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# Approximation on the reciprocal-cubic and reciprocal-quartic functional equations in non-Archimedean fields

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**Abstract**
The aim of this paper is to study the generalized Hyers-Ulam stability of a form of reciprocal-cubic and reciprocal-quartic functional equations in non-Archimedean fields. Some related examples for the singular cases of these new functional equations on an Archimedean field are indicated.

**MSC:** 39B82; 39B72

**Keywords:** reciprocal functional equation; reciprocal-cubic functional equation; reciprocal-quartic functional equation; generalized Hyers-Ulam stability; non-Archimedean field

## 1 Introduction

The study of the stability of functional equations was instigated by the famous question of Ulam [1] during a Mathematical Colloquium at the University of Wiskonsin in the year 1940. In the successive year, Hyers [2] provided a partial answer to the question of Ulam. Later, Hyers's result was extended and generalized for a Cauchy functional equation by Bourgin [3], Th.M. Rassias [4], Gruber [5], Aoki [6], J.M. Rassias [7] and Găvruta [8] in various adaptations. After that several stability articles, many textbooks and research monographs have investigated the result for various functional equations, also for mappings with more general domains and ranges; for instance, see [9–16] and [17].

In 2010, Ravi and Senthil Kumar [18] obtained Ulam-Găvruta-Rassias stability for the Rassias reciprocal functional equation

$$r(x+y)=\frac{r(x)r(y)}{r(x)+r(y)}, \tag{1.1}$$

where $r: X \longrightarrow \mathbb{R}$ is a mapping with X as the space of non-zero real numbers. The reciprocal function $r(x)=\frac{c}{x}$ is a solution of the functional equation (1.1). The functional equation (1.1) holds good for the 'reciprocal formula' of any electric circuit with two resistors connected in parallel [19]. Ravi et al. [20] obtained the solution of a new generalized reciprocal-type functional equation in two variables of the form

$$r(x+y)=\frac{kr(x+(k-1)y)r((k-1)x+y)}{r(x+(k-1)y)+r((k-1)x+y)}, \tag{1.2}$$

where $k > 2$ is a positive integer, and investigated its generalized Hyers-Ulam stability in non-Archimedean fields. Then Senthil Kumar et al. [21] found a general solution of a reciprocal-type functional equation

$$f(x+y)=\frac{f(\frac{k_1x+k_2y}{k})f(\frac{k_2x+k_1y}{k})}{f(\frac{k_1x+k_2y}{k})+f(\frac{k_2x+k_1y}{k})} \tag{1.3}$$

and investigated its generalized Hyers-Ulam-Rassias stability in non-Archimedean fields, where $k > 2$, $k_1$ and $k_2$ are positive integers with $k = k_1 + k_2$ and $k_1 \neq k_2$. The other results pertaining to the stability of different reciprocal-type functional equations can be found in [22–24] and [25].

For the first time, Kim and Bodaghi [26] introduced and studied the Ulam-Găvruta-Rassias stability for the quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2}. \tag{1.4}$$

Then the functional equation (1.4) was generalized in [27] as

$$f\big((a+1)x+ay\big)+f\big((a+1)x-ay\big)=\frac{2f(x)f(y)[(a+1)^2f(y)+a^2f(x)]}{((a+1)^2f(y)-a^2f(x))^2}, \tag{1.5}$$

where $a \in \mathbb{Z}$ with $a \neq 0, -1$. In [27], the authors established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5) in non-Archimedean fields. Since then Ravi et al. [28] investigated the generalized Hyers-Ulam-Rassias stability of a reciprocal-quadratic functional equation of the form

$$r(x+2y)+r(2x+y)=\frac{r(x)r(y)[5r(x)+5r(y)+8\sqrt{r(x)r(y)}]}{[2r(x)+2r(y)+5\sqrt{r(x)r(y)}]^2} \tag{1.6}$$

in intuitionistic fuzzy normed spaces; for another form of a reciprocal-quadratic functional equation, see [29].

