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Solvability and Mann iterative approximations for a higher order nonlinear neutral delay differential equation

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Abstract

The purpose of this paper is to study solvability of the higher order nonlinear neutral delay differential equation

$$\frac{d^n}{dt^n}\big[x(t)+c(t)x(t-\tau)\big]+(-1)^{n+1}f\big(t,x(\sigma_1(t)),x(\sigma_2(t)),\ldots,x(\sigma_k(t))\big)$$
$$=g(t),\quad t\geq t_0,$$

where n and k are positive integers, $\tau>0$, $t_0\in\mathbb{R}$, $f\in C([t_0,+\infty)\times\mathbb{R}^k,\mathbb{R})$, $c,g,\sigma_i\in C([t_0,+\infty),\mathbb{R})$ and $\lim_{t\to+\infty}\sigma_i(t)=+\infty$ for $i\in\{1,2,\ldots,k\}$. Under suitable conditions, several existence results of uncountably many nonoscillatory solutions and convergence of Mann iterative approximations for the above equation are shown. Three nontrivial examples are given to demonstrate the advantage of our results over the existing ones in the literature.

MSC: 34K07; 34K11; 34C15

Keywords: higher order nonlinear neutral delay differential equation; uncountably many nonoscillatory solutions; contraction mapping; Mann iterative sequence; error estimate

1 Introduction and preliminaries

In the past twenty years or so, the existence of oscillatory and nonoscillatory solutions for a lot of neutral delay linear and nonlinear differential equations has been studied and discussed by many researchers, for example, see [1–9] and the references therein.

In 1989, Chuanxi and Ladas [4] investigated the first order neutral delay differential equation

$$\frac{d}{dt}\big[x(t)+p(t)x(t-\tau)\big]+Q(t)x(t-\sigma)=0,\quad t\geq t_0.\tag{1.1}$$

In 1998, 2001 and 2004, Kulenović and Hadžiomerspahić [6, 7] and Cheng and Annie [3] investigated, respectively, the first and second order neutral delay differential equations with positive and negative coefficients:

$$\frac{d}{dt}\big[x(t)+cx(t-\tau)\big]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,\quad t\geq t_0\tag{1.2}$$

and

$$\frac{d^2}{dt^2}\left[x(t) + cx(t-\tau)\right] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \quad t \ge t_0. \tag{1.3}$$

Under $c \neq \pm 1$ and other conditions, they obtained sufficient conditions for the existence of nonoscillatory solutions of Eqs. (1.2) and (1.3), respectively. In 2002, Zhou and Zhang [9] extended the result in [7] to the nth order neutral functional differential equation with positive and negative coefficients

$$\frac{d^n}{dt^n}\left[x(t) + cx(t-\tau)\right] + (-1)^{n+1}\left[Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2)\right] = 0, \quad t \ge t_0, \tag{1.4}$$

where $c \neq \pm 1$. In 2007, Liu et al. [8] extended and improved the results in [3, 6, 7, 9] to the following nth order neutral delay nonlinear differential equation:

$$\begin{aligned}&\frac{d^n}{dt^n}\left[x(t) + cx(t-\tau)\right] + (-1)^{n+1}f\big(t, x(t-\sigma_1), x(t-\sigma_2), \dots, x(t-\sigma_k)\big)\\ &\quad = g(t), \quad t \ge t_0,\end{aligned} \tag{1.5}$$

where $c \neq -1$. They gave sufficient conditions for the existence of nonoscillatory solutions for Eq. (1.5), constructed the Mann iterative approximations for these nonoscillatory solutions and established the error estimates between these nonoscillatory solutions and the Mann iterative approximations. At the same time, they also proved the existence of infinitely many nonoscillatory solutions for Eq. (1.5).

Inspired and motivated by the results in [1–9], in this paper, we study the following higher order nonlinear neutral delay differential equation:

$$\begin{aligned}&\frac{d^n}{dt^n}\left[x(t) + c(t)x(t-\tau)\right] + (-1)^{n+1}f\big(t, x(\sigma_1(t)), x(\sigma_2(t)), \dots, x(\sigma_k(t))\big)\\ &\quad = g(t), \quad t \ge t_0,\end{aligned} \tag{1.6}$$

where n and k are positive integers, $\tau > 0$, $t_0 \in \mathbb{R}$, $c, g, \sigma_i \in C([t_0, +\infty), \mathbb{R})$, $\lim_{t\to+\infty}\sigma_i(t) = +\infty$ for $i \in \{1, 2, \dots, k\}$, and $f \in C([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$ satisfies the following condition:

(H_1) there exist constants $M > N > 0$ and functions $p, q \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying

$$\begin{aligned}&\left|f(t, u_1, \dots, u_k) - f(t, \bar{u}_1, \dots, \bar{u}_k)\right|\\ &\quad \le p(t)\max\left\{|u_i - \bar{u}_i| : 1 \le i \le k\right\}, \quad t \in [t_0, +\infty), u_i, \bar{u}_i \in [N, M], 1 \le i \le k\end{aligned}$$

and

$$\left|f(t, u_1, \dots, u_k)\right| \le q(t), \quad t \in [t_0, +\infty), u_i \in [N, M], 1 \le i \le k,$$

where $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$.

