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On the nonexistence of global solutions for a class of fractional integro-differential problems

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Abstract

We study the nonexistence of (nontrivial) global solutions for a class of fractional integro-differential problems in an appropriate underlying space. Integral conditions on the kernel, and for some degrees of the involved parameters, ensuring the nonexistence of global solutions are determined. Unlike the existing results, the source term considered is, in general, a convolution and therefore nonlocal in time. The class of problems we consider includes problems with sources that are polynomials and fractional integrals of polynomials in the state as special cases. Singular kernels illustrating interesting cases in applications are provided and discussed. Our results are obtained by considering a weak formulation of the problem with an appropriate test function and several appropriate estimations.

Keywords: nonexistence; global solution; fractional integro-differential equation; Riemann-Liouville fractional derivative; nonlocal source

1 Introduction

We consider the initial value problem

$$\begin{cases} (D_{0^+}^{\alpha}x)(t) + \sigma(D_{0^+}^{\beta}x)(t) \geq \int_0^t k(t-s)|x(s)|^q\,ds, & t>0, q>1, \\ (I_{0^+}^{1-\alpha}x)(0^+) = c_0, \quad c_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where D_{0+}^{α} and D_{0+}^{β} are the Riemann-Liouville fractional derivatives of orders α and β, respectively, $0 \leq \beta < \alpha \leq 1$ (see (8)-(10)), and $\sigma = 0, 1$.

Problem (1) includes many interesting special cases. When $\alpha = 1$, $\sigma = 0$ and $k(t) = \delta(t)$ (the Dirac delta function), the equality in (1) reduces to the initial value for the Bernoulli differential equation

$$\begin{cases} x'(t) + x(t) = x^q(t), & t>0, q>1, \\ x(0) = x_0, \end{cases} \tag{2}$$

for which the solution is

$$x(t) = \left(\left(x_0^{1-q} - 1\right)e^{(q-1)t} + 1\right)^{\frac{1}{1-q}}.$$

This solution blows up in the finite time

$$T_b = \frac{1}{1-q}\ln\bigl(1 - x_0^{1-q}\bigr)$$

if and only if the initial data $x_0 > 1$ (see e.g. [1]).

The nonlinear Volterra integro-differential equation

$$x'(t) = -c + \int_0^t x^q(s)\,ds, \quad t > 0, q > 1, \tag{3}$$

can be transformed by differentiation into the second-order ordinary differential equation

$$x''(t) = x^q(t). \tag{4}$$

When $c = \sqrt{\frac{2}{q+1}x_0^{q+1}}$, $x_0 > 0$, the solution of (4) is given by

$$x(t) = \left(\frac{1-q}{2}\sqrt{\frac{2}{q+1}}t + x_0^{\frac{1-q}{2}}\right)^{\frac{2}{1-q}},$$

and it blows up in the finite time

$$T_b = \frac{2}{q-1}\sqrt{\frac{q+1}{2}}x_0^{\frac{1-q}{2}}.$$

When $\alpha = \sigma = 1$, $\beta = 0$, $x(0) = x_0 \geq 0$ and $k(t)$ is a positive and locally integrable function with $\lim_{t\to\infty}\int_0^t k(s)\,ds = \infty$, Ma showed in [2] that the solution of

$$x'(t) + x(t) = \int_0^t k(t-s)f\bigl(x(s)\bigr)\,ds, \quad t > 0, \tag{5}$$

blows up in finite time if and only if, for some $\beta > 0$,

$$\int_\nu^\infty \left(\frac{s}{f(s)}\right)^{\frac{1}{\beta}}\frac{ds}{s} < \infty \quad \text{for any } \nu > 0. \tag{6}$$

Here $f(t)$ is assumed to be continuous, nonnegative and nondecreasing for $t > 0$, $f \equiv 0$ for $t \leq 0$, and $\lim_{t\to\infty}\frac{f(t)}{t} = \infty$. Clearly, if $f(x(s)) = |x(s)|^q$ in (5), condition (6) simply means that $q > 1$.

Recently, Kassim *et al.* showed in [3] that the problem

$$\begin{cases} (D_{0^+}^{\alpha}x)(t) + (D_{0^+}^{\beta}x)(t) \geq t^{\theta}|x(t)|^q, & t > 0, 0 < \beta \leq \alpha \leq 1, \\ (I_{0^+}^{1-\alpha}x)(0^+) = c_0, & c_0 \in \mathbb{R}, \end{cases}$$

has no global solution when $c_0 \geq 0$, $1 < q \leq \frac{\theta+1}{1-\beta}$ and $\theta > -\beta$.