In this paper, we introduce the reciprocal-cubic functional equation

$$c(2x+y)+c(x+2y)=\frac{9c(x)c(y)[c(x)+c(y)+2c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}(c(x)^{\frac{1}{3}}+c(y)^{\frac{1}{3}})]}{[2c(x)^{\frac{2}{3}}+2c(y)^{\frac{2}{3}}+5c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}]^3} \tag{1.7}$$

and the reciprocal-quartic functional equation

$$q(2x+y)+q(2x-y)=\frac{2q(x)q(y)[q(x)+16q(y)+24\sqrt{q(x)q(y)}]}{[4\sqrt{q(y)}-\sqrt{q(x)}]^4}. \tag{1.8}$$

It can be verified that the reciprocal-cubic function $c(x)=\frac{1}{x^3}$ and the reciprocal-quartic function $q(x)=\frac{1}{x^4}$ are solutions of the functional equations (1.7) and (1.8), respectively. Then we investigate the generalized Hyers-Ulam stability of these new functional equations in the framework of non-Archimedean fields. We extend the results concerning Hyers-Ulam stability, Hyers-Ulam-Rassias stability and Ulam-Găvruta-Rassias stability controlled by the mixed product-sum of powers of norms for equations (1.7) and (1.8). We also provide related examples that the functional equations (1.7) and (1.8) are not stable for the singular cases.

## 2 Preliminaries

In this section, we recall the basic concepts of a non-Archimedean field.

**Definition 2.1** By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|p| = 0$ if and only if $p = 0$, $|pq| = |p||q|$ and $|p+q| \leq \max\{|p|,|q|\}$ for all $p,q \in \mathbb{K}$.

Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0,1$. Due to the fact that

$$|p_n - p_m| \leq \max\{|p_{j+1} - p_j| : m \leq j \leq n-1\} \quad (n > m),$$

a sequence $\{p_n\}$ is Cauchy if and only if $\{p_{n+1} - p_n\}$ converges to zero in a non-Archimedean field. By a complete non-Archimedean field, we mean that every Cauchy sequence is convergent in the field.

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{A}$ is the mapping

$$|k| = \begin{cases} 0 & \text{if } k = 0, \\ \frac{1}{k} & \text{if } k > 0, \\ -\frac{1}{k} & \text{if } k < 0 \end{cases}$$

for any $k \in \mathbb{A}$.

Let $p$ be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which $m$ and $n$ are co-prime to the prime number $p$, consider the $p$-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$, which is denoted by $\mathbb{Q}_p$, is said to be the $p$-adic number field. Note that if $p > 2$, then $|2^n|_p = 1$ for all integers $n$.

Throughout this paper, we consider that $\mathbb{X}$ and $\mathbb{Y}$ are a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, for a non-Archimedean field $\mathbb{X}$, we put $\mathbb{X}^* = \mathbb{X} \setminus \{0\}$. For the purpose of simplification, let us define the difference operators $\Delta_1 c, \Delta_2 q : \mathbb{X}^* \times \mathbb{X}^* \longrightarrow \mathbb{Y}$ by

$$\Delta_1 c(x,y) = c(2x+y) + c(x+2y) - \frac{9c(x)c(y)[c(x) + c(y) + 2c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}(c(x)^{\frac{1}{3}} + c(y)^{\frac{1}{3}})]}{[2c(x)^{\frac{2}{3}} + 2c(y)^{\frac{2}{3}} + 5c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}]^3}$$

and

$$\Delta_2 q(x,y) = q(2x+y) + q(2x-y) - \frac{2q(x)q(y)[q(x) + 16q(y) + 24\sqrt{q(x)q(y)}]}{[4\sqrt{q(y)} - \sqrt{q(x)}]^4}$$

for all $x, y \in \mathbb{X}^*$.

**Definition 2.2** A mapping $c : \mathbb{X}^* \longrightarrow \mathbb{Y}$ is called a reciprocal-cubic mapping if $c$ satisfies equation (1.7). Also, a mapping $q : \mathbb{X}^* \longrightarrow \mathbb{Y}$ is called a reciprocal-quartic mapping if $q$ satisfies equation (1.8).

## 3 Hyers-Ulam stability for equations (1.7) and (1.8)

In this section, we investigate the generalized Hyers-Ulam stability of equations (1.7) and (1.8) in non-Archimedean fields. We also establish the results pertaining to Hyers-Ulam stability, Hyers-Ulam-Rassias stability and Ulam-Găvruta-Rassias stability controlled by product-sum of powers of norms.

**Theorem 3.1** *Let $l \in \{1,-1\}$ be fixed, and let $F : \mathbb{X}^* \times \mathbb{X}^* \longrightarrow [0,\infty)$ be a mapping such that*

$$\lim_{n\to\infty}\left|\frac{1}{27}\right|^{ln} F\left(\frac{x}{3^{ln+\frac{l+1}{2}}}, \frac{y}{3^{ln+\frac{l+1}{2}}}\right) = 0 \tag{3.1}$$