It is easy to see that Eq. (1.6) includes Eqs. (1.1)-(1.5) as special cases. Our aim in this paper is to establish a few existence results of uncountably many nonoscillatory solutions for Eq. (1.6), to suggest Mann iterative approximations for these nonoscillatory solutions and to discuss the error estimates between the approximate solutions and the nonoscillatory solutions. These results obtained in this paper extend, improve and unify the corresponding results in [3, 6–9].

Throughout this paper, we assume that

(H_2) $\int_{t_0}^{+\infty} s^n \max\{p(s), q(s), |g(s)|\}\,ds < +\infty$;
(H_3) $\int_{t_0}^{+\infty} s^{n-1} \max\{p(s), q(s), |g(s)|\}\,ds < +\infty$;
(H_4) $\{\lambda_n\}_{n\geq 0}$ is an arbitrary sequence in $[0,1]$ satisfying

$$\sum_{m=0}^{\infty} \lambda_m = +\infty.$$

By a solution of Eq. (1.6), we mean a function $x \in C([t_1 - \tau, \infty), \mathbb{R})$ for some $t_1 \geq t_0$ such that $x(t) + cx(t - \tau)$ is n-times continuously differentiable in $[t_1, \infty)$ and such that Eq. (1.6) is satisfied for $t \geq t_1$. As is customary, a solution of Eq. (1.6) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise.

Let X denote the Banach space of all continuous and bounded functions on $[t_0, +\infty)$ with norm $\|x\| = \sup_{t\geq t_0} |x(t)|$, and $A(N, M) = \{x \in X : N \leq x(t) \leq M, t \geq t_0\}$ for $M > N > 0$. It is easy to see that $A(N, M)$ is a bounded closed and convex subset of X.

2 Main results

Now we study those conditions under which Eq. (1.6) possesses uncountably many nonoscillatory solutions, and the Mann-type iterative sequences converge to these nonoscillatory solutions.

Theorem 2.1 *Let* (H_1), (H_2) *and* (H_4) *be fulfilled and*

$$c(t) = -1, \quad t \geq t_0. \tag{2.1}$$

Then

(a) *for any* $L \in (N, M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for any* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m\geq 0}$ *generated by the following iterative scheme*:

$$x_{m+1}(t) = \begin{cases} (1-\lambda_m)x_m(t) + \lambda_m[L - \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty} \\ \quad\cdots\int_{s_1}^{+\infty} f(s_0, x_m(\sigma_1(s_0)), \ldots, x_m(\sigma_k(s_0)))\,ds_0 \\ \quad\cdots\,ds_{n-2}\,ds_{n-1} - (-1)^n\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty} \\ \quad\cdots\int_{s_1}^{+\infty} g(s_0)\,ds_0\cdots ds_{n-2}\,ds_{n-1}], & t \geq T, m \geq 0, \\ x_{m+1}(T), & t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.2}$$

converges to a nonoscillatory solution $x \in A(N, M)$ *of Eq.* (1.6) *and has the following error estimate*:

$$\|x_{m+1} - x\| \leq e^{-(1-\theta)\sum_{i=0}^{m}\lambda_i}\|x_0 - x\|, \quad m \geq 0; \tag{2.3}$$

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable.*

Proof First of all we prove that (a) holds. Notice that for any $t \ge t_0$,

$$\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty} s^{n-1}p(s)\,ds < +\infty \quad \Longleftrightarrow \quad \int_{t}^{+\infty} s^{n}p(s)\,ds < +\infty.$$

It follows from (H_2) that there exist $\theta \in (0,1)$ and $T > t_0 + \tau$ satisfying

$$\frac{1}{(n-1)!}\sum_{i=1}^{\infty}\int_{T+i\tau}^{+\infty} s^{n-1}p(s)\,ds = \theta \tag{2.4}$$

and

$$\frac{1}{(n-1)!}\sum_{i=1}^{\infty}\int_{T+i\tau}^{+\infty} s^{n-1}\bigl[q(s) + \bigl|g(s)\bigr|\bigr]\,ds \le \min\{L-N, M-L\}. \tag{2.5}$$