The authors in [4] proved that all nontrivial solutions $u \in C([0, T_{\max}, C_0(\mathbb{R}^N)))$ of the initial value problem

$$\begin{cases} u_t - \Delta u = \int_0^t (t-s)^{-\gamma}|u|^{q-1}u(s)\,ds, & \text{in } (0,T)\times\mathbb{R}^N, \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{7}$$

where $q > 1$, $0 \leq \gamma < 1$ and $u_0 \in C_0(\mathbb{R}^N)$, blow up in finite time when $u_0 \geq 0$ and $q \leq \max\{\frac{1}{\gamma}, 1 + \frac{4-2\gamma}{(N-2+2\gamma)^+}\} \in (0, \infty]$. See Remark 10 for the main difference of this result with the present one. This will clarify our main contribution.

It is known from the definition of Riemann-Liouville fractional derivative that it uses in someway all the history of the state through a convolution with a singular kernel. Moreover, in the case of fractional integro-differential equations, the source term may involve additional singularities in the kernel. Because of all these issues, it is difficult to apply the approaches and methods for integer order existing in the literature to the non-integer case.

As is well known, studying the nonexistence of solutions for differential equations is as important as studying the existence of solutions. The sufficient conditions for the nonexistence of solutions provide necessary conditions for the existence of solutions. Investigating the nonexistence of solutions for differential equations provides very important and necessary information on limiting behaviors of many physical systems. It is also interesting to know what could happen to these solutions in cases such as blowing up in finite time or at infinity. In industry, knowing the blow-up in finite time can prevent accidents and malfunction. It helps also improve the performance of machines and extend their lifespan.

There are many results in the literature on the existence of solutions for various classes of fractional differential equations and fractional integro-differential equations (see [5–14]). Agarwal *et al.* surveyed many of these results in [5]. They focused on initial and boundary value problems for fractional differential equations with Caputo fractional derivatives of orders between 0 and 1.

For the issue of nonexistence of local solutions and global solutions for fractional differential equations, we refer to [15–21] and the references therein. However, to the best of our knowledge, there are no investigations on the nonexistence of solutions for fractional integro-differential inequalities of type (1).

In this paper, we prove the nonexistence of (nontrivial) global solutions for the initial value problem (1) under some integral conditions on the kernel $k(t)$. The proof is based on the test function method due to Mitidieri and Pohozaev [22] adopted here to the fractional case, see also [15, 18, 21]. For the purpose of studying the effect of considering one or two fractional derivatives, we choose σ to be either 0 or 1.

Our results could be utilized to identify the limitations of many physical systems and to analyze the behavior of solutions of some nonlinear fractional differential equations and inequalities for which the explicit solution may not be available. Also, our results will extend the abundant results on integer-order problems to the (limited results available for) fractional-order problems.

The rest of this paper is organized as follows. In the next section we briefly recall some necessary material from fractional calculus that we use in this paper. Section 3 is devoted to the statements and proofs of our results. Some applications and special cases are given in Section 4.

2 Preliminaries

In this section we introduce some notation, definitions and preliminary results from fractional calculus.

Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$. The Riemann-Liouville left-sided and right-sided fractional derivatives of order $0 \leq \alpha \leq 1$ are defined by

$$\left(D_{a^+}^{\alpha}x\right)(t) = D\left(I_{a^+}^{1-\alpha}x\right)(t), \tag{8}$$

$$\left(D_{b^-}^{\alpha}x\right)(t) = -D\left(I_{b^-}^{1-\alpha}x\right)(t), \tag{9}$$

respectively, where $D = \frac{d}{dt}$. $I_{a^+}^{\alpha}$ and $I_{b^-}^{\alpha}$ are the Riemann-Liouville left-sided and right-sided fractional integrals of order $\alpha > 0$ defined by

$$\begin{aligned} \left(I_{a+}^{\alpha}x\right)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\,ds, \quad t > a, \\ \left(I_{b^-}^{\alpha}x\right)(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}x(s)\,ds, \quad t < b, \end{aligned} \tag{10}$$

respectively, provided the right-hand sides exist. We define $I_{a+}^0 x = I_{b^-}^0 x = x$. The function Γ is the Euler gamma function. In particular, when $\alpha = 0$ and $\alpha = 1$, it follows from the definition that

$$D_{a^+}^0 x = D_{b^-}^0 x = x \quad \text{and} \quad D_{a^+}^1 x = -D_{b^-}^1 x = Dx.$$

For more details about fractional integrals and fractional derivatives, the reader is referred to the books [23–26].