*for all $x, y \in \mathbb{X}^*$. Suppose that $c : \mathbb{X}^* \longrightarrow \mathbb{Y}$ is a mapping satisfying the inequality*

$$\left|\Delta_1 c(x,y)\right| \leq F(x,y) \tag{3.2}$$

*for all $x, y \in \mathbb{X}^*$. Then there exists a unique reciprocal-cubic mapping $C : \mathbb{X}^* \longrightarrow \mathbb{Y}$ such that*

$$\left|c(x) - C(x)\right| \leq \sup\left\{\left|\frac{1}{27}\right|^{jl+\frac{l-1}{2}} F\left(\frac{x}{3^{jl+\frac{l+1}{2}}}, \frac{x}{3^{jl+\frac{l+1}{2}}}\right) : j \in \mathbb{N}\cup\{0\}\right\} \tag{3.3}$$

*for all $x \in \mathbb{X}^*$.*

*Proof* Interchanging $(x,y)$ into $(x,x)$ in (3.2), we obtain

$$\left|c(x) - \frac{1}{27^l} c\left(\frac{x}{3^l}\right)\right| \leq |27|^{\frac{|l-1|}{2}} F\left(\frac{x}{3^{\frac{l+1}{2}}}, \frac{x}{3^{\frac{l+1}{2}}}\right) \tag{3.4}$$

for all $x \in \mathbb{X}^*$. Replacing $x$ by $\frac{x}{3^{ln}}$ in (3.4) and multiplying by $|\frac{1}{27}|^{ln}$, we have

$$\left|\frac{1}{27^{ln}} c\left(\frac{x}{3^{ln}}\right) - \frac{1}{27^{(n+1)l}} c\left(\frac{x}{3^{(n+1)l}}\right)\right| \leq \left|\frac{1}{27}\right|^{ln+\frac{l-1}{2}} F\left(\frac{x}{3^{ln+\frac{l+1}{2}}}, \frac{x}{3^{ln+\frac{l+1}{2}}}\right) \tag{3.5}$$

for all $x \in \mathbb{X}^*$. It follows from relations (3.1) and (3.5) that the sequence $\{\frac{1}{27^{ln}} c(\frac{x}{3^{ln}})\}$ is Cauchy. Since $\mathbb{Y}$ is complete, this sequence converges to a mapping $C : \mathbb{X}^* \longrightarrow \mathbb{Y}$ defined by

$$C(x) = \lim_{n\to\infty} \frac{1}{27^{ln}} c\left(\frac{x}{3^{ln}}\right). \tag{3.6}$$

On the other hand, for each $x \in \mathbb{X}^*$ and non-negative integers $n$, we have

$$\begin{aligned}
\left|\frac{1}{27^{ln}} c\left(\frac{x}{3^{ln}}\right) - c(x)\right| &= \left|\sum_{j=0}^{n-1}\left\{\frac{1}{27^{(j+1)l}} c\left(\frac{x}{3^{(j+1)l}}\right) - \frac{1}{27^{jl}} c\left(\frac{x}{3^{jl}}\right)\right\}\right| \\
&\leq \max\left\{\left|\frac{1}{27^{(j+1)l}} c\left(\frac{x}{3^{(j+1)l}}\right) - \frac{1}{27^{jl}} c\left(\frac{x}{3^{jl}}\right)\right| : 0 \leq i < n\right\} \\
&\leq \max\left\{\left|\frac{1}{27}\right|^{jl+\frac{l-1}{2}} F\left(\frac{x}{3^{jl+\frac{l+1}{2}}}, \frac{x}{3^{jl+\frac{l+1}{2}}}\right) : 0 \leq j < n\right\}.
\end{aligned} \tag{3.7}$$

Applying (3.6) and letting $n \to \infty$ in inequality (3.7), we find that inequality (3.3) holds. Using (3.1), (3.2) and (3.6), for all $x, y \in \mathbb{X}^*$, we have

$$\left|\Delta_1 C(x,y)\right| = \lim_{n\to\infty}\left|\frac{1}{27}\right|^{ln}\left|\Delta_1 c\left(\frac{x}{3^{ln}}, \frac{y}{3^{ln}}\right)\right| \le \lim_{n\to\infty}\left|\frac{1}{27}\right|^{ln} F\left(\frac{x}{3^{ln}}, \frac{y}{3^{ln}}\right) = 0.$$

