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# On a hybrid spectral exponential Chebyshev method for time-fractional coupled Burgers equations on a semi-infinite domain

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## Abstract

In this study we propose a hybrid spectral exponential Chebyshev method (HSECM) for solving time-fractional coupled Burgers equations (TFCBEs). The method is based upon a spectral collection method, utilizing exponential Chebyshev functions in space and trapezoidal quadrature formula (TQF), and also a finite difference method (FDM) for time-fractional derivative. Some test examples are included to demonstrate the efficiency and validity of the proposed method.

**Keywords:** exponential Chebyshev; fractional coupled Burgers equation; trapezoidal quadrature; finite difference; Chebyshev polynomials; spectral collection method

## 1 Introduction

Several computational problems in various research areas such as mathematics, fluid dynamics, chemistry, biology, viscoelasticity, engineering and physics have arisen in semi-infinite domains [1–9]. Subsequently, many researchers have utilized various transformations on orthogonal polynomials to map  $[-1, 1]$  into  $[0, \infty)$  maintaining their orthogonal property [10–15].

Spectral methods provide a computational approach that has become better known over the last decade and has become the topic of study for many researchers [16–26], especially when linked with the fractional calculus [9, 27–38] which is an important branch of applied mathematics. This type of differentiation and integration could be considered as a generalization of the usual definition of differentiation and integration to non-integer order.

In this paper, we study coupled Burgers equations with time-fractional derivatives given by

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + 2u \frac{\partial u}{\partial x} - \frac{\partial(uv)}{\partial x}, \quad 0 < \alpha < 1, \quad (1)$$

$$\frac{\partial^\beta v(x, t)}{\partial t^\beta} = \frac{\partial^2 v}{\partial x^2} + 2v \frac{\partial v}{\partial x} - \frac{\partial(uv)}{\partial x}, \quad 0 < \beta < 1 \quad (2)$$

on the semi-infinite domain  $[0, \infty)$ .

The coupled Burgers equations have recently been applied to different areas of science, in particular in physical problems such as the phenomena of turbulence flow through a shock wave traveling in a viscous fluid (see [39, 40]).

The study of coupled Burgers equations is very important because the system is a basic model of sedimentation or evolution of scaled volume concentrations of two sorts of particles in liquid suspensions or colloids under the impact of gravity [40]. It has been studied by many authors using various techniques [41–46].

In this paper, we introduce the exponential Chebyshev functions collocation method based upon orthogonal Chebyshev polynomials to solve a time-fractional coupled Burgers equation. The fractional derivative is defined in the Caputo sense for time variable which is discretized utilizing a trapezoidal quadrature formula (TQF) and a finite difference method (FDM).

The justification of this paper is to apply the Chebyshev exponential method for efficient applicable in unbounded domains with steady state property ( $u(\infty) = constant$ ), i.e., the solution to be regular at  $\infty$ . In fact, many problems in mathematical physics and astrophysics which occur on a semi-infinite interval are related to the diffusion equations such as Burgers, KdV and heat equations. Furthermore, many methods based on polynomials basis, such as Legendre, Chebyshev, Laguerre spectral methods and also semi-analytic methods such as Adomian decomposition, variational iteration and differential transform methods, can not justify the steady state property of fluid  $u(\infty) = constant$ . In this study we will show that such difficulty can be surmounted by our proposed method.

The error analysis of exponential Chebyshev functions expansion has also been investigated, which confirms the efficiency of the method.

## 2 Definitions and basic properties

In this section, we give some definitions and basic properties of fractional calculus and Chebyshev polynomials which are required for our subsequent development.

### 2.1 Definition of fractional calculus

Here we recall definition and basic results of fractional calculus; for more details, we refer to [32].

**Definition 1** A real function  $u(t)$ ,  $t > 0$  is said to be in the space  $C_\mu$ ,  $\mu \in \mathbb{R}$  if there exists a real number  $p > \mu$  such that  $u(t) = t^p u_1(t)$ , where  $u_1(t) \in C(0, \infty)$ , and it is said to be in the space  $C_\mu^n$  if and only if  $u^{(n)} \in C_\mu$ ,  $n \in \mathbb{N}$ .

**Definition 2** The Riemann-Liouville fractional integral operator of order  $\alpha > 0$ , of a function  $f \in C_\mu$ ,  $\mu \geq -1$ , is defined as

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad \alpha > 0,$$

$$I^0 u(t) = u(t),$$

where  $\Gamma(\cdot)$  is the well-known gamma function.

**Definition 3** The fractional derivative of  $u(t)$  in the Caputo sense is defined as

$$D^\alpha u(t) = I^{m-\alpha} D^m u(t),$$

for  $m - 1 < \alpha \leq m, m \in \mathbb{N}, t > 0$  and  $u \in C_{-1}^m$ . Also it can be rewritten in the following form:

$$D^\alpha u(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - x)^{m-\alpha-1} u^{(m)}(x) dx.$$

Similar to integer-order differentiation, Caputo fractional differential has the linear property:

$$D^\alpha (c_1 f_1(t) + c_2 f_2(t)) = c_1 D^\alpha f_1(t) + c_2 D^\alpha f_2(t),$$

where  $c_1$  and  $c_2$  are constants. If so, for the Caputo derivative, we have the following basic properties:

$$(i) \quad D^\alpha t^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{for } \gamma \in \mathbb{N}_0 \text{ and } \gamma \geq \lceil \alpha \rceil \text{ or } \gamma \notin \mathbb{N} \text{ and } \gamma > \lfloor \alpha \rfloor, \\ 0, & \text{for } \gamma \in \mathbb{N}_0, \end{cases} \quad (3)$$

$$(ii) \quad D^\alpha(c) = 0,$$

$$(iii) \quad I^\alpha D^\alpha u(t) = u(t) - \sum_{i=0}^{m-1} \frac{u^{(i)}(0)}{i!} t^i, \quad (4)$$

where  $c$  is constant,  $\lfloor \alpha \rfloor$  and  $\lceil \alpha \rceil$  are floor and ceiling functions, respectively,  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$  and  $\mathbb{N} = \{1, 2, \dots\}$ .