We denote by $L^p(a,b)$, $1 \leq p < \infty$, the set of Lebesgue real-valued measurable functions f on $[a,b]$ for which $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} = \left(\int_a^b \left|f(t)\right|^p dt\right)^{1/p}, \quad 1 \leq p < \infty.$$

We denote by $C_\gamma[a,b]$ and $C_\gamma^\mu[a,b]$ *the following two weighted spaces of continuous functions:*

$$\begin{aligned} C_\gamma[a,b] &= \left\{f : (a,b] \to \mathbb{R} \mid (t-a)^\gamma f(t) \in C[a,b]\right\}, \\ C_\gamma^\mu[a,b] &= \left\{f : (a,b] \to \mathbb{R} \mid f, D_{0^+}^\mu f \in C_\gamma[a,b]\right\}, \end{aligned} \tag{11}$$

respectively, where $0 \leq \gamma < 1$, $\mu \geq 0$ and $C[a,b]$ is the space of continuous functions.

The next lemma shows that the Riemann-Liouville fractional integral and derivative of the power functions yield power functions multiplied by certain coefficients and with the order of the fractional derivative added or subtracted from the power.

Lemma 1 ([23]) *If* $\alpha \geq 0$, $\beta > 0$, *then*

$$\left(I_{b^-}^{\alpha}(b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1},$$

$$\left(D_{b^-}^{\alpha}(b-s)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}.$$

Now we consider a useful property of the Riemann-Liouville fractional integral I_{a+}^{α} in the space $C_\gamma[a,b]$ defined in (11).

Lemma 2 ([27]) *Let $0 \leq \gamma < 1$ and $\alpha > \gamma$. If $u \in C_\gamma[a,b]$, then*

$$\left(I_{a+}^{\alpha} u\right)\left(a^{+}\right) = \lim_{t \to a^{+}} \left(I_{a+}^{\alpha} u\right)(t) = 0.$$

A formula for the fractional integration by parts is given in the next lemma.

Lemma 3 ([26]) *Let $\alpha \geq 0$, $m_1 \geq 1$, $m_2 \geq 1$ and $\frac{1}{m_1} + \frac{1}{m_2} \leq 1 + \alpha$ ($m_1 \neq 1$ and $m_2 \neq 1$ in the case when $\frac{1}{m_1} + \frac{1}{m_2} = 1 + \alpha$). If $\varphi_1 \in L^{m_1}(a,b)$ and $\varphi_2 \in L^{m_2}(a,b)$, then*

$$\int_a^b \varphi_1(t)\left(I_{a+}^{\alpha}\varphi_2\right)(t)\,dt = \int_a^b \varphi_2(t)\left(I_{b-}^{\alpha}\varphi_1\right)(t)\,dt.$$

In this paper, we use the test function

$$\varphi(t) := \begin{cases} T^{-\lambda}(T-t)^{\lambda}, & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \tag{12}$$

This test function has the following property.

Lemma 4 *Let φ be as in (12) and $p > 1$, then for $\lambda > p-1$,*

$$\int_0^T \varphi^{1-p}(t)\left|\varphi'(t)\right|^p\,dt = \frac{\lambda^p}{(\lambda - p + 1)} T^{1-p}, \quad T > 0.$$

Proof

$$\begin{aligned} &\int_0^T \varphi^{1-p}(t)\left|\varphi'(t)\right|^p\,dt \\ &\quad = \int_0^T T^{-\lambda+\lambda p}(T-t)^{\lambda-\lambda p}\left|-\lambda T^{-\lambda}(T-t)^{\lambda-1}\right|^p\,dt \\ &\quad = \lambda^p T^{-\lambda}\int_0^T (T-t)^{\lambda-p}\,dt = \frac{\lambda^p}{(\lambda-p+1)}T^{1-p}. \end{aligned}$$ □

3 The nonexistence results

In this section we study the nonexistence of a global solution for the initial value problem (1). We start with the following lemma.