Define a mapping $S : A(N,M) \to X$ by

$$Sx(t) = \begin{cases} L - \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty} \\ \quad \cdots \int_{s_1}^{+\infty} f(s_0, x(\sigma_1(s_0)),\dots,x(\sigma_k(s_0)))\,ds_0 \\ \quad \cdots ds_{n-2}\,ds_{n-1} - (-1)^n \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty} \\ \quad \cdots \int_{s_1}^{+\infty} g(s_0)\,ds_0 \cdots ds_{n-2}\,ds_{n-1}, & t \ge T, \\ Sx(T), & t_0 \le t < T \end{cases} \tag{2.6}$$

for any $x \in A(N,M)$. Put $x, y \in A(N,M)$. It follows from (2.6) and (H_1) that, for any $t \ge T$,

$$\begin{aligned}
&\bigl|Sx(t) - Sy(t)\bigr| \\
&\quad = \Biggl|\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_1}^{+\infty}\bigl[f\bigl(s_0, x\bigl(\sigma_1(s_0)\bigr),\dots,x\bigl(\sigma_k(s_0)\bigr)\bigr) \\
&\qquad - f\bigl(s_0, y\bigl(\sigma_1(s_0)\bigr),\dots,y\bigl(\sigma_k(s_0)\bigr)\bigr)\bigr]\,ds_0\cdots ds_{n-2}\,ds_{n-1}\Biggr| \\
&\quad \le \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_1}^{+\infty} p(s_0) \\
&\qquad \times \max\bigl\{\bigl|x\bigl(\sigma_i(s_0)\bigr) - y\bigl(\sigma_i(s_0)\bigr)\bigr| : 1 \le i \le k\bigr\}\,ds_0\cdots ds_{n-2}\,ds_{n-1} \\
&\quad \le \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_2}^{+\infty}\int_{s_1}^{+\infty} p(s_0)\,ds_0\,ds_1\cdots ds_{n-2}\,ds_{n-1}\|x-y\| \\
&\quad = \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_2}^{+\infty} p(s_0)\,ds_0\int_{s_2}^{s_0} ds_1\cdots ds_{n-2}\,ds_{n-1}\|x-y\| \\
&\quad = \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_3}^{+\infty}\int_{s_2}^{+\infty} p(s_0) \\
&\qquad \times (s_0 - s_2)\,ds_0\,ds_2\cdots ds_{n-2}\,ds_{n-1}\|x-y\| \\
&\quad = \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_4}^{+\infty}\int_{s_3}^{+\infty} p(s_0)\,ds_0
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \int_{s_3}^{s_0} (s_0 - s_2)\, ds_2\, ds_3 \cdots ds_{n-2}\, ds_{n-1} \|x - y\| \\
&= \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_4}^{+\infty} \int_{s_3}^{+\infty} \frac{1}{2} p(s_0) \\
&\quad \times (s_0 - s_3)^2\, ds_0\, ds_3 \cdots ds_{n-2}\, ds_{n-1} \|x - y\| \\
&= \cdots \\
&= \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \frac{1}{(n-2)!} p(s_0)(s_0 - s_{n-1})^{n-2}\, ds_0\, ds_{n-1} \|x - y\| \\
&= \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \frac{1}{(n-2)!}\, ds_0 \int_{t+i\tau}^{s_0} p(s_0)(s_0 - s_{n-1})^{n-2}\, ds_{n-1} \|x - y\| \\
&= \sum_{i=1}^{\infty} -\frac{1}{(n-1)!} \int_{t+i\tau}^{+\infty} \bigl[p(s_0)(s_0 - s_{n-1})^{n-1}\bigr]\Big|_{s_{n-1}=t+i\tau}^{s_{n-1}=s_0}\, ds_0 \|x - y\| \\
&= \sum_{i=1}^{\infty} \frac{1}{(n-1)!} \int_{t+i\tau}^{+\infty} p(s_0)\bigl(s_0 - (t + i\tau)\bigr)^{n-1}\, ds_0 \|x - y\|.
\end{aligned} \tag{2.7}
$$