Thus, the mapping $C$ satisfies (1.7) and hence it is a reciprocal-cubic mapping. In order to prove the uniqueness of $C$, let us consider another reciprocal-cubic mapping $C' : \mathbb{X}^* \longrightarrow \mathbb{Y}$ satisfying (3.3). Then

$$\begin{aligned}
&\left|C(x) - C'(x)\right| \\
&\quad = \lim_{m\to\infty}\left|\frac{1}{27}\right|^{lm}\left|C\left(\frac{x}{3^{lm}}x\right) - C'\left(\frac{x}{3^{lm}}\right)\right| \\
&\quad \le \lim_{m\to\infty}\left|\frac{1}{27}\right|^{lm}\max\left\{\left|C\left(\frac{x}{3^{lm}}\right) - c\left(\frac{x}{3^{lm}}\right)\right|, \left|c\left(\frac{x}{3^{lm}}\right) - C'\left(\frac{x}{3^{lm}}\right)\right|\right\} \\
&\quad \le \lim_{m\to\infty}\lim_{n\to\infty}\max\left\{\max\left\{\left|\frac{1}{27}\right|^{(j+m)l+\frac{l-1}{2}} F\left(\frac{x}{3^{(j+m)l+\frac{l+1}{2}}}, \frac{x}{3^{(j+m)l+\frac{l+1}{2}}}\right) : m \le j \le n+m\right\}\right\} \\
&\quad = 0
\end{aligned}$$

for all $x \in \mathbb{X}^*$, which shows that $C$ is unique. This finishes the proof. □

From now on, we assume that $|2| < 1$. The following corollaries are immediate consequences of Theorem 3.1 concerning the stability of (1.7).

**Corollary 3.2** *Let $\epsilon > 0$ be a constant. If $c : \mathbb{X}^* \longrightarrow \mathbb{Y}$ satisfies $|\Delta_1 c(x,y)| \le \epsilon$ for all $x, y \in \mathbb{X}^*$, then there exists a unique reciprocal-cubic mapping $C : \mathbb{X}^* \longrightarrow \mathbb{Y}$ satisfying* (1.7) *and $|c(x) - C(x)| \le \epsilon$ for all $x \in \mathbb{X}^*$.*

*Proof* Defining $F(x,y) = \epsilon$ and applying Theorem 3.1 for the case $l = -1$, we get the desired result. □

**Corollary 3.3** *Let $\epsilon \ge 0$ and $r \ne -3$ be fixed constants. If $c : \mathbb{X}^* \longrightarrow \mathbb{Y}$ satisfies $|\Delta_1 c(x,y)| \le \epsilon(|x|^r + |y|^r)$ for all $x, y \in \mathbb{X}^*$, then there exists a unique reciprocal-cubic mapping $C : \mathbb{X}^* \longrightarrow \mathbb{Y}$ satisfying* (1.7) *and*

$$\left|c(x) - C(x)\right| \le \begin{cases} \frac{2\epsilon}{|3|^r}|x|^r, & r > -3, \\ 2\epsilon|3|^3|x|^r, & r < -3 \end{cases}$$

*for all $x \in \mathbb{X}^*$.*

*Proof* The result follows immediately from Theorem 3.1 by taking $F(x,y) = \epsilon(|x|^r + |y|^r)$. □

**Corollary 3.4** *Let $c : \mathbb{X}^* \longrightarrow \mathbb{Y}$ be a mapping, and let there exist real numbers $p$, $q$, $r = p + q \ne -3$ and $\epsilon \ge 0$ such that $|\Delta_1 c(x,y)| \le \epsilon|x|^p|y|^q$ for all $x, y \in \mathbb{X}^*$. Then there ex-*

*ists a unique reciprocal-cubic mapping* $C : \mathbb{X}^* \longrightarrow \mathbb{Y}$ *satisfying* (1.7) *and*

$$\left|c(x) - C(x)\right| \leq \begin{cases} \frac{\epsilon}{|3|^r}|x|^r, & r > -3, \\ \epsilon|3|^3||x|^r, & r < -3 \end{cases}$$

*for all* $x \in \mathbb{X}^*$.

*Proof* The required result is obtained by choosing $F(x,y) = \epsilon|x|^p|y|^q$ for all $x, y \in \mathbb{X}^*$ in Theorem 3.1. □

**Corollary 3.5** *Let* $\epsilon \geq 0$ *and* $r \neq -3$ *be real numbers and* $c : \mathbb{X}^* \longrightarrow \mathbb{Y}$ *be a mapping satisfying the functional inequality*

$$\left|\Delta_1 c(x,y)\right| \leq \epsilon\left(|x|^{\frac{r}{2}}|y|^{\frac{r}{2}} + \left(|x|^r + |y|^r\right)\right)$$

*for all* $x, y \in \mathbb{X}^*$. *Then there exists a unique reciprocal-cubic mapping* $C : \mathbb{X}^* \longrightarrow \mathbb{Y}$ *satisfying* (1.7) *and*

$$\left|c(x) - C(x)\right| \leq \begin{cases} \frac{3\epsilon}{|3|^r}|x|^r, & r > -3, \\ 3\epsilon|3|^3|x|^r, & r < -3 \end{cases}$$

*for all* $x \in \mathbb{X}^*$.