### 2.2 Exponential Chebyshev functions

The well-known first kind Chebyshev polynomials of degree  $n$ , defined on the interval  $[-1, 1]$ , are given by

$$T_n(s) = \cos(n \cos^{-1}(s)),$$

where  $s = \cos(\theta)$ , and thus the following property is immediately obtained:

$$T_n(s) = \cos(n\theta) \leq 1. \quad (5)$$

Also, we have the relation

$$T_{n+1}(s) = 2sT_n(s) - T_{n-1}(s), \quad n = 1, 2, 3, \dots, \quad (6)$$

where  $T_0(s) = 1$  and  $T_1(s) = s$ .  $T_n(s)$  is the eigenfunction of the singular Sturm-Liouville problem

$$\sqrt{1 - s^2} \partial_s (\sqrt{1 - s^2} \partial_s T_n(s)) + n^2 T_n(s) = 0. \quad (7)$$

The first kind Chebyshev polynomials are orthogonal in the interval  $[-1, 1]$  with respect to the weight function

$$w(s) = \frac{1}{\sqrt{1 - s^2}}.$$

The analytic form of the first kind Chebyshev polynomials of degree  $n$  is given by

$$T_n(s) = n \sum_{k=0}^n (-2)^k \frac{(n+k-1)!}{(n-k)!(2k)!} (1-s)^k, \quad n > 0. \tag{8}$$

From Eq. (5), the zeroes of  $T_n(s)$  are obtained as follows:

$$s_k = \cos\left(\pi \frac{2k+1}{2n}\right), \quad k = 0, 1, \dots, n-1. \tag{9}$$

**Definition 4** (Exponential Chebyshev functions) We can use exponential transformation to have new functions which are defined on the semi-infinite interval. The  $n$ th exponential Chebyshev functions can be defined by the one-to-one transformation

$$s = 1 - 2e^{-\frac{x}{L}}, \quad L > 0,$$

as

$$E_n(x) = T_n(s) = T_n\left(1 - 2e^{-\frac{x}{L}}\right). \tag{10}$$

According to (6) and (10), we may deduce the recurrences relation for  $E_n(x)$  in the form

$$E_{n+1}(x) = 2\left(1 - 2e^{-\frac{x}{L}}\right)E_n(x) - E_{n-1}(x),$$

with starting values  $E_0(x) = 1, E_1(x) = 1 - 2e^{-\frac{x}{L}}$ . The first few exponential Chebyshev functions of the first kind are as follows:

$$\begin{cases} E_0(x) = 1, \\ E_1(x) = 1 - 2e^{-\frac{x}{L}}, \\ E_2(x) = 1 - 8e^{-\frac{x}{L}} + 8e^{-\frac{2x}{L}}, \\ E_3(x) = 1 - 18e^{-\frac{x}{L}} + 48e^{-\frac{2x}{L}} - 32e^{-\frac{3x}{L}}. \end{cases} \tag{11}$$

According to (7),  $E_n(x)$  is the  $n$ th eigenfunction of the singular Sturm-Liouville problem

$$L^2 \sqrt{\exp(x/L) - 1} \partial_x \left( \sqrt{\exp(x/L) - 1} \partial_x E_n(x) \right) + n^2 E_n(x) = 0. \tag{12}$$

Also, from formula (8), we can directly construct the  $n$ th exponential Chebyshev functions as

$$E_n(x) = n \sum_{k=0}^n (-4)^k \frac{(n+k-1)!}{(n-k)!(2k)!} \exp\left(-k \frac{x}{L}\right), \quad n > 0. \tag{13}$$

The roots of  $E_n(x)$  are immediately obtained from (9) as follows:

$$x_k = -L \ln\left(\frac{1-s_k}{2}\right), \quad k = 0, 1, \dots, n-1. \tag{14}$$

### 3 Function approximation

Let

$$\rho(x) = \frac{1}{L\sqrt{\exp(\frac{x}{L}) - 1}},$$

which denotes a non-negative, integrable, real-valued function over the semi-infinite interval  $\Lambda = [0, \infty)$ . Subsequently, we define

$$L^2_\rho(\Lambda) = \{v : \Lambda \rightarrow \mathbb{R} \mid v \text{ is measurable and } \|v\|_\rho < \infty\},$$

where

$$\|v\|_\rho^2 = \int_0^{+\infty} v^2(x)\rho(x) dx$$

is the norm induced by the inner product of the space  $L^2_\rho(\Lambda)$ ,

$$\langle u, v \rangle_\rho = \int_0^{+\infty} u(x)v(x)\rho(x) dx. \tag{15}$$

It is easily seen that  $\{E_j(x)\}_{j \geq 0}$  denotes a system which is mutually orthogonal under (15), i.e.,

$$\langle E_n(x), E_m(x) \rangle_\rho = c_n \delta_{nm}, \quad c_0 = \pi, c_n = \frac{\pi}{2}, n \geq 1.$$

The classical Weierstrass theorem implies that such a system is complete in the space  $L^2_\rho(\Lambda)$ . Thus, for any function  $u(x) \in L^2_\rho(\Lambda)$ , the following expansion holds:

$$u(x) = \sum_{j=0}^{+\infty} a_j E_j(x), \tag{16}$$

where

$$a_j = c_j^{-1} \int_0^{+\infty} u(x)E_j(x)\rho(x) dx = c_j^{-1} \langle u(x), E_j(x) \rangle_\rho. \tag{17}$$

If  $u(x)$  in (16) is truncated up to the  $m$ th terms, then it can be written as

$$u(x) \simeq u_m(x) = \sum_{j=0}^m a_j E_j(x). \tag{18}$$

Now, we can estimate an upper bound for function approximation in a special case. Firstly, the error function  $e_m(x)$  can be defined in the following form:

$$e_m(x) = u(x) - u_m(x), \quad x \in \Lambda. \tag{19}$$

The completeness of the system  $\{E_i(x)\}_{i \geq 0}$  is equivalent to the following property as  $m$  tends to infinity:

$$u_m(x) \rightarrow u(x), \quad e_{m,w} = \|e_m(x)\|_\rho \rightarrow 0.$$

Accordingly, the  $L_\infty$  bound for  $e_m(x)$  will be

$$\begin{aligned}
 e_{m,\infty} &= \|e_m(x)\|_\infty = \max_{x \in \Lambda} \left| \sum_{j=m+1}^\infty a_j E_j(x) \right| = \max_{s \in [-1,1]} \left| \sum_{j=m+1}^\infty a_j T_j(s) \right| \\
 &= \max_{0 \leq \theta \leq 2\pi} \left| \sum_{j=m+1}^\infty a_j \cos(j\theta) \right| \leq \sum_{j=m+1}^\infty |a_j|.
 \end{aligned} \tag{20}$$