In light of (2.4) and (2.7), we get that

$$
\bigl|Sx(t) - Sy(t)\bigr| \le \frac{1}{(n-1)!} \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} s_0^{n-1} p(s_0)\, ds_0 \|x - y\| \le \theta \|x - y\|, \quad t \ge T,
$$

which yields that

$$
\|Sx - Sy\| \le \theta \|x - y\|, \quad x, y \in A(N, M). \tag{2.8}
$$

Using (2.5) and (2.6), we gain that, for $t \ge T$,

$$
\begin{aligned}
Sx(t) &= L - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} f\bigl(s_0, x\bigl(\sigma_1(s_0)\bigr), \\
&\quad \ldots, x\bigl(\sigma_k(s_0)\bigr)\bigr)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\
&\quad - (-1)^n \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} g(s_0)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\
&\le L + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} \bigl(q(s_0) + \bigl|g(s_0)\bigr|\bigr)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\
&\le L + \frac{1}{(n-1)!} \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \bigl(s_0 - (t + i\tau)\bigr)^{n-1} \bigl(q(s_0) + \bigl|g(s_0)\bigr|\bigr)\, ds_0 \\
&\le L + \frac{1}{(n-1)!} \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} s_0^{n-1} \bigl(q(s_0) + \bigl|g(s_0)\bigr|\bigr)\, ds_0 \\
&\le L + \min\{L - N, M - L\} \\
&\le M
\end{aligned}
$$

and

$$\begin{aligned} Sx(t) &= L - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} f\big(s_0, x\big(\sigma_1(s_0)\big), \\ &\quad \ldots, x\big(\sigma_k(s_0)\big)\big)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\ &\quad - (-1)^n \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} g(s_0)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\ &\geq L - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} \big(q(s_0) + \big|g(s_0)\big|\big)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\ &\geq L - \frac{1}{(n-1)!} \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \big(s_0 - (t+i\tau)\big)^{n-1} \big(q(s_0) + \big|g(s_0)\big|\big)\, ds_0 \\ &\geq L - \frac{1}{(n-1)!} \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} s_0^{n-1} \big(q(s_0) + \big|g(s_0)\big|\big)\, ds_0 \\ &\geq L - \min\{L-N, M-L\} \\ &\geq N, \end{aligned}$$

which imply that $S(A(N,M)) \subseteq A(N,M)$. Consequently, (2.8) means that $S: A(N,M) \to A(N,M)$ is a contraction mapping. Hence S has a unique fixed point $x \in A(N,M)$, that is,

$$x(t) = \begin{cases} L - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} f(s_0, x(\sigma_1(s_0)), \ldots, x(\sigma_k(s_0)))\, ds_0 \\ \quad \cdots ds_{n-2} d_{n-1} - (-1)^n \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \\ \quad \cdots \int_{s_1}^{+\infty} g(s_0)\, ds_0 \cdots ds_{n-2}\, ds_{n-1}, & t \geq T, \\ x(T), & t_0 \leq t < T. \end{cases}$$

It follows that, for $t \geq T + \tau$,

$$\begin{aligned} &x(t-\tau) \\ &\quad = L - \sum_{i=1}^{\infty} \int_{t+(i-1)\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} f\big(s_0, x\big(\sigma_1(s_0)\big), \ldots, x\big(\sigma_k(s_0)\big)\big)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\ &\qquad - (-1)^n \sum_{i=1}^{\infty} \int_{t+(i-1)\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} g(s_0)\, ds_0 \cdots ds_{n-2}\, ds_{n-1}. \end{aligned}$$

Consequently, we know that, for $t \geq T + \tau$,

$$\begin{aligned} &x(t) - x(t-\tau) \\ &\quad = \int_{t}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} f\big(s_0, x\big(\sigma_1(s_0)\big), \ldots, x\big(\sigma_k(s_0)\big)\big)\, ds_0 \cdots ds_{n-2}\, ds_{n-1} \\ &\qquad + (-1)^n \int_{t}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} g(s_0)\, ds_0 \cdots ds_{n-2}\, ds_{n-1}. \end{aligned}$$

In view of (2.1) and the above equation, we easily verify that x is a nonoscillatory solution of Eq. (1.6). It follows from (2.2), (2.6) and (2.8) that, for each $t \geq T$,

$$\begin{aligned}
&\bigl|x_{m+1}(t) - x(t)\bigr| \\
&\quad = \Biggl|(1-\lambda_m)x_m(t) + \lambda_m\Biggl[L - \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_1}^{+\infty} f\bigl(s_0, x_m\bigl(\sigma_1(s_0)\bigr), \\
&\qquad \ldots, x_m\bigl(\sigma_k(s_0)\bigr)\bigr)\,ds_0\cdots ds_{n-2}\,ds_{n-1} \\
&\qquad - (-1)^n\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_1}^{+\infty} g(s_0)\,ds_0\cdots ds_{n-2}\,ds_{n-1}\Biggr] - x(t)\Biggr| \\
&\quad \leq (1-\lambda_m)\bigl|x_m(t) - x(t)\bigr| + \lambda_m\bigl|Sx_m(t) - Sx(t)\bigr| \\
&\quad \leq \bigl[(1-\lambda_m) + \lambda_m\theta\bigr]\|x_m - x\| \\
&\quad \leq e^{-(1-\theta)\sum_{i=0}^{m}\lambda_i}\|x_0 - x\|, \quad m \geq 0,
\end{aligned}$$

which yields that (2.3) holds. Thus (2.3) and (H_4) ensure that $x_m \to x$ as $m \to +\infty$.