*Proof* Considering $F(x,y) = \epsilon(|x|^{\frac{r}{2}}|y|^{\frac{r}{2}} + (|x|^r + |y|^r))$ in Theorem 3.1, one can find the result. □

We have the following result which is analogous to Theorem 3.1 for the functional equation (1.8). We include the proof for the sake of completeness.

**Theorem 3.6** *Let* $l \in \{1, -1\}$ *be fixed, and let* $G : \mathbb{X}^* \times \mathbb{X}^* \longrightarrow [0, \infty)$ *be a mapping such that*

$$\lim_{n\to\infty}\left|\frac{1}{81}\right|^{ln} G\left(\frac{x}{3^{ln+\frac{l+1}{2}}}, \frac{y}{3^{ln+\frac{l+1}{2}}}\right) = 0 \tag{3.8}$$

*for all* $x, y \in \mathbb{X}^*$. *Suppose that* $q : \mathbb{X}^* \longrightarrow \mathbb{Y}$ *is a mapping satisfying the inequality*

$$\left|\Delta_2 q(x,y)\right| \leq G(x,y) \tag{3.9}$$

*for all* $x, y \in \mathbb{X}^*$. *Then there exists a unique reciprocal-quartic mapping* $Q : \mathbb{X}^* \longrightarrow \mathbb{Y}$ *such that*

$$\left|q(x) - Q(x)\right| \leq \sup\left\{\left|\frac{1}{81}\right|^{jl+\frac{l-1}{2}} F\left(\frac{x}{3^{jl+\frac{l+1}{2}}}, \frac{x}{3^{jl+\frac{l+1}{2}}}\right) : j \in \mathbb{N} \cup \{0\}\right\} \tag{3.10}$$

*for all* $x \in \mathbb{X}^*$.

*Proof* Replacing $(x,y)$ by $(x,x)$ in (3.9), we get

$$\left|q(x)-\frac{1}{81^l}q\left(\frac{x}{3^l}\right)\right|\le|81|^{\frac{|l-1|}{2}}G\left(\frac{x}{3^{\frac{l+1}{2}}},\frac{x}{3^{\frac{l+1}{2}}}\right)\tag{3.11}$$

for all $x\in\mathbb{X}^*$. Switching $x$ into $\frac{x}{3^{ln}}$ in (3.11) and multiplying by $|\frac{1}{81}|^{ln}$, we arrive at

$$\left|\frac{1}{81^{ln}}c\left(\frac{x}{3^{ln}}\right)-\frac{1}{81^{(n+1)l}}c\left(\frac{x}{3^{(n+1)l}}\right)\right|\le\left|\frac{1}{81}\right|^{ln+\frac{l-1}{2}}G\left(\frac{x}{3^{ln+\frac{l+1}{2}}},\frac{x}{3^{ln+\frac{l+1}{2}}}\right)\tag{3.12}$$

for all $x\in\mathbb{X}^*$. Relations (3.8) and (3.12) imply that $\{\frac{1}{81^{ln}}q(\frac{x}{3^{ln}})\}$ is a Cauchy sequence. Due to the completeness of $\mathbb{Y}$, there is a mapping $Q:\mathbb{X}^*\longrightarrow\mathbb{Y}$ so that

$$Q(x)=\lim_{n\to\infty}\frac{1}{81^{ln}}q\left(\frac{x}{3^{ln}}\right)\tag{3.13}$$

for all $x\in\mathbb{X}^*$. The rest of the proof is similar to the proof of Theorem 3.1. □

Here, we bring some corollaries regarding the stability of functional equation (1.8) which are a direct consequence of Theorem 3.6.

**Corollary 3.7** *Let $\delta>0$ be a constant, and let $q:\mathbb{X}^*\longrightarrow\mathbb{Y}$ satisfy $|\Delta_2q(x,y)|\le\delta$ for all $x,y\in\mathbb{X}^*$. Then there exists a unique reciprocal-quartic mapping $Q:\mathbb{X}^*\longrightarrow\mathbb{Y}$ satisfying* (1.8) *and $|q(x)-Q(x)|\le\delta$ for all $x\in\mathbb{X}^*$.*