**Lemma 1** *The  $L_\infty$  and  $L_\rho$  errors for a function  $u \in L^2_\rho(\Lambda)$ , defined by (19), satisfy the following relations:*

$$e_{m,\rho}^2 = \frac{2}{\pi} \sum_{i=m+1}^\infty \langle u(x), E_i(x) \rangle_\rho^2, \tag{21}$$

$$e_{m,\infty} = \|e_m(x)\|_\infty \leq \frac{2}{\pi} \sum_{i=m+1}^\infty \langle u(x), E_i(x) \rangle_\rho. \tag{22}$$

*Proof* The completeness of the system  $\{E_i(x)\}_{i \geq 0}$  helped us to consider the error as

$$e_{m,\rho}^2 = \left\| \sum_{i=m+1}^\infty a_i E_i(x) \right\|_\rho^2.$$

Using the definition of  $\|\cdot\|_\rho$ , one has

$$e_{m,\rho}^2 = \sum_{i=m+1}^\infty \sum_{j=m+1}^\infty a_i a_j \langle E_i(x), E_j(x) \rangle_\rho = \frac{\pi}{2} \sum_{i=m+1}^\infty \sum_{j=m+1}^\infty a_i a_j \delta_{ij} = \frac{\pi}{2} \sum_{i=m+1}^\infty a_i^2.$$

Now by using Eq. (17) the first relation can be proved. Also,

$$\begin{aligned}
 e_{m,\infty} &= \|e_m(x)\|_\infty = \max_{x \in \Lambda} \left| \sum_{i=m+1}^\infty a_i E_i(x) \right| = \max_{s \in [-1,1]} \left| \sum_{i=m+1}^\infty a_i T_i(s) \right| \\
 &= \max_{0 \leq \theta \leq 2\pi} \left| \sum_{i=m+1}^\infty a_i \cos(i\theta) \right| \leq \sum_{i=m+1}^\infty |a_i|.
 \end{aligned} \tag{23}$$

Consequently, (17) completes the proof. □

This lemma shows that the convergence of exponential Chebyshev functions approximation is involved with the function  $u(x)$ . Now, by knowing that the function  $u(x) \in L^2_\rho(\Lambda)$  has some good properties, we can present an upper bound for estimating the error of function approximation by these basis functions.

**Theorem 1** *Let  $u_m(x)$  be function approximation of  $u(x) \in L^2_\rho(\Lambda)$ , obtained by (18), and  $\mathcal{U}(s) = u(-L \ln(\frac{1-s}{2}))$  be analytic on  $(-1, 1)$ , then an error bound for this approximation can be presented as follows:*

$$e_{m,\infty} \leq M_\infty \frac{1}{(m+1)!} \left(\frac{1}{2}\right)^m, \quad e_{m,\rho} \leq \sqrt{\frac{\pi}{3}} M_\infty \frac{1}{(m+1)!} \left(\frac{1}{2}\right)^{m+\frac{1}{2}},$$

where  $M_\infty \geq 2 \max_i |\mathcal{U}^{(i)}(s)|, s \in (-1, 1)$ .

*Proof* Defining  $x = -L \ln(\frac{1-s}{2})$  gives

$$\langle u(x), E_i(x) \rangle_\rho = \int_{-1}^1 \frac{\mathcal{U}(s) T_i(s)}{\sqrt{1-s^2}} ds.$$

Knowing that  $\mathcal{U}(s)$  is analytic, we have

$$\langle u(x), E_i(x) \rangle_\rho = \sum_{j=0}^{i-1} \frac{\mathcal{U}^{(j)}(0)}{j!} \int_{-1}^1 s^j T_i(s) w(s) ds + \frac{\mathcal{U}^{(i)}(\eta_i)}{i!} \int_{-1}^1 s^i T_i(s) w(s) ds, \quad \eta_i \in (-1, 1).$$

Using the following properties of Chebyshev polynomials

$$\int_{-1}^1 s^j T_i(s) w(s) ds = 0, \quad j < i, \quad \int_{-1}^1 s^i T_i(s) w(s) ds = \frac{\pi}{2^i},$$

yields

$$\langle u(x), E_i(x) \rangle_\rho = \frac{\pi \mathcal{U}^{(i)}(\eta_i)}{i! 2^i}.$$

Now, assuming  $M_\infty \geq 2 \max_i |\mathcal{U}^{(i)}(x)|, s \in (-1, 1)$  and using (23), we get

$$e_{m,\infty} \leq M_\infty \sum_{i=m+1}^\infty \frac{1}{i! 2^i} \leq M_\infty \frac{1}{(m+1)! 2^m}.$$

Now, according to Lemma 1, we can prove the theorem as follows:

$$e_{m,\rho}^2 \leq \frac{\pi}{2} M_\infty^2 \sum_{i=m+1}^\infty \frac{1}{(i!)^2 2^{2i}} \leq \pi M_\infty^2 \frac{1}{((m+1)!)^2} \sum_{i=m+1}^\infty \frac{1}{2^{2i+1}},$$

$$e_{m,\rho} \leq \sqrt{\frac{\pi}{3}} M_\infty \frac{1}{(m+1)!} \left(\frac{1}{2}\right)^{m+\frac{1}{2}}. \quad \square$$

From the previous theorem, any real function defined in  $L^2_\rho(\Lambda)$ , whose mapping under the transformation  $-L \ln(\frac{1-s}{2})$  is analytic, has a convergence series solution in the form (18). Furthermore, we can show that the error defined in (19) has superlinear convergence defined below.

**Definition 5**  $x_m$  tends to  $\bar{x}$  with superlinear convergence if there exist a positive sequence  $\lambda_m \rightarrow 0$  and an integer number  $N$  such that

$$|x_{m+1} - \bar{x}| \leq \lambda_m |x_m - \bar{x}|, \quad m \geq N. \tag{24}$$

**Theorem 2** In Theorem 1, let  $M \geq M_i$  for any integer  $i$ , then the error is superlinear convergence to zero.

*Proof* Choosing the positive sequence

$$\lambda_m = \frac{1}{2m}$$

for Theorem 1 gives  $e_{m+1} \leq \lambda_m e_m$ , and consequently, Definition 5 completes the proof.  $\square$

According to Theorem 2, any function  $u(x) \in L^2_\rho(\Lambda)$  that is analytic under the transformation  $x = -L \ln(\frac{1-s}{2})$  has a superlinear convergence series in the form (16).