Next we prove that (b) holds. It follows from (a) that for any distinct L_1 and $L_2 \in (N, M)$, there exist $S_{L_j} : A(N,M) \to A(N,M)$, $\theta_j \in (0,1)$ and $T_j > t_0 + \tau$ satisfying

$$S_{L_j}x(t) = \begin{cases} L_j - \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_1}^{+\infty} f(s_0, x(\sigma_1(s_0)), \\ \quad \ldots, x(\sigma_k(s_0)))\,ds_0\cdots ds_{n-2}\,ds_{n-1} \\ \quad - (-1)^n\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_1}^{+\infty} g(s_0)\,ds_0\cdots ds_{n-2}\,ds_{n-1}, & t \geq T_j, \\ S_{L_j}x(T_j), & t_0 \leq t < T_j, \end{cases}$$

$$\frac{1}{(n-1)!}\sum_{i=1}^{\infty}\int_{T_j+i\tau}^{+\infty} s^{n-1}p(s)\,ds = \theta_j$$

and

$$\frac{1}{(n-1)!}\sum_{i=1}^{\infty}\int_{T_j+i\tau}^{+\infty} s^{n-1}\bigl(q(s) + \bigl|g(s)\bigr|\bigr)\,ds \leq \min\{L_j - N, M - L_j\}$$

for each $x \in A(N,M)$ and $j \in \{1,2\}$. Note that the contraction mappings S_{L_1} and S_{L_2} have fixed points x and $y \in A(N,M)$, respectively, that is, x and y are two nonoscillatory solutions of Eq. (1.6). Put $T = \max\{T_1, T_2\}$, $\theta = \max\{\theta_1, \theta_2\}$. It is clear that

$$\frac{1}{(n-1)!}\sum_{i=1}^{\infty}\int_{T+i\tau}^{+\infty} s^{n-1}p(s)\,ds \leq \theta$$

and

$$\begin{aligned}
&\bigl|x(t) - y(t)\bigr| \\
&\quad = \Biggl|L_1 - L_2 - \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{s_{n-1}}^{+\infty}\cdots\int_{s_1}^{+\infty}\bigl[f\bigl(s_0, x\bigl(\sigma_1(s_0)\bigr), \ldots, x\bigl(\sigma_k(s_0)\bigr)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&- f\bigl(s_0, y(\sigma_1(s_0)), \ldots, y(\sigma_k(s_0))\bigr)\bigr] \, ds_0 \cdots ds_{n-2} \, ds_{n-1} \Bigr| \\
&\geq |L_1 - L_2| - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{s_{n-1}}^{+\infty} \cdots \int_{s_1}^{+\infty} p(s_0) \, ds_0 \cdots ds_{n-2} \, ds_{n-1} \|x - y\| \\
&\geq |L_1 - L_2| - \frac{1}{(n-1)!} \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} s_0^{n-1} p(s_0) \, ds_0 \|x - y\| \\
&\geq |L_1 - L_2| - \theta \|x - y\|, \quad t \geq T,
\end{aligned}
$$

which yields that

$$
\|x - y\| \geq |L_1 - L_2| - \theta \|x - y\|.
$$

That is,

$$
\|x - y\| \geq \frac{|L_1 - L_2|}{1 + \theta} > 0.
$$

Hence $x \neq y$. It follows that for any distinct L_1 and $L_2 \in (N, M)$, the corresponding nonoscillatory solutions x and $y \in A(N, M)$ of Eq. (1.6) are distinct. Consequently, the set of nonoscillatory solutions of Eq. (1.6) is uncountable. This completes the proof. □

The proofs of Theorems 2.2-2.8 are similar to the proof of Theorem 2.1, hence are omitted.