Lemma 5 *Let $0 \leq \nu \leq 1$ and $p > 1$. Let φ be as in (12) with $\lambda > p-1$. Suppose that k is a nonnegative function which is different from zero almost everywhere and $t^{-\nu p}k^{1-p}(t) \in L^1_{\mathrm{loc}}[0,+\infty)$. Then, for any $T > 0$,*

$$\int_0^T \left(I_{T^-}^{1-\nu}\left|\varphi'\right|\right)^p(t)\left(\int_t^T k(s-t)\varphi(s)\,ds\right)^{1-p}\,dt \leq \Lambda_{\nu,p}T^{1-p}\int_0^T t^{-\nu p}k^{1-p}(t)\,dt,$$

where $\Lambda_{\nu,p} = \frac{\lambda^p}{(\lambda-p+1)(\Gamma(1-\nu))^p}$.

Proof Since

$$\begin{aligned}\left(I_{T^-}^{1-\nu}|\varphi'|\right)(t) &= \frac{1}{\Gamma(1-\nu)}\int_t^T (s-t)^{-\nu}\left|\varphi'(s)\right| ds \\ &= \frac{1}{\Gamma(1-\nu)}\int_t^T (s-t)^{-\nu}k^{\frac{1}{p'}}(s-t)\varphi^{\frac{1}{p'}}(s)k^{-\frac{1}{p'}}(s-t)\varphi^{-\frac{1}{p'}}(s)\left|\varphi'(s)\right| ds\end{aligned}$$

for all $0 \leq t < T$. Using Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$, we find

$$\begin{aligned}\left(I_{T^-}^{1-\nu}|\varphi'|\right)(t) \leq{}& \frac{1}{\Gamma(1-\nu)}\left(\int_t^T k(s-t)\varphi(s)\, ds\right)^{\frac{1}{p'}} \\ &\times \left(\int_t^T (s-t)^{-\nu p}k^{-\frac{p}{p'}}(s-t)\varphi^{-\frac{p}{p'}}(s)\left|\varphi'(s)\right|^p ds\right)^{\frac{1}{p}}. \end{aligned}\tag{13}$$

Therefore,

$$\begin{aligned}&\int_0^T \left(I_{T^-}^{1-\nu}|\varphi'|\right)^p(t)\left(\int_t^T k(s-t)\varphi(s)\, ds\right)^{1-p} dt \\ &\leq \frac{1}{(\Gamma(1-\nu))^p}\int_0^T\int_t^T (s-t)^{-\nu p}k^{-\frac{p}{p'}}(s-t)\varphi^{-\frac{p}{p'}}(s)\left|\varphi'(s)\right|^p ds\, dt \\ &= \frac{1}{(\Gamma(1-\nu))^p}\int_0^T\int_0^s (s-t)^{-\nu p}k^{1-p}(s-t)\varphi^{1-p}(s)\left|\varphi'(s)\right|^p dt\, ds \\ &= \frac{1}{(\Gamma(1-\nu))^p}\int_0^T \varphi^{1-p}(s)\left|\varphi'(s)\right|^p\left(\int_0^s (s-t)^{-\nu p}k^{1-p}(s-t)\, dt\right) ds. \end{aligned}\tag{14}$$

Let $\tau = s-t$ in the inner integral, then we obtain the uniform bound

$$\int_0^s \tau^{-\nu p}k^{1-p}(\tau)\, d\tau \leq \int_0^T \tau^{-\nu p}k^{1-p}(\tau)\, d\tau.$$

Now the result follows from Lemma 4. □

Definition 6 By a global nontrivial solution to problem (1), we mean a nonzero function $x(t)$ defined for all $t > 0$ such that $x \in C_{1-\alpha}^{\alpha}[0,T]$ for all $T > 0$ that satisfies the inequality and initial conditions in (1).

In what follows we provide the conditions under which problem (1) cannot have global nontrivial solutions.

Theorem 7 *Let* $0 \leq \beta < \alpha \leq 1$ *and* k *be a nonnegative function which is different from zero almost everywhere. Assume that* $(t^{-\alpha q'}+\sigma^{q'}t^{-\beta q'})k^{1-q'}(t) \in L^1_{\mathrm{loc}}[0,\infty)$ *and*

$$\lim_{T\to\infty} T^{1-q'}\left(\int_0^T t^{-\alpha q'}k^{1-q'}(t)\, dt + \sigma^{q'}\int_0^T t^{-\beta q'}k^{1-q'}(t)\, dt\right) = 0, \tag{15}$$

where $q' = \frac{q}{q-1}$. *Then problem* (1) *does not admit any global nontrivial solution when* $c_0 \geq 0$.

