_7\|u_2\|_{X_p}. \tag{3.2}$$

Let $\rho_0 = \min\{\frac{\min\{\delta_0,\delta_1\}}{D_0(q)}, \frac{\min\{\delta_0,\delta_1\}}{D_0(p)}\}$. For any $\rho \in (0, \rho_0)$, if $\|u\|_X = \rho$, then $\|u_1\|_\infty \leq D_0(q)\|u_1\|_{X_q} \leq D_0(q)\rho \leq \min\{\delta_0, \delta_1\}$ and $\|u_2\|_\infty \leq D_0(p)\|u_2\|_{X_q} \leq D_0(p)\rho \leq \min\{\delta_0, \delta_1\}$. Thus it follows from (HIK2), (HF3), (3.2), and Hölder's inequality that

$$\begin{aligned}
\Theta(u) &= -\varphi(u_1, u_2) \\
&= -\frac{1}{q}\int_0^T \left|\dot{u}_1(t)\right|^q dt - \frac{1}{p}\int_0^T \left|\dot{u}_2(t)\right|^p dt + \lambda\int_0^T F\bigl(t, u_1(t), u_2(t)\bigr)\,dt \\
&\quad - \frac{1}{q}\int_0^T \left|u_1(t)\right|^q dt - \frac{1}{p}\int_0^T \left|u_2(t)\right|^p dt \\
&\quad - \sum_{j=1}^{l} I_j\bigl(u_1(t_j)\bigr) - \sum_{m=1}^{k} K_m\bigl(u_2(s_m)\bigr) \\
&\quad - \frac{1}{q}\left|u_1(0)\right|^q - \frac{1}{p}\left|u_2(0)\right|^p \\
&\geq -\frac{1}{q}\|u_1\|_{X_q}^q - \frac{1}{p}\|u_2\|_{X_p}^p + \lambda d_5\int_0^T \left(\left|u_1(t)\right|^{\mu_1} + \left|u_2(t)\right|^{\mu_2}\right) dt \\
&\quad - \sum_{j=1}^{l} d_3\left|u_1(t_j)\right|^{\nu_1} - \sum_{m=1}^{k} d_4\left|u_2(s_m)\right|^{\nu_2} - \frac{1}{q}\|u_1\|_\infty^q - \frac{1}{p}\|u_2\|_\infty^p \\
&\geq -\frac{1}{q}\|u_1\|_{X_q}^q - \frac{1}{p}\|u_2\|_{X_p}^p + \lambda d_5 d_6^{\mu_1}\|u_1\|_{X_q}^{\mu_1} + \lambda d_5 d_7^{\mu_2}\|u_2\|_{X_p}^{\mu_2} \\
&\quad - l d_3\|u_1\|_\infty^{\nu_1} - k d_4\|u_2\|_\infty^{\nu_2} \\
&\quad - \frac{1}{q}\bigl(D_0(q)\bigr)^q\|u_1\|_{X_q}^q - \frac{1}{p}\bigl(D_0(p)\bigr)^p\|u_2\|_{X_p}^p \\
&\geq -\frac{1}{q}\|u_1\|_{X_q}^q - \frac{1}{p}\|u_2\|_{X_p}^p + \lambda d_5 d_6^{\mu_1}\|u_1\|_{X_q}^{\mu_1} + \lambda d_5 d_7^{\mu_2}\|u_2\|_{X_p}^{\mu_2} \\
&\quad - l d_3\bigl(D_0(q)\bigr)^{\nu_1}\|u_1\|_{X_q}^{\nu_1} - k d_4\bigl(D_0(p)\bigr)^{\nu_2}\|u_2\|_{X_p}^{\nu_2} \\
&\quad - \frac{1}{q}\bigl(D_0(q)\bigr)^q\|u_1\|_{X_q}^q - \frac{1}{p}\bigl(D_0(p)\bigr)^p\|u_2\|_{X_p}^p.
\end{aligned}$$

Observing that $\mu_1 \in (1,q)$ and $\mu_2 \in (1,p)$, we take sufficiently small $\rho \in (0,\rho_0)$ such that $\Theta(u) \geq 0$ on $B_\rho \cap \tilde{X}$ and $\Theta(u) > 0$ on $\partial B_\rho \cap \tilde{X}$. Therefore, Θ satisfies (φ_2) in Lemma 2.3. Then, according to Lemma 2.3, Θ has infinitely many critical points in X so that (1.1) has infinitely many solutions in X. □

Competing interests
The author declares that she has no competing interests.

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