*Proof* It is enough to put $G(x,y)=\delta$ in Theorem 3.6 when $l=-1$. □

**Corollary 3.8** *Let $\delta\ge0$ and $\alpha\ne-4$ be fixed constants. If $q:\mathbb{X}^*\longrightarrow\mathbb{Y}$ satisfies $|\Delta_2q(x,y)|\le\delta(|x|^\alpha+|y|^\alpha)$ for all $x,y\in\mathbb{X}^*$, then there exists a unique reciprocal-quartic mapping $Q:\mathbb{X}^*\longrightarrow\mathbb{Y}$ satisfying* (1.8) *and*

$$\left|q(x)-Q(x)\right|\le\begin{cases}\frac{2\delta}{|3|^\alpha}|x|^\alpha, & \alpha>-4,\\ 2\delta|3|^4|x|^\alpha, & \alpha<-4\end{cases}$$

*for all $x\in\mathbb{X}^*$.*

*Proof* Considering $G(x,y)=\delta(|x|^\alpha+|y|^\alpha)$ for all $x,y\in\mathbb{X}^*$ in Theorem 3.6, we reach the result. □

**Corollary 3.9** *Let $q:\mathbb{X}^*\longrightarrow\mathbb{Y}$ be a mapping, and let there exist real numbers $a$, $b$, $\alpha=a+b\ne-4$ and $\delta\ge0$ such that*

$$\left|D_2q(x,y)\right|\le\delta|x|^a|y|^b$$

*for all $x,y\in\mathbb{X}^*$. Then there exists a unique reciprocal-quartic mapping $Q:\mathbb{X}^*\longrightarrow\mathbb{Y}$ satisfying* (1.8) *and*

$$\left|q(x)-Q(x)\right|\le\begin{cases}\frac{\delta}{|3|^\alpha}|x|^\alpha, & \alpha>-4,\\ \delta|3|^4|x|^\alpha, & \alpha<-4\end{cases}$$

*for all $x\in\mathbb{X}^*$.*

*Proof* Choosing $G(x,y) = \delta|x|^{\alpha}|y|^{\alpha}$ in Theorem 3.6, one can derive the desired result. □

**Corollary 3.10** *Let $\delta \geq 0$ and $\alpha \neq -4$ be real numbers and $q : \mathbb{X}^* \longrightarrow \mathbb{Y}$ be a mapping satisfying the functional inequality*

$$\left|D_2 q(x,y)\right| \leq \delta\left(|x|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}} + \left(|x|^{\alpha} + |y|^{\alpha}\right)\right)$$

*for all $x, y \in \mathbb{X}^*$. Then there exists a unique reciprocal-quartic mapping $Q : \mathbb{X}^* \longrightarrow \mathbb{Y}$ satisfying* (1.8) *and*

$$\left|q(x) - Q(x)\right| \leq \begin{cases} \frac{3\delta}{|3|^{\alpha}}|x|^{\alpha}, & \alpha > -4, \\ 3\delta|3|^{4}|x|^{\alpha}, & \alpha < -4 \end{cases}$$

*for all $x \in \mathbb{X}^*$.*

*Proof* The proof follows immediately by taking $G(x,y) = \delta(|x|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}} + (|x|^{\alpha} + |y|^{\alpha}))$ in Theorem 3.6. □

## 4 Related examples

In this section, applying the idea of the well-known counter-example provided by Gajda [30], we show that Corollary 3.3 for $r = -3$ and Corollary 3.8 for $\alpha = -4$ do not hold in $\mathbb{R}$ with usual $|\cdot|$. Note that $(\mathbb{R}, |\cdot|)$ is an Archimedean field.

Consider the function

$$\varphi(x) = \begin{cases} \frac{\delta}{x^3} & \text{for } x \in (0,\infty), \\ \delta, & \text{otherwise}, \end{cases} \tag{4.1}$$

where $\varphi : \mathbb{R}^* \longrightarrow \mathbb{R}$. Let $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 27^{-n}\varphi\left(3^{-n}x\right) \tag{4.2}$$

for all $x \in \mathbb{R}^*$.

**Theorem 4.1** *If the function $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined in* (4.2) *satisfies the functional inequality*

$$\left|\Delta_1 f(x,y)\right| \leq \frac{28\delta}{13}\left(|x|^{-3} + |y|^{-3}\right) \tag{4.3}$$

*for all $x, y \in X$, then there do not exist a reciprocal-cubic mapping $c : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $\mu > 0$ such that*

$$\left|f(x) - c(x)\right| \leq \mu|x|^{-3} \tag{4.4}$$

*for all $x \in \mathbb{R}^*$.*

*Proof* First, we are going to show that $f$ satisfies (4.3). By computation, we have

$$\left|f(x)\right| = \left|\sum_{n=0}^{\infty} 27^{-n}\varphi\left(3^{-n}x\right)\right| \leq \sum_{n=0}^{\infty} \frac{\delta}{27^n} = \frac{27\delta}{26}.$$