Proof Assume, on the contrary, that a solution $x \in C^{\alpha}_{1-\alpha}[0,T]$ exists for all $T > 0$. Multiplying both sides of the inequality in (1) by the test function φ defined in (12) with $\lambda > 2q' - 1$ and integrating, we obtain

$$J \leq \int_0^T \varphi(t)\bigl(D^{\alpha}_{0^+}x\bigr)(t)\,dt + \sigma \int_0^T \varphi(t)\bigl(D^{\beta}_{0^+}x\bigr)(t)\,dt, \tag{16}$$

where

$$J := \int_0^T \varphi(t)\biggl(\int_0^t k(t-s)\bigl|x(s)\bigr|^q\,ds\biggr)\,dt.$$

An integration by parts for each integral on the right-hand side of (16) gives

$$\begin{aligned}\int_0^T \varphi(t)\bigl(D^{\alpha}_{0^+}x\bigr)(t)\,dt &= \int_0^T \varphi(t)\bigl(DI^{1-\alpha}_{0^+}x\bigr)(t)\,dt \\ &= \bigl[\varphi(t)\bigl(I^{1-\alpha}_{0^+}x\bigr)(t)\bigr]_{t=0}^T - \int_0^T \varphi'(t)\bigl(I^{1-\alpha}_{0^+}x\bigr)(t)\,dt\end{aligned} \tag{17}$$

and

$$\int_0^T \varphi(t)\bigl(D^{\beta}_{0^+}x\bigr)(t)\,dt = \bigl[\varphi(t)\bigl(I^{1-\beta}_{0^+}x\bigr)(t)\bigr]_{t=0}^T - \int_0^T \varphi'(t)\bigl(I^{1-\beta}_{0^+}x\bigr)(t)\,dt. \tag{18}$$

As $\varphi(T) = 0$, $\varphi(0) = 1$ and $(I^{1-\alpha}_{0^+}x)(0^+) = c_0$, we can write (17) as

$$\int_0^T \varphi(t)\bigl(D^{\alpha}_{0^+}x\bigr)(t)\,dt = -c_0 - \int_0^T \varphi'(t)\bigl(I^{1-\alpha}_{0^+}x\bigr)(t)\,dt.$$

Also, since $I^{1-\beta}_{0^+}x = I^{\alpha-\beta}_{0^+}I^{1-\alpha}_{0^+}x$, $x \in C_{1-\alpha}[0,T]$ and $\beta < \alpha$, we see from Lemma 2 that $(I^{1-\beta}_{0^+}x)(0^+) = 0$. Hence (18) reduces to

$$\int_0^T \varphi(t)\bigl(D^{\beta}_{0^+}x\bigr)(t)\,dt = -\int_0^T \varphi'(t)\bigl(I^{1-\beta}_{0^+}x\bigr)(t)\,dt,$$

and (16) becomes

$$J \leq -c_0 - \int_0^T \varphi'(t)\bigl(I^{1-\alpha}_{0^+}x\bigr)(t)\,dt - \sigma \int_0^T \varphi'(t)\bigl(I^{1-\beta}_{0^+}x\bigr)(t)\,dt. \tag{19}$$

Having in mind that $c_0 \geq 0$ and φ' is negative, we entail that

$$\begin{aligned}J &\leq \int_0^T \bigl(-\varphi'(t)\bigr)\bigl(I^{1-\alpha}_{0^+}x\bigr)(t)\,dt + \sigma \int_0^T \bigl(-\varphi'(t)\bigr)\bigl(I^{1-\beta}_{0^+}x\bigr)(t)\,dt \\ &\leq \int_0^T \bigl(-\varphi'(t)\bigr)\bigl(I^{1-\alpha}_{0^+}|x|\bigr)(t)\,dt + \sigma \int_0^T \bigl(-\varphi'(t)\bigr)\bigl(I^{1-\beta}_{0^+}|x|\bigr)(t)\,dt.\end{aligned} \tag{20}$$

Applying Lemma 3 to each integral on the right-hand side of (20), we obtain

$$J \leq \int_0^T \bigl|x(t)\bigr|\bigl(I^{1-\alpha}_{T^-}\bigl(-\varphi'\bigr)\bigr)(t)\,dt + \sigma \int_0^T \bigl|x(t)\bigr|\bigl(I^{1-\beta}_{T^-}\bigl(-\varphi'\bigr)\bigr)(t)\,dt. \tag{21}$$