Theorem 2.2 *Let* (H$_1$), (H$_3$) *and* (H$_4$) *hold. Assume that there exists* $C \in (0,1)$ *such that* $0 \leq c(t) \leq C$ *for* $t \geq t_0$ *and* $M > \frac{1}{1-C} N$. *Then*

(a) *for any* $L \in (CM + N, M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for each* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m \geq 0}$ *generated by the following iterative scheme*:

$$
x_{m+1}(t) = \begin{cases} (1 - \lambda_m) x_m(t) + \lambda_m \{L - c(t) x_m(t - \tau) \\ \quad + \frac{1}{(n-1)!} \int_t^{+\infty} (s - t)^{n-1} f(s, x_m(\sigma_1(s)), \\ \quad \ldots, x_m(\sigma_k(s))) \, ds \\ \quad + \frac{(-1)^n}{(n-1)!} \int_t^{+\infty} (s - t)^{n-1} g(s) \, ds\}, & t \geq T, m \geq 0, \\ x_{m+1}(T), & t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.9}
$$

converges to a nonoscillatory solution $x \in A(N, M)$ *of Eq.* (1.6), *and the error estimate* (2.3) *holds*;

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable.*

Theorem 2.3 *Let* (H$_1$), (H$_3$) *and* (H$_4$) *hold. Assume that there exists* $C \in (0,1)$ *such that* $-C \leq c(t) \leq 0$ *for* $t \geq t_0$ *and* $M > \frac{1}{1-C} N$. *Then*

(a) *for any* $L \in (N, (1 - C)M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for each* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m \geq 0}$ *generated by* (2.9) *converges to a nonoscillatory solution* $x \in A(N, M)$ *of Eq.* (1.6), *and the error estimate* (2.3) *holds*;

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable.*

Theorem 2.4 *Let* (H_1), (H_3) *and* (H_4) *hold. Assume that there exists* $C > 1$ *such that* $c(t) \geq C$ *for* $t \geq t_0$ *and* $\frac{C}{C-1}N < M$. *Then*

(a) *for any* $L \in (\frac{1}{C}M + N, M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for each* $x_0 \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\geq 0}$ *generated by the following iterative scheme*:

$$x_{m+1}(t) = \begin{cases} (1-\lambda_m)x_m(t) + \lambda_m\{L - \frac{1}{c(t+\tau)}x_m(t+\tau) \\ \quad + \frac{1}{c(t+\tau)(n-1)!}\int_{t+\tau}^{+\infty}(s-t-\tau)^{n-1} \\ \quad \times f(s, x_m(\sigma_1(s)), \dots, x_m(\sigma_k(s)))\,ds \\ \quad + \frac{(-1)^n}{c(t+\tau)(n-1)!}\int_{t+\tau}^{+\infty}(s-t-\tau)^{n-1}g(s)\,ds\}, & t \geq T, m \geq 0, \\ x_{m+1}(T), & t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.10}$$

converges to a nonoscillatory solution $x \in A(N,M)$ *of Eq.* (1.6), *and the error estimate* (2.3) *holds*;

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable.*

Theorem 2.5 *Let* (H_1), (H_3) *and* (H_4) *hold. Assume that there exists* $C > 1$ *such that* $c(t) \leq -C$ *for* $t \geq t_0$ *and* $\frac{C}{C-1}N < M$. *Then*

(a) *for any* $L \in (N, (1 - \frac{1}{C})M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for each* $x_0 \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\geq 0}$ *generated by* (2.10) *converges to a nonoscillatory solution* $x \in A(N,M)$ *of Eq.* (1.6), *and the error estimate* (2.3) *holds*;

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable.*

Theorem 2.6 *Let* (H_1), (H_3) *and* (H_4) *hold. Assume that there exists* $C \in (0, \frac{1}{2})$ *such that* $|c(t)| \leq C$ *for* $t \geq t_0$ *and* $N < (1-2C)M$. *Then*

(a) *for any* $L \in (CM + N, (1-C)M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for each* $x_0 \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\geq 0}$ *generated by* (2.9) *converges to a nonoscillatory solution* $x \in A(N,M)$ *of Eq.* (1.6), *and the error estimate* (2.3) *holds*;

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable.*

Theorem 2.7 *Let* $n = 1$, (H_1), (H_3) *and* (H_4) *hold and* $c(t) = 1$ *for* $t \geq t_0$. *Then*

(a) *for any* $L \in (N, M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for every* $x_0 \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\geq 0}$ *generated by the following iterative scheme*:

$$x_{m+1}(t) = \begin{cases} (1-\lambda_m)x_m(t) + \lambda_m\{L + \sum_{j=1}^{+\infty}\int_{t+(2j-1)\tau}^{t+2j\tau} \\ \quad \times f(s, x_m(\sigma_1(s)), \dots, x_m(\sigma_k(s)))\,ds \\ \quad - \sum_{j=1}^{+\infty}\int_{t+(2j-1)\tau}^{t+2j\tau} g(s)\,ds\}, & t \geq T, m \geq 0, \\ x_{m+1}(T), & t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.11}$$

converges to a nonoscillatory solution $x \in A(N,M)$ *of Eq.* (1.6), *and the error estimate* (2.3) *holds*;