Therefore, we see that $f$ is bounded by $\frac{27\delta}{26}$ on $\mathbb{R}$. If $|x|^{-3} + |y|^{-3} \geq 1$, then the left-hand side of (4.3) is less than $\frac{28\delta}{13}$. Now, suppose that $0 < |x|^{-3} + |y|^{-3} < 1$. Hence, there exists a positive integer $k$ such that

$$\frac{1}{27^{k+1}} \leq |x|^{-3} + |y|^{-3} < \frac{1}{27^k}. \tag{4.5}$$

Thus, relation (4.5) requires $27^k(|x|^{-3} + |y|^{-3}) < 1$ or, equivalently, $27^k x^{-3} < 1$, $27^k y^{-3} < 1$. So, $\frac{x^3}{27^k} > 1$, $\frac{y^3}{27^k} > 1$. The last inequalities imply that $\frac{x^3}{27^{k-1}} > 27 > 1$, $\frac{y^3}{27^{k-1}} > 27 > 1$; and consequently,

$$\frac{1}{3^{k-1}}(x) > 1, \qquad \frac{1}{3^{k-1}}(y) > 1, \qquad \frac{1}{3^{k-1}}(2x + y) > 1, \qquad \frac{1}{3^{k-1}}(x + 2y) > 1.$$

Therefore, for each value of $n = 0, 1, 2, \dots, k - 1$, we obtain

$$\frac{1}{3^n}(x) > 1, \qquad \frac{1}{3^n}(y) > 1, \qquad \frac{1}{3^n}(2x + y) > 1, \qquad \frac{1}{3^n}(x + 2y) > 1$$

and $\Delta_1\varphi(3^{-n}x, 3^{-n}y) = 0$ for $n = 0, 1, 2, \dots, k-1$. Using (4.1) and the definition of $f$, we obtain

$$\begin{aligned}\left|\Delta_1 f(x,y)\right| &\leq \sum_{n=k}^{\infty} \frac{\delta}{27^n} + \sum_{n=k}^{\infty} \frac{\delta}{27^n} + \frac{54}{729}\sum_{n=k}^{\infty} \frac{\delta}{27^n} \leq 2\delta \sum_{n=k}^{\infty} \frac{1}{27^n} + \frac{2\delta}{27}\sum_{n=k}^{\infty} \frac{1}{27^n} \\ &\leq \frac{56\delta}{27}\frac{1}{27^k}\left(1 - \frac{1}{27}\right)^{-1} \leq \frac{28\delta}{13}\frac{1}{27^k} \leq \frac{28\delta}{13}\frac{1}{27^{k+1}} \leq \frac{28\delta}{13}\left(|x|^{-3} + |y|^{-3}\right)\end{aligned}$$

for all $x, y \in \mathbb{R}^*$. Therefore, inequality (4.3) holds. We claim that the reciprocal-cubic functional equation (1.7) is not stable for $r = -3$ in Corollary 3.3. Assume that there exists a reciprocal-cubic mapping $c : \mathbb{R}^* \longrightarrow \mathbb{R}$ satisfying (4.4). Therefore,

$$\left|f(x)\right| \leq (\mu + 1)|x|^{-3}. \tag{4.6}$$

However, we can choose a positive integer $m$ with $m\delta > \mu + 1$. If $x \in (1, 3^{m-1})$, then $3^{-n}x \in (1, \infty)$ for all $n = 0, 1, 2, \dots, m - 1$, and thus

$$\left|f(x)\right| = \sum_{n=0}^{\infty} \frac{\varphi(3^{-n}x)}{27^n} \geq \sum_{n=0}^{m-1} \frac{\frac{27^n\delta}{x^3}}{27^n} = \frac{m\delta}{x^3} > (\mu + 1)x^{-3},$$

which contradicts (4.6). This completes the proof. □

Now, we consider the function $\phi : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined via

$$\phi(x) = \begin{cases} \frac{\lambda}{x^4} & \text{for } x \in (0, \infty), \\ \lambda, & \text{otherwise.} \end{cases} \tag{4.7}$$

Also, let $g : \mathbb{R}^* \longrightarrow \mathbb{R}$ be defined by

$$g(x) = \sum_{n=0}^{\infty} 81^{-n}\phi\left(3^{-n}x\right) \tag{4.8}$$

for all $x \in \mathbb{R}^*$. In analogy with Theorem 4.1, we show that Corollary 3.8 does not hold for $\alpha = -4$ in $\mathbb{R}$ with usual $|\cdot|$.