To obtain a bound for the expression J, we rewrite J as

$$J = \int_0^T |x(s)|^q \left(\int_s^T k(t-s)\varphi(t)\,dt \right) ds = \int_0^T |x(s)|^q K(s)\,ds, \tag{22}$$

where

$$K(s) := \int_s^T k(t-s)\varphi(t)\,dt, \quad 0 \le s < t \le T. \tag{23}$$

Next, we insert $K^{\frac{1}{q}}(t)K^{-\frac{1}{q}}(t)$ inside each integral on the right-hand side of (21) and apply Hölder's inequality

$$J \le J^{\frac{1}{q}} \left(\int_0^T K^{1-q'}(t) \left(I_{T^-}^{1-\alpha}(-\varphi') \right)^{q'}(t)\,dt \right)^{\frac{1}{q'}} + \sigma \left(\int_0^T K^{\frac{-q'}{q}}(t) \left(I_{T^-}^{1-\beta}(-\varphi') \right)^{q'}(t)\,dt \right)^{\frac{1}{q'}}$$

or

$$J \le 2^{q'-1} \left(\int_0^T K^{1-q'}(t) \left(I_{T^-}^{1-\alpha}(-\varphi') \right)^{q'}(t)\,dt + \sigma^{q'} \int_0^T K^{\frac{-q'}{q}}(t) \left(I_{T^-}^{1-\beta}(-\varphi') \right)^{q'}(t)\,dt \right).$$

Using Lemma 5, we get

$$J \le \Lambda_1 T^{1-q'} \left(\int_0^T t^{-\alpha q'} k^{1-q'}(t)\,dt + \sigma^{q'} \int_0^T t^{-\beta q'} k^{1-q'}(t)\,dt \right), \tag{24}$$

where $\Lambda_1 = 2^{q'-1} \max\{\Lambda_{\alpha,q'}, \Lambda_{\beta,q'}\}$. Assumption (15) leads to a contradiction since the solution is supposed to be nontrivial. □

Our Theorem 7 shows that the fractional damping is not able to remove the effect of nonlinearity. It provides sufficient conditions on the exponent q and on the family of kernels, which leads to the nonexistence of global solutions.

As a corollary of Theorem 7, we have the following result.

Corollary 8 *Let $0 \le \beta < \alpha \le 1$ and k be a nonnegative function which is different from zero almost everywhere with $t^{-\alpha q'} k^{1-q'}$, $t^{-\beta q'} k^{1-q'}(t) \in L^1_{\mathrm{loc}}[0,+\infty)$. Suppose that, for any $T > 0$, there are some positive constants c_1, c_2, θ_1, θ_2 with*

$$0 < \theta_1, \theta_2 < \frac{1}{q-1} \tag{25}$$

such that

$$\int_0^T t^{-\alpha q'} k^{1-q'}(t)\,dt \le c_1 T^{\theta_1} \quad \text{and} \quad \int_0^T t^{-\beta q'} k^{1-q'}(t)\,dt \le c_2 T^{\theta_2}, \tag{26}$$

where $q' = \frac{q}{q-1}$. Then problem (1) *does not admit any global nontrivial solution when $c_0 \ge 0$.*

Proof To prove this corollary, it suffices to notice that conditions (25) and (26) imply that hypothesis (15) is fulfilled. Indeed, in virtue of (26), we have

$$0 \le T^{1-q'} \left(\int_0^T t^{-\alpha q'} k^{1-q'}(t)\,dt + \sigma^{q'} \int_0^T t^{-\beta q'} k^{1-q'}(t)\,dt \right) \le c_1 T^{1-q'+\theta_1} + \sigma^{q'} c_2 T^{1-q'+\theta_2}.$$

We find from (25) that $1 - q' + \theta_1$ and $1 - q' + \theta_2$ are both negative and condition (15) follows. □

4 Applications

Our results can be applied to a variety of kernels that appear in the literature. The following corollary of Theorem 7 is concerned with the Riemann-Liouville fractional integral kernel.