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable.*

Theorem 2.8 *Let* $n \geq 2$, (H_1), (H_3) *and* (H_4) *hold and* $c(t) = 1$ *for* $t \geq t_0$. *Then*

(a) *for any* $L \in (N, M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0 + \tau$ *such that for arbitrary* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m\geq 0}$ *generated by the following iterative scheme*:

$$x_{m+1}(t) = \begin{cases} (1-\lambda_m)x_m(t) \\ \quad + \lambda_m\{L + \frac{1}{(n-2)!}\sum_{j=1}^{+\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}(s-u)^{n-2} \\ \quad \times f(s, x_m(\sigma_1(s)), \dots, x_m(\sigma_k(s)))\,ds\,du \\ \quad + \frac{(-1)^n}{(n-2)!}\sum_{j=1}^{+\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty} g(s)\,ds\,du\}, & t \geq T, m \geq 0, \\ x_{m+1}(T), & t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.12}$$

converges to a nonoscillatory solution $x \in A(N, M)$ *of Eq.* (1.6), *and the error estimate* (2.3) *holds*;

(b) *the set of nonoscillatory solutions of Eq.* (1.6) *is uncountable*.

Remark 2.1 Theorems 2.1-2.8 extend, improve and unify a few known results due to Cheng and Annie [3], Kulenović and Hadžiomerspahić [6, 7], Liu et al. [8] and Zhou and Zhang [9] and others.

3 Examples

In this section, in order to illustrate the advantage of our results, we consider the following three examples.

Example 3.1 Consider the *n*th order neutral delay differential equation

$$\begin{aligned} &\frac{d^n}{dt^n}\big[x(t) - x(t-\tau)\big] \\ &\quad + (-1)^{n+1}\left\{e^{-t^2}\cos^5(\sqrt{t} - t) + \frac{\sin\sqrt{t}}{1+t^{n+2}}x^3\big(t^2 - 4\tau\big)x^4(\sqrt{t})\right\} \\ &= t^{10}e^{1-t}\sin\big(t^6 - 3t^5 + 1\big), \quad t \geq 0, \end{aligned} \tag{3.1}$$

where τ is a positive number. Let $\{\lambda_n\}_{n\geq 0}$ be an arbitrary sequence in $[0,1]$ satisfying (H_4), M and N be constants with $M > N > 0$. Put $t_0 = 0$, $k = 2$,

$$\begin{aligned} &\sigma_1(t) = t^2 - 4\tau, \qquad \sigma_2(t) = \sqrt{t}, \\ &f(t, u_1, u_2) = e^{-t^2}\cos^5(\sqrt{t} - t) + \frac{\sin\sqrt{t}}{1+t^{n+2}}u_1^3u_2^4, \\ &c(t) = -1, \qquad g(t) = t^{10}e^{1-t}\sin\big(t^6 - 3t^5 + 1\big), \\ &p(t) = \frac{7M^6}{1+t^{n+2}}, \qquad q(t) = e^{-t^2} + \frac{M^7}{1+t^{n+2}} \end{aligned}$$

for $t \geq 0$, $u_i \in \mathbb{R}$ and $i \in \{1,2\}$. It is easy to verify that (H_1) and (H_2) hold. It follows from Theorem 2.1 that Eq. (3.1) possesses uncountably many nonoscillatory solutions, and for each $L \in (N, M)$ the Mann iterative sequence $\{x_n\}_{n\geq 0}$ generated by (2.2) converges to some nonoscillatory solution of Eq. (3.1), and the error estimate (2.3) holds. But the results in [3, 6–9] are inapplicable for Eq. (3.1).

Example 3.2 Consider the nth order neutral delay differential equation

$$\begin{aligned} &\frac{d^n}{dt^n}\bigl[x(t)+t^3x(t-\tau)\bigr]+(-1)^{n+1}\biggl\{\frac{t^9-2}{t^{n+10}}\ln\bigl(1+\bigl|x(\sqrt{t}-5\tau)\bigr|\bigr)\\ &\quad+\frac{1-t^6\cos^3 t}{t^2+t^{n+7}}x^2(2t-9)-\frac{x^8(t^3-\sqrt{t}-10)}{t^{n+1}+x^2(t^2-400)}\biggr\}\\ &=\frac{\sqrt{t}+t\sin(t^5)}{t^{n+3}+\ln^2 t},\quad t\geq 2, \end{aligned}\tag{3.2}$$

where τ is a positive number. Let $\{\lambda_n\}_{n\geq 0}$ be an arbitrary sequence in $[0,1]$ satisfying (H_4), M and N be constants with $0<\frac{8}{7}N<M$. Put $t_0=2$, $k=4$, $C=8$,