#### 4 Spectral collection method to solve TFCBEs

In this section, we discuss the spectral collection method to solve the following time-fractional coupled Burgers equations:

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= L_1[u(x,t), v(x,t)], \quad 0 < \alpha < 1, \\ \frac{\partial^\beta v(x,t)}{\partial t^\beta} &= L_2[u(x,t), v(x,t)], \quad 0 < \beta < 1, \end{aligned} \tag{25}$$

where  $L_1$  and  $L_2$  are some derivative operators. The initial and boundary conditions are

$$\begin{aligned} u(x,0) &= I_u(x), & v(x,0) &= I_v(x), \\ u(0,t) &= B_1(t), & u(\infty,t) &= B_2(t), \\ v(0,t) &= B_3(t), & v(\infty,t) &= B_4(t). \end{aligned}$$

The functions  $u(x,t)$  and  $v(x,t)$  are discretized in time  $t = t_n$ , and then they can be expanded by the exponential Chebyshev functions as follows:

$$u(x,t_n) \simeq u_m(x,t_n) = \sum_{i=0}^m a_i^n E_i(x), \quad v(x,t_n) \simeq v_m(x,t_n) = \sum_{i=0}^m b_i^n E_i(x). \tag{26}$$

Also, the time-fractional derivative can be discretized by TQF and FDM as well.

#### 4.1 Trapezoidal quadrature formula

Now we consider the following fractional differential equation:

$$D_*^\alpha u(t) = f(u(t), t), \quad u(0) = u_0, \quad 0 < \alpha < 1, \tag{27}$$

which, by applying (4), converts to the Volterra integral equation

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(u(s), s) ds. \tag{28}$$

For the numerical computation of (28), the integral is replaced by TQF at point  $t_n$

$$\int_0^{t_n} (t_n-s)^{\alpha-1} g(s) ds \approx \int_0^{t_n} (t_n-s)^{\alpha-1} \tilde{g}_n(s) ds, \tag{29}$$



where  $g(s) = f(s, u(s))$  and  $\tilde{g}_n(s)$  is the piecewise linear interpolation of  $g$  with nodes and knots chosen at  $t_j, j = 0, 1, 2, \dots, n$ . After some elementary calculations, the right-hand side of (29) gives [47]

$$\int_0^{t_n} (t_n - s)^{\alpha-1} \tilde{g}_n(s) ds = \frac{\tau^\alpha}{\alpha(\alpha + 1)} \sum_{j=0}^n \gamma_{j,n}^{(\alpha)} g(t_j), \tag{30}$$

where

$$\gamma_{j,n}^{(\alpha)} = \begin{cases} (n-1)^{\alpha+1} - (n-1-\alpha)n^\alpha, & \text{if } j = 0, \\ (n-j+1)^{\alpha+1} + (n-j-1)^{\alpha+1} - 2(n-j)^{\alpha+1}, & \text{if } 1 \leq j \leq n-1, \\ 1, & \text{if } j = n \end{cases} \tag{31}$$

and  $\gamma_{j,n}^{(\alpha)}$  is a positive number bounded by  $(0 < \gamma_{j,n}^{(\alpha)} \leq 1)$ .

From (29) we immediately get

$$\begin{aligned} & \left| \int_0^{t_n} (t_n - s)^{\alpha-1} g(s) ds - \int_0^{t_n} (t_n - s)^{\alpha-1} \tilde{g}_n(s) ds \right| \\ & \leq \max_{0 \leq t \leq t_n} |g(t) - \tilde{g}_n(t)| \int_0^{t_n} |(t_n - s)^{\alpha-1}| ds, \end{aligned} \tag{32}$$

so that error bounds and orders of convergence for product integration follow from standard results of approximation theory. For a piecewise linear approximation to a smooth function  $g(t)$ , the produced TQF is of second order [48].

Accordingly, the time-fractional derivative for Eqs. (25) can be converted to the following singular integro-partial differential equations:

$$\begin{aligned} u(x, t) &= u(x, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_1[u(x, s), v(x, s)] ds, \\ v_m(x, t) &= v(x, 0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} L_2[u(x, s), v(x, s)] ds. \end{aligned}$$

Then TQF (30) together with (26) gives

$$u_m(x, t_n) = I_u(x) + s_\alpha \sum_{j=0}^n \gamma_{j,n}^{(\alpha)} L_1[u_m(x, t_j), v_m(x, t_j)] ds, \tag{33}$$

$$v_m(x, t_n) = I_v(x) + s_\beta \sum_{j=0}^n \gamma_{j,n}^{(\beta)} L_2[u_m(x, t_j), v_m(x, t_j)] ds, \tag{34}$$

where  $s_\alpha = \tau^\alpha / \Gamma(\alpha + 2)$ . From the above equations, the unknown coefficients  $a_i^n$  and  $b_i^n, i = 0, 1, \dots, m$ , should be determined for any step of time. To do so, we use  $m - 1$  collocation nodes  $x_k$ , the roots of  $E_{m-1}(x)$ , together with the boundary conditions as follows:

$$u_m(x_k, t_n) = I_u(x_k) + s_\alpha \sum_{j=0}^n \gamma_{j,n}^{(\alpha)} L_1[u_m(x_k, t_j), v_m(x_k, t_j)] ds,$$

$$v_m(x_k, t_n) = I_v(x_k) + s_\beta \sum_{j=0}^n \gamma_{j,n}^{(\beta)} L_2[u_m(x_k, t_j), v_m(x_k, t_j)] ds,$$

$$\sum_{i=0}^m (-1)^i a_i^n = B_1(t_n), \quad \sum_{i=0}^m a_i^n = B_2(t_n), \tag{35}$$

$$\sum_{i=0}^m (-1)^i b_i^n = B_3(t_n), \quad \sum_{i=0}^m b_i^n = B_4(t_n). \tag{36}$$

For any time step, the above equations form an algebraic system of nonlinear equations with  $2m + 2$  unknowns which can be solved by the fixed point iterative method.

### 4.2 Finite difference approximations for time-fractional derivative

In this section, a fractional order finite difference approximation [27, 49] for the time-fractional partial differential equations is proposed.