Corollary 9 *Let $0 \leq \beta < \alpha \leq 1$ and $q > 1$. Suppose that $k(t) \geq at^{-\gamma}$, $t > 0$, for some constant $a > 0$, where $1 - q(1 - \alpha) < \gamma < 2 + q(\beta - 1)$. Then problem* (1) *does not admit a global nontrivial solution when $c_0 \geq 0$.*

Proof It suffices to show that the function k satisfies (15). Indeed, since $k(t) \geq at^{-\gamma}$; $a > 0$, then $k^{1-q'}(t) \leq a^{1-q'} t^{\gamma(q'-1)}$, $q' = \frac{q}{q-1}$ and

$$\int_0^T t^{-\alpha q'} k^{1-q'}(t)\,dt \leq a^{1-q'} \int_0^T t^{\gamma(q'-1)-\alpha q'}\,dt = \frac{a^{1-q'}}{\gamma(q'-1) - \mu q' + 1} T^{\gamma(q'-1)-\mu q'+1},$$

$$\int_0^T t^{-\beta q'} k^{1-q'}(t)\,dt \leq \frac{a^{1-q'}}{\gamma(q'-1) - \beta q' + 1} T^{\gamma(q'-1)-\beta q'+1}.$$

Hence,

$$T^{1-q'} \left(\int_0^T t^{-\alpha q'} k^{1-q'}(t)\,dt + \sigma^{q'} \int_0^T t^{-\beta q'} k^{1-q'}(t)\,dt \right)$$
$$\leq \frac{a^{1-q'}}{\gamma(q'-1) - \alpha q' + 1} T^{2-\gamma+q'(\gamma-\alpha-1)} + \frac{a^{1-q'}\sigma^{q'}}{\gamma(q'-1) - \beta q' + 1} T^{2-\gamma+q'(\gamma-\beta-1)}.$$

It follows from $1 - q(1 - \alpha) < \gamma < 2 + q(\beta - 1)$ that (15) is satisfied. □

Remark 10 Notice that the kernel treated in Problem 7 fits into the special case considered in Corollary 9. Treating a more general kernel is not the main difference with the work in [4]. The problems, the results and the arguments are different. Indeed, we treated a fractional equation (or inequality) and proved a 'nonexistence' result, whereas in [4] they studied the heat equation (order one) and proved a 'blow-up' result. Even in the fractional context, introducing a fractional damping presents a challenge as it is known that damping competes with the nonlinear force. It tends to annihilate (or at least reduce) the destabilizing effect produced by the nonlinear source.

Remark 11 Corollary 9 can be considered also as a consequence of Corollary 8 with

$$c_1 = \frac{a^{1-q'}}{\gamma(q'-1) - \alpha q' + 1}, \qquad c_2 = \frac{a^{1-q'}\sigma^{q'}}{\gamma(q'-1) - \beta q' + 1},$$
$$\theta_1 = \gamma\left(q' - 1\right) - \alpha q' + 1 = \frac{q(1-\alpha) + \gamma - 1}{q-1}, \quad 0 < \alpha \leq 1,$$
$$\theta_2 = \gamma\left(q' - 1\right) - \beta q' + 1 = \frac{q(1-\beta) + \gamma - 1}{q-1}, \quad 0 \leq \beta < \alpha \leq 1.$$

It is clear from $1 - q(1 - \alpha) < \gamma < 2 + q(\beta - 1)$ that $0 < \theta_1, \theta_2 < \frac{1}{q-1}$.

Remark 12 Observe that the upper bound of the exponent γ is controlled by the order β of the lower derivative.

As an example of the kernels in Corollary 9, we have the following case when the right-hand side of (1) is the Riemann-Liouville fractional integral of $|x(t)|^q$.

Example 13 The problem

$$\begin{aligned} &\left(D_{0+}^{\alpha}x\right)(t) + \left(D_{0+}^{\beta}x\right)(t) \geq \left(I_{0^+}^{1-\gamma}\left|x(s)\right|^q\right)(t), \quad t > 0, q > 1, \\ &\left(I_{0^+}^{1-\alpha}x\right)\left(0^+\right) = c_0, \quad c_0 \in \mathbb{R}, \end{aligned} \tag{27}$$

is a special case of problem (1) with

$$k(t) = t^{-\gamma}, \qquad 1 - q(1-\alpha) < \gamma < 2 + q(\beta - 1), \quad 0 \leq \beta < \alpha < 1.$$

Therefore, as a consequence of Corollary 9, problem (27) does not admit a global nontrivial solution when $c_0 \geq 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All three authors have worked on all parts of the paper equally.

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