$$\begin{aligned} &\sigma_1(t)=\sqrt{t}-5\tau,\qquad \sigma_2(t)=2t-9,\\ &\sigma_3(t)=t^3-\sqrt{t}-10,\qquad \sigma_4(t)=t^2-400,\\ &f(t,u_1,u_2,u_3,u_4)=\frac{t^9-2}{t^{n+10}}\ln\bigl(1+|u_1|\bigr)+\frac{1-t^6\cos^3 t}{t^2+t^{n+7}}u_2^2-\frac{u_3^8}{t^{n+1}+u_4^2},\\ &c(t)=t^3,\qquad g(t)=\frac{\sqrt{t}+t\sin(t^5)}{t^{n+3}+\ln^2 t},\\ &p(t)=\frac{t^9-2}{t^{n+10}}+\frac{2M|1-t^6\cos^3 t|}{t^2+t^{n+7}}+\frac{2M^7(5M^2+4t^{n+1})}{(N^2+t^{n+1})^2},\\ &q(t)=\frac{t^9-2}{t^{n+10}}\ln(1+M)+\frac{M^2|1-t^6\cos^3 t|}{t^2+t^{n+7}}+\frac{M^8}{N^2+t^{n+1}} \end{aligned}$$

for $t\geq 2$, $u_i\in\mathbb{R}$ and $i\in\{1,2,3,4\}$. Clearly, (H_1) and (H_3) hold. It follows from Theorem 2.4 that Eq. (3.2) possesses uncountably many nonoscillatory solutions, and for any $L\in(\frac{1}{8}M+N,M)$, the Mann iterative sequence $\{x_n\}_{n\geq 0}$ generated by (2.10) converges to some nonoscillatory solution of Eq. (3.2), and the error estimate (2.3) holds. However, the results in [3, 6–9] are not applicable for Eq. (3.2).

Example 3.3 Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned} &\frac{d^n}{dt^n}\biggl[x(t)-\frac{t-\sin t}{4t}x(t-\tau)\biggr]+(-1)^{n+1}\biggl\{\frac{1-4t^{2n+3}}{t^{3n+5}}x^2(\sqrt{t})\\ &\quad+\frac{t^2(t^3-10)}{t^{n+6}+\cos t}\sin^2\bigl(x^3(2t)\bigr)+\frac{1-2t-t^7}{t^{n+8}+|x^3(3t)|}\biggr\}\\ &=\frac{2+t^n\cos t^n}{t^{2n+1}},\quad t\geq 1, \end{aligned}\tag{3.3}$$

where τ is a positive number. Let $\{\lambda_n\}_{n\geq 0}$ be an arbitrary sequence in $[0,1]$ satisfying (H_4), M and N be constants with $M>2N>0$. Put $t_0=1$, $k=3$, $C=-\frac{1}{2}$,

$$\begin{aligned} &\sigma_1(t)=\sqrt{t},\qquad \sigma_2(t)=2t,\qquad \sigma_3(t)=3t,\\ &f(t,u_1,u_2,u_3)=\frac{1-4t^{2n+3}}{t^{3n+5}}u_1^2+\frac{t^2(t^3-10)}{t^{n+6}+\cos t}\sin^2 u_2^3+\frac{1-2t-t^7}{t^{n+8}+|u_3^3|},\\ &c(t)=-\frac{t-\sin t}{4t},\qquad g(t)=\frac{2+t^n\cos t^n}{t^{2n+1}}, \end{aligned}$$

$$p(t) = \frac{2M(4t^{2n+3}-1)}{t^{3n+5}} + \frac{6M^2t^2|t^3-10|}{t^{n+6}+\cos t} + \frac{3M^2(t^7+2t-1)}{(t^{n+8}+N^3)^2},$$
$$q(t) = \frac{M^2(4t^{2n+3}-1)}{t^{3n+5}} + \frac{t^2|t^3-10|}{t^{n+6}+\cos t} + \frac{t^7+2t-1}{t^{n+8}+N^3}$$

for $t \geq 1$, $u_i \in \mathbb{R}$ and $i \in \{1,2,3\}$. Obviously, (H_1) and (H_3) hold. It follows from Theorem 2.3 that Eq. (3.3) possesses uncountably many nonoscillatory solutions, and for any $L \in (N, \frac{1}{2}M)$, the Mann iterative sequence $\{x_n\}_{n\geq 0}$ generated by (2.9) converges to some nonoscillatory solution of Eq. (3.3), and the error estimate (2.3) holds. However, the results in [3, 6–9] are not applicable for Eq. (3.3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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