Define  $t_j = j\tau$ ,  $j = 0, 1, 2, \dots, n$ , where  $\tau = T/n$ . The time-fractional derivative term of order  $0 < \alpha \leq 1$  with respect to time at  $t = t_n$  is approximated by the following scheme:

$$\begin{aligned} \frac{\partial^\alpha u(x, t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_n} (t_n - s)^{-\alpha} \frac{\partial u(x, s)}{\partial s} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{-\alpha} \frac{\partial u(x, s)}{\partial s} ds \\ &\simeq \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{-\alpha} \frac{u(x, t_{j+1}) - u(x, t_j)}{\tau} ds \\ &= \sum_{j=0}^{n-1} w_{n-j-1}^{(\alpha)} (u(x, t_{j+1}) - u(x, t_j)) \\ &= w_0^{(\alpha)} u(x, t_n) - w_{n-1}^{(\alpha)} u(x, t_0) + \sum_{j=1}^{n-1} (w_{n-j}^{(\alpha)} - w_{n-j-1}^{(\alpha)}) u(x, t_j). \end{aligned} \tag{37}$$

Similarly,

$$\frac{\partial^\beta v(x, t_n)}{\partial t^\beta} \simeq w_0^{(\beta)} v(x, t_n) - w_{n-1}^{(\beta)} v(x, t_0) + \sum_{j=1}^{n-1} (w_{n-j}^{(\beta)} - w_{n-j-1}^{(\beta)}) v(x, t_j), \tag{38}$$

where

$$w_j^{(\alpha)} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} ((j+1)^{1-\alpha} - j^{1-\alpha}),$$

$$w_j^{(\beta)} = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} ((j+1)^{1-\beta} - j^{1-\beta}).$$

We apply this formula to discretize the time variable. The rate of convergence of this formula is  $O(\tau^{2-\alpha})$ .

Accordingly, Eqs. (25), using the initial conditions, are converted to

$$w_0^{(\alpha)} u(x, t_n) - L_1[u(x, t_n), v(x, t_n)] = w_{n-1}^{(\alpha)} I_u(x) - \sum_{j=1}^{n-1} (w_{n-j}^{(\alpha)} - w_{n-j-1}^{(\alpha)}) u(x, t_j), \tag{39}$$

$$w_0^{(\beta)} v(x, t_n) - L_2[u(x, t_n), v(x, t_n)] = w_{n-1}^{(\beta)} I_v(x) - \sum_{j=1}^{n-1} (w_{n-j}^{(\beta)} - w_{n-j-1}^{(\beta)}) v(x, t_j). \tag{40}$$

Again, similar to the last subsection, we use  $m - 1$  collocation nodes  $x_k$ , which are the roots of  $E_{m-1}(x)$ , together with the boundary conditions (35) and (36) to obtain the unknown coefficients  $a_i^n$  and  $b_i^n$  at any time step.

### 5 Numerical experiments

In this section, we present four examples to illustrate the numerical results.

**Example 1** Consider the following time-fractional coupled Burgers equations:

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= \frac{\partial^2 u}{\partial x^2} + 2u \frac{\partial u}{\partial x} - \frac{\partial(uv)}{\partial x} + f(x, t), \quad 0 < \alpha < 1, \\ \frac{\partial^\beta v(x, t)}{\partial t^\beta} &= \frac{\partial^2 v}{\partial x^2} + 2v \frac{\partial v}{\partial x} - \frac{\partial(uv)}{\partial x} + g(x, t), \quad 0 < \beta < 1 \end{aligned}$$

with the initial conditions

$$u(x, 0) = 1, \quad v(x, 0) = 1,$$

and the boundary conditions

$$\begin{aligned} u(0, t) &= 0.841t^3 + 1, \quad v(0, t) = 0.841t^3 + 1, \\ u(x, t), v(x, t) &\rightarrow 1 \quad \text{when } x \rightarrow \infty. \end{aligned}$$

Also,  $f(x, t)$  and  $g(x, t)$  are given by

$$\begin{aligned} f(x, t) &= \frac{3! \sin(e^{-x}) t^{3-\alpha}}{\Gamma(4 - \alpha)} + t^3 e^{-2x} \sin(e^{-x}) - t^3 e^{-x} \cos(e^{-x}), \\ g(x, t) &= \frac{3! \sin(e^{-x}) t^{3-\beta}}{\Gamma(4 - \beta)} + t^3 e^{-2x} \sin(e^{-x}) - t^3 e^{-x} \cos(e^{-x}). \end{aligned}$$

Exact solution for this problem is  $u(x, t) = v(x, t) = t^3 \sin(e^{-x}) + 1$ .

In the first problem, we explain the proposed method with more details. Firstly, we approximate  $u(x, t_n)$ ,  $v(x, t_n)$  and their derivatives by ECFs as follows:

$$\begin{aligned} u_m(x, t_n) &= \sum_{i=0}^m a_i^n E_i(x), & v_m(x, t_n) &= \sum_{i=0}^m b_i^n E_i(x), \\ \frac{\partial u_m(x, t_n)}{\partial x} &= \sum_{i=0}^m a_i^n E'_i(x), & \frac{\partial v_m(x, t_n)}{\partial x} &= \sum_{i=0}^m b_i^n E'_i(x), \end{aligned}$$

$$\frac{\partial^2 u_m(x, t_n)}{\partial x^2} = \sum_{i=0}^m a_i^n E_i''(x), \quad \frac{\partial^2 v_m(x, t_n)}{\partial x^2} = \sum_{i=0}^m b_i^n E_i''(x).$$

For this problem, the operators  $L_1$  and  $L_2$  defined in (25) after substituting the collocation nodes  $x_k$  are obtained as follows:

$$\begin{aligned} L_{1k}^j &= L_1[u_m(x_k, t_j), v_m(x_k, t_j)] \\ &= \sum_{i=0}^m a_i^j E_i''(x_k) + 2 \sum_{i=0}^m a_i^j E_i'(x_k) \sum_{i=0}^m a_i^j E_i'(x_k) \\ &\quad - \sum_{i=0}^m a_i^j E_i(x_k) \sum_{i=0}^m b_i^j E_i'(x_k) - \sum_{i=0}^m b_i^j E_i(x_k) \sum_{i=0}^m a_i^j E_i'(x_k) + f(x_k, t_j), \\ L_{2k}^j &= L_2[u_m(x_k, t_j), v_m(x_k, t_j)] \\ &= \sum_{i=0}^m b_i^j E_i''(x_k) + 2 \sum_{i=0}^m b_i^j E_i'(x_k) \sum_{i=0}^m b_i^j E_i'(x_k) \\ &\quad - \sum_{i=0}^m a_i^j E_i(x_k) \sum_{i=0}^m b_i^j E_i'(x_k) - \sum_{i=0}^m b_i^j E_i(x_k) \sum_{i=0}^m a_i^j E_i'(x_k) + g(x_k, t_j). \end{aligned}$$

Note that the values of  $E_i(x_k)$  and its derivatives can be obtained from Eq. (13) as well.