**Theorem 4.2** *If the function $g : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined in* (4.8) *satisfies the functional inequality*

$$\left|\Delta_2 g(x,y)\right| \le \frac{61\lambda}{20}\left(|x|^{-4} + |y|^{-4}\right) \tag{4.9}$$

*for all $x, y \in X$, then there do not exist a reciprocal-quartic mapping $q : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ such that*

$$\left|g(x) - q(x)\right| \le \beta|x|^{-4} \tag{4.10}$$

*for all $x \in \mathbb{R}^*$.*

*Proof* Let us first prove that $g$ satisfies (4.9).

$$\left|g(x)\right| = \left|\sum_{n=0}^{\infty} 81^{-n}\phi\left(3^{-n}x\right)\right| \le \sum_{n=0}^{\infty} \frac{\lambda}{81^n} = \frac{81\lambda}{80}.$$

Hence, we find that $g$ is bounded by $\frac{81\lambda}{80}$ on $\mathbb{R}$. If $|x|^{-4} + |y|^{-4} \ge 1$, then the left-hand side of (4.9) is less than $\frac{61\lambda}{20}$. Now, suppose that $0 < |x|^{-4} + |y|^{-4} < 1$. Then there exists a positive integer $m$ such that

$$\frac{1}{81^{m+1}} \le |x|^{-4} + |y|^{-4} < \frac{1}{81^m}.$$

By arguments similar to those in Theorem 4.1, the relation $|x|^{-4} + |y|^{-4} < \frac{1}{81^m}$ implies

$$\frac{1}{3^{m-1}}(x) > 1, \qquad \frac{1}{3^{m-1}}(y) > 1, \qquad \frac{1}{3^{m-1}}(2x+y) > 1, \qquad \frac{1}{3^{m-1}}(2x-y) > 1.$$

Therefore, for any $n = 0, 1, 2, \dots, m-1$, we get

$$\frac{1}{3^n}(x) > 1, \qquad \frac{1}{3^n}(y) > 1, \qquad \frac{1}{3^n}(2x+y) > 1, \qquad \frac{1}{3^n}(2x-y) > 1$$

and $\Delta_2\phi(3^{-n}x, 3^{-n}y) = 0$ for $n = 0, 1, 2, \dots, m-1$. Using (4.7) and the definition of $g$, we find

$$\begin{aligned}
\left|\Delta_2 g(x,y)\right| &\le \sum_{n=m}^{\infty} \frac{\lambda}{81^n} + \sum_{n=m}^{\infty} \frac{\lambda}{81^n} + \frac{82}{81}\sum_{n=k}^{\infty} \frac{\lambda}{81^n} \le 2\lambda \sum_{n=m}^{\infty} \frac{1}{81^n} + \frac{82\lambda}{81}\sum_{n=m}^{\infty} \frac{1}{81^n} \\
&\le \frac{244\lambda}{81}\frac{1}{81^m}\left(1 - \frac{1}{81}\right)^{-1} \le \frac{244\lambda}{80}\frac{1}{81^m} \le \frac{244\lambda}{80}\frac{1}{81^{k+1}} \\
&\le \frac{61\lambda}{20}\left(|x|^{-4} + |y|^{-4}\right)
\end{aligned}$$

for all $x, y \in \mathbb{R}^*$. This shows that inequality (4.9) holds. Here, we prove that the reciprocal-quartic functional equation (1.8) is not stable for $\alpha = -4$ in Corollary 3.8. Assume that there exists a reciprocal-quartic mapping $q : \mathbb{R}^* \longrightarrow \mathbb{R}$ satisfying (4.10). Hence

$$\left|g(x)\right| \le (\beta + 1)|x|^{-4}. \tag{4.11}$$

On the other hand, we can choose a positive integer $k$ with $k\lambda > \beta + 1$. If $x \in (1, 3^{k-1})$, then $3^{-n}x \in (1, \infty)$ for all $n = 0, 1, 2, \ldots, k-1$, and so

$$\left|g(x)\right| = \sum_{n=0}^{\infty} \frac{\phi(3^{-n}x)}{81^n} \ge \sum_{n=0}^{k-1} \frac{\frac{81^n\lambda}{x^4}}{81^n} = \frac{k\lambda}{x^4} > (\beta + 1)x^{-4},$$

which contradicts (4.11). Therefore, the reciprocal-quartic functional equation (1.8) is not stable in the case $\alpha = -4$ in Corollary 3.8 for $(\mathbb{R}, |\cdot|)$. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The study presented here was carried out in collaboration between all authors. BVSK suggested writing the current article. All authors read and approved the final manuscript.

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