*TQF implementation*

Now TQF gives the following  $2m - 2$  equations at any step of time  $t_n$

$$\begin{aligned} \sum_{i=0}^m a_i^n E_i(x_k) - s_\alpha L_{1k}^n &= I_u(x_k) + s_\alpha \sum_{j=0}^{n-1} \gamma_{j,n}^{(\alpha)} L_{1k}^j, \quad k = 1, \dots, m - 1, \\ \sum_{i=0}^m b_i^n E_i(x_k) - s_\beta L_{2k}^n &= I_v(x_k) + s_\beta \sum_{j=0}^{n-1} \gamma_{j,n}^{(\beta)} L_{2k}^j, \quad k = 1, \dots, m - 1, \end{aligned}$$

where for this problem  $I_u(x_k) = I_v(x_k) = 1$ . Also it should be noted that the second hand sides of the above equations are known since they are obtained in the last steps of time. The above equations together with the boundary conditions (35) and (36)

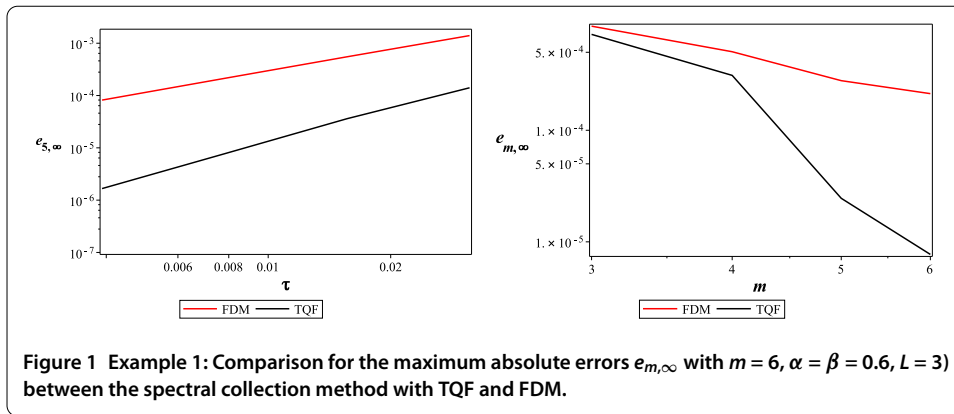
$$\sum_{i=0}^m (-1)^i a_i^n = 0.841 t_n^3 + 1, \quad \sum_{i=0}^m a_i^n = 1, \tag{41}$$

$$\sum_{i=0}^m (-1)^i b_i^n = 0.841 t_n^3 + 1, \quad \sum_{i=0}^m b_i^n = 1, \tag{42}$$

construct a system of nonlinear equations which can be solved by the Newton method (or fsolve command) to find the coefficients  $a_i^n$  and  $b_i^n$  at any step of time.

*FDM implementation*

After substituting the collocation nodes  $x_k$  in Eqs. (39) and (40), and knowing that  $I_u(x_k) = I_v(x_k) = 1$ , the following  $2m - 2$  equations at any step of time  $t_n$  are generated



as

$$w_0^{(\alpha)} u(x_k, t_n) - L_{1k}^n = w_{n-1}^{(\alpha)} - \sum_{j=1}^{n-1} (w_{n-j}^{(\alpha)} - w_{n-j-1}^{(\alpha)}) \sum_{i=0}^m a_i^j E_i(x_k), \tag{43}$$

$$w_0^{(\beta)} v(x_k, t_n) - L_{2k}^n = w_{n-1}^{(\beta)} - \sum_{j=1}^{n-1} (w_{n-j}^{(\beta)} - w_{n-j-1}^{(\beta)}) \sum_{i=0}^m b_i^j E_i(x_k). \tag{44}$$

Now these equations along with four boundary conditions that appear in Eqs. (41) and (42) give a nonlinear system of equations which can be solved by the Newton method (or fsolve command) to find the coefficients  $a_j^n$  and  $b_j^n$  at any step of time.

The maximum errors  $e_{m,\infty}(u)$  and  $e_{m,\infty}(v)$  obtained via the proposed methods are shown in Figure 1 with parameter  $L = 3$ . A comparison between TQF and FDM reveals that TQF approach is superior to FDM.

**Example 2** We consider the time-fractional coupled Burgers equation with the initial condition

$$u(x, 0) = 0, \quad v(x, 0) = 0,$$

and the boundary conditions

$$\begin{aligned} u(0, t) &= \frac{1}{3}t^3, & u &\rightarrow \frac{1}{2}t^3 \quad \text{as } x \rightarrow \infty, \\ v(0, t) &= \frac{1}{3}t^3, & v &\rightarrow \frac{1}{2}t^3 \quad \text{as } x \rightarrow \infty, \end{aligned}$$

where  $f(x, t)$  and  $g(x, t)$  are given by

$$\begin{aligned} f(x, t) &= \frac{3!t^{3-\alpha}}{(e^{-x} + 2)\Gamma(4 - \alpha)} - \frac{2t^3 e^{-2x}}{(e^{-x} + 2)^3} + \frac{t^3 e^{-x}}{(e^{-x} + 2)^2}, \\ g(x, t) &= \frac{3!t^{3-\beta}}{(e^{-x} + 2)\Gamma(4 - \beta)} - \frac{2t^3 e^{-2x}}{(e^{-x} + 2)^3} + \frac{t^3 e^{-x}}{(e^{-x} + 2)^2}. \end{aligned}$$

Exact solution for this problem is  $u(x, t) = v(x, t) = \frac{t^3}{(e^{-x} + 2)}$ .

**Table 1 Example 2: Maximum absolute errors  $e_{m,\infty}$  with  $m = 5, \alpha = 0.4, \beta = 0.4$  and  $L = 3$**

$\tau$	TQF $e_{5,\infty}(u) = e_{5,\infty}(v)$	$O(\tau^2)$	FDM $e_{5,\infty}(u) = e_{5,\infty}(v)$	$O(\tau^{2-\alpha})$
0.015625	$1.62969572 \times 10^{-3}$		$1.58537183 \times 10^{-2}$	
0.0078125	$4.07935306 \times 10^{-4}$	1.99	$5.42638453 \times 10^{-3}$	1.54
0.00390625	$9.40924121 \times 10^{-5}$	2.01	$1.82754471 \times 10^{-3}$	1.57
0.001953125	$1.58381076 \times 10^{-5}$	2.07	$6.04886113 \times 10^{-4}$	1.55

**Table 2 Example 2: Maximum absolute errors  $e_{m,\infty}$  with  $\tau = 1/128, \alpha = 0.4, \beta = 0.4$  and  $L = 3$**

$m$	TQF $e_{m,\infty}(u) = e_{m,\infty}(v)$	FDM $e_{m,\infty}(u) = e_{m,\infty}(v)$
3	$3.21997018 \times 10^{-4}$	$3.02999632 \times 10^{-3}$
4	$2.32880457 \times 10^{-4}$	$2.48938916 \times 10^{-3}$
5	$1.24955778 \times 10^{-4}$	$1.51170703 \times 10^{-3}$

**Table 3 Example 3: Maximum absolute errors  $e_{m,\infty}$  with  $m = 5, \alpha = 0.5, \beta = 0.5$  and  $L = 3$**

$\tau$	TQF $e_{5,\infty}(u) = e_{5,\infty}(v)$	$O(\tau^2)$	FDM $e_{5,\infty}(u) = e_{5,\infty}(v)$	$O(\tau^{2-\alpha})$
0.015625	$1.21322402 \times 10^{-4}$		$1.23475331 \times 10^{-3}$	
0.0078125	$3.17697252 \times 10^{-5}$	1.95	$4.67009965 \times 10^{-4}$	1.46
0.00390625	$8.15383818 \times 10^{-6}$	1.96	$1.71676167 \times 10^{-4}$	1.45
0.001953125	$2.07020075 \times 10^{-6}$	1.97	$6.21304589 \times 10^{-5}$	1.48

The maximum absolute errors for time-fractional coupled Burgers equation for this problem with  $(\alpha = 0.4, \beta = 0.4, L = 3)$  are reported in Tables 1 and 2.

**Example 3** We consider the time-fractional coupled Burgers equation of the first example with the initial condition

$$u(x, 0) = 1, \quad v(x, 0) = 1,$$

and the boundary conditions

$$u(0, t) = t^6 + 1, \quad u \rightarrow 1 \quad \text{as } x \rightarrow \infty,$$

$$v(0, t) = t^6 + 1, \quad v \rightarrow 1 \quad \text{as } x \rightarrow \infty,$$

where  $f(x, t)$  and  $g(x, t)$  are given by

$$f(x, t) = \frac{6!e^{-x}t^{6-\alpha}}{\Gamma(7-\alpha)} - t^6 e^x$$

$$g(x, t) = \frac{6!e^{-x}t^{6-\beta}}{\Gamma(7-\beta)} + t^6 e^x.$$

Exact solution of this problem is  $u(x, t) = v(x, t) = t^6 e^{-x} + 1$ .

The maximum absolute errors for a time-fractional coupled Burgers equation for this problem are reported in Tables 3 and 4.

**Table 4 Example 3: Maximum absolute errors  $e_{m,\infty}$  with  $\tau = 1/128, \alpha = 0.5, \beta = 0.5$  and  $L = 3$**

$m$	TQF		FDM	
	$e_{m,\infty}(u) = e_{m,\infty}(v)$		$e_{m,\infty}(u) = e_{m,\infty}(v)$	
3	$4.04277852 \times 10^{-5}$		$5.75567882 \times 10^{-4}$	
4	$2.66167043 \times 10^{-5}$		$4.15010485 \times 10^{-4}$	
5	$1.85584712 \times 10^{-5}$		$4.08775869 \times 10^{-4}$	

**Table 5 Example 4: Maximum absolute errors  $e_{m,\infty}$  with  $\tau = 1/64, \alpha = \beta = 1$  and  $L = 3$**

$m$	TQF		FDM	
	$u(x, t)$	$v(x, t)$	$u(x, t)$	$v(x, t)$
5	$2.56389687 \times 10^{-3}$	$2.56389687 \times 10^{-3}$	$2.12780543 \times 10^{-3}$	$2.12780543 \times 10^{-3}$
7	$1.19107232 \times 10^{-3}$	$1.19107232 \times 10^{-3}$	$1.06246855 \times 10^{-4}$	$1.06246855 \times 10^{-4}$
9	$6.89082786 \times 10^{-4}$	$6.89082786 \times 10^{-4}$	$1.71691547 \times 10^{-6}$	$1.71691547 \times 10^{-6}$
11	$5.71482080 \times 10^{-4}$	$5.71482080 \times 10^{-4}$	$1.98386814 \times 10^{-8}$	$1.98386814 \times 10^{-8}$

**Table 6 Example 4: Absolute errors  $|u_7 - u_6|$  and  $|v_7 - v_6|$  with  $\alpha = \beta = 0.5$ , in the final time**

$x$	TQF		FDM	
	$ u_7 - u_6 $	$ v_7 - v_6 $	$ u_7 - u_6 $	$ v_7 - v_6 $
0.1	$1.920638996 \times 10^{-2}$	$1.920638996 \times 10^{-2}$	$1.775781437 \times 10^{-2}$	$1.775781437 \times 10^{-2}$
0.2	$3.057241885 \times 10^{-3}$	$3.057241885 \times 10^{-3}$	$2.819522500 \times 10^{-3}$	$2.819522500 \times 10^{-3}$
0.3	$3.102165273 \times 10^{-3}$	$3.102165273 \times 10^{-3}$	$2.850429678 \times 10^{-3}$	$2.850429678 \times 10^{-3}$
0.4	$2.564731408 \times 10^{-3}$	$2.564731408 \times 10^{-3}$	$2.345949628 \times 10^{-3}$	$2.345949628 \times 10^{-3}$
0.5	$1.882240640 \times 10^{-3}$	$1.882240640 \times 10^{-3}$	$1.712524225 \times 10^{-3}$	$1.712524225 \times 10^{-3}$
0.6	$1.281492551 \times 10^{-3}$	$1.281492551 \times 10^{-3}$	$1.158901227 \times 10^{-3}$	$1.158901227 \times 10^{-3}$
0.7	$8.355109022 \times 10^{-4}$	$8.355109022 \times 10^{-4}$	$7.506877573 \times 10^{-4}$	$7.506877573 \times 10^{-4}$
0.8	$5.381590409 \times 10^{-4}$	$5.381590409 \times 10^{-4}$	$4.805394710 \times 10^{-4}$	$4.805394710 \times 10^{-4}$
0.9	$3.545769785 \times 10^{-4}$	$3.545769785 \times 10^{-4}$	$3.151826781 \times 10^{-4}$	$3.151826781 \times 10^{-4}$

**Example 4** Considering the following homogeneous TFCBEs:

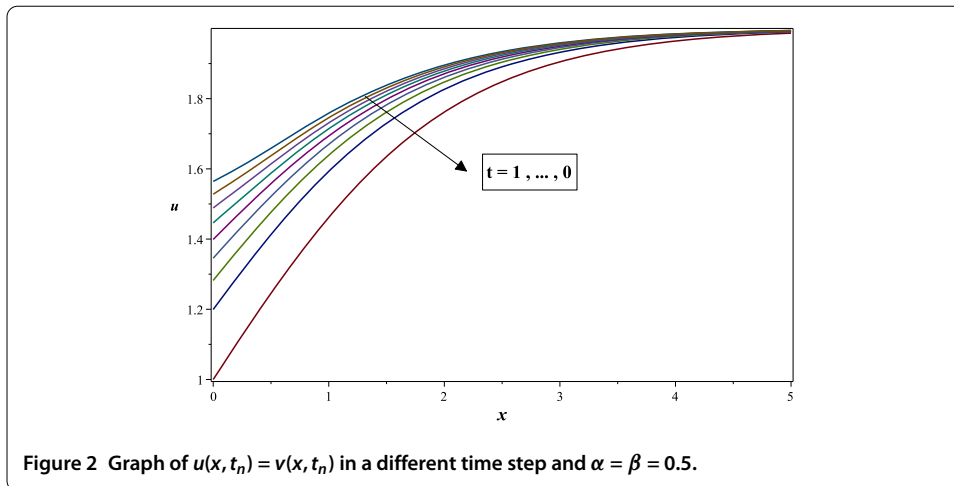
$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + v \frac{\partial u}{\partial x}, \quad 0 < \alpha < 1, \tag{45}$$

$$\frac{\partial^\beta v(x, t)}{\partial t^\beta} = \frac{\partial^2 v}{\partial x^2} + u \frac{\partial v}{\partial x}, \quad 0 < \beta < 1 \tag{46}$$

with the initial and boundary conditions

$$\begin{aligned}
 u(x, 0) &= \frac{2}{1 + e^{-x}}, & v(x, 0) &= \frac{2}{1 + e^{-x}}, \\
 u(0, t) &= \frac{2}{1 + e^{-t}}, & u(\infty, t) &= 2, \\
 v(0, t) &= \frac{2}{1 + e^{-x}}, & v(\infty, t) &= 2.
 \end{aligned}$$

For this problem, only for the case  $\alpha = \beta = 1$ , the exact solution is  $u(x, t) = v(x, t) = \frac{2}{1 + e^{-(x+t)}}$ . Table 5 illustrates the maximum error for this case when  $\tau = 1/64$ . When  $\alpha = \beta = 0.5$ , we report the difference between the values of  $u_7$  and  $u_6$  at the final time  $T = 1$  in Table 6. Also, the graph of  $u(x, t_n) = v(x, t_n)$ , for different time steps, for this case is displayed in Figure 2.



**Example 5** Consider the following homogeneous time-fractional Burgers equation [50, 51]:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = v \frac{\partial^2 u}{\partial x^2} - v \frac{\partial u}{\partial x}, \quad 0 < \alpha < 1 \tag{47}$$

$$\frac{\partial^\beta v(x, t)}{\partial t^\beta} = v \frac{\partial^2 v}{\partial x^2} - u \frac{\partial v}{\partial x}, \quad 0 < \beta < 1 \tag{48}$$

with the following initial and boundary conditions:

$$u(x, 0) = v(x, 0) = \frac{\mu + \sigma + (\sigma - \mu)e^{\frac{\mu}{v}(x-\lambda)}}{1 + e^{\frac{\mu}{v}(x-\lambda)}},$$

$$u(0, t) = v(0, t) = \frac{\mu + \sigma + (\sigma - \mu)e^{\frac{\mu}{v}(-\sigma t-\lambda)}}{1 + e^{\frac{\mu}{v}(-\sigma t-\lambda)}},$$

$$u(\infty, t) = v(\infty, t) = 0,$$

where  $\mu, \sigma, \lambda$  and  $v$  are arbitrary constants. For this problem, the exact solution only exists in the case  $\alpha = \beta = 1$  as follows:

$$u(x, t) = v(x, t) = \frac{\mu + \sigma + (\sigma - \mu)e^{\frac{\mu}{v}(x-\sigma t-\lambda)}}{1 + e^{\frac{\mu}{v}(x-\sigma t-\lambda)}}.$$

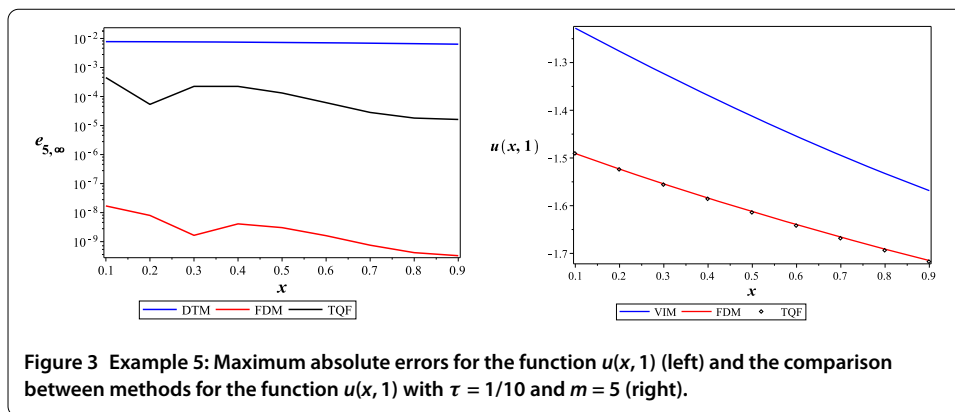
We can compare the results obtained by the proposed method and three-term solution of the differential transform method (DTM) [50] for  $\alpha = \beta = 1$ . Figure 3 (left) displays the maximum error for these methods with  $v = 1, \mu = -1, \lambda = 0$  and  $\sigma = -1$ .

Also, we can compare our results by the variational iteration method (VIM) [52] for different  $\alpha$  and  $\beta$ . We report the results obtained by the proposed method and VIM [52] for  $u(x, t)$  at the final time  $T = 1$  while  $\alpha = \beta = 0.5$  in Figure 3 (right).

### 6 Conclusion

In this paper we presented a numerical method for solving the time-fractional Burgers equation by utilizing the exponential Chebyshev functions and TQF and FDM as well. Numerical results illustrate the validity and efficiency of the method and comparison for





the maximum absolute errors between spectral collection method with TQF and FDM. This technique can be used to solve fractional time partial differential equations.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

BA and HA designed the study, developed the methodology, collected the data, performed the analysis and wrote the manuscript.

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