Zhao and Liang *Advances in Difference Equations* (2017) 2017:50
DOI 10.1186/s13662-017-1099-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Solvability of triple-point integral boundary value problems for a class of impulsive fractional differential equations

Kaihong Zhao* and Jiangyan Liang

*Correspondence:
zhaokaihongs@126.com
Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650093, P.R. China

Abstract

This paper is concerned with a class of triple-point integral boundary value problems for impulsive fractional differential equations involving the Riemann-Liouville fractional derivative of order α ($2 < \alpha \leq 3$). Some sufficient criteria for the existence of solutions are obtained by applying the contraction mapping principle and the fixed point theorem. As an application, one example is given to demonstrate the validity of our main results.

MSC: 34B10; 34B15; 34B37

Keywords: impulsive fractional differential equations; existence of solutions; integral boundary value problems; fixed point theorem

1 Introduction

Towards the end of the 19th century Liouville and Riemann mentioned the definition of the fractional derivative which is the generalization of the traditional integer order differential and integral calculus. The fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The subject of fractional differential equations is gaining much importance and attention because of its extensive applications in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electroanalytical chemistry, biology, control theory, fitting of experimental data, and so forth. For more details of the basic theory of fractional differential equations, refer to [1–6] and the references therein. In recent decades, the boundary value problems of fractional differential equations have received a great deal of attention. There are a large number of papers dealing with the existence, nonexistence, multiplicity of solutions of boundary value problem for some nonlinear fractional differential equations (see [7–27]).

As we know, many evolutionary processes experience short-time rapid change after undergoing relatively long smooth variation. In order to describe the dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so on, some authors have used an impulsive differential system to describe these kinds of phenomena since the last century. For the theory of impulsive differential equations, the reader can refer to [28–30]. Recently, the boundary value problems of impulsive fractional differential equations have been studied extensively in the literature (see [31–45]). To the best of our knowledge, there are few articles involving the impulsive fractional order differential equations. Therefore, we will study the existence and uniqueness of solutions for the following impulsive integral boundary value problems (BVPs for short) of fractional order differential equations:

$$\begin{cases} {}_{t_k}D_t^{\alpha}u(t) = f(t,u,u',D^{\alpha-1}u), \quad t \neq t_k, \\ \Delta D^{\alpha-1}u(t_k) = I_k(u(t_k)), \quad k = 1,\dots,m, \\ u(0) = u'(0) = 0, \qquad u'(1) = \int_0^{\eta} g(s,u(s))\,ds, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $J = [0,1]$, $J_0 = [t_0,t_1]$, $J_k = (t_k,t_{k+1}] \subset J$ $(k = 1,2,\dots,m)$. ${}_{t_k}D_t^{\alpha}$ is the Riemann-Liouville fractional derivative of order $2 < \alpha \leq 3$. $f \in C(J \times \mathbb{R}^3,\mathbb{R})$, $I_k \in C(\mathbb{R},\mathbb{R})$, $0 < \eta < 1$, $g \in C(J \times \mathbb{R},\mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. $u(t_k^+) = \lim_{h\to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h\to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. $u(t_k^-) = u(t_k)$, ${}_{t_k}D_t^{\alpha-1}u(t_k^-) = {}_{t_k}D_t^{\alpha-1}u(t_k)$. The right-hand limits $u(t_k^+)$ and ${}_{t_k}D_t^{\alpha-1}u(t_k^+)$ all exist. $\Delta D^{\alpha-1}u(t_k) = {}_{t_k}D_t^{\alpha-1}u(t_k^+) - {}_{t_{k-1}}D_t^{\alpha-1}u(t_k^-)$.

The rest of this paper is organized as follows. In Section 2, we shall introduce some definitions and lemmas to prove our main results. In Section 3, we give some sufficient conditions for the existence of single positive solutions for boundary value problem (1.1). As an application, one interesting example is presented to illustrate the main results in Section 4. Finally, the conclusion is given to simply recall our studied contents and obtained results in Section 5.

2 Preliminaries

Let $C(J,\mathbb{R})$ be the Banach space of continuous functions from J to $\mathbb{R}$ with the norm $\|u\|_C = \sup_{0\leq t\leq 1}|u(t)|$. Now let us to introduce the useful Banach space $PC^1(J,\mathbb{R})$ defined by

$$\begin{aligned} PC^1(J,\mathbb{R}) = \{ & u \in C(J,\mathbb{R}) : {}_{t_k}D_t^{\alpha-1}u(t_k^+) \text{ and } {}_{t_k}D_t^{\alpha-1}u(t_k^-) \text{ exist with} \\ & {}_{t_k}D_t^{\alpha-1}u(t_k) = {}_{t_k}D_t^{\alpha-1}u(t_k^-), k = 0,1,\dots,m \} \end{aligned} \tag{2.1}$$

equipped with the norm $\|u\|_{PC^1} = \max\{\|u\|_C, \|u'\|_C, \|{}_{t_k}D_t^{\alpha-1}u\|_C\}$.

Definition 2.1 A function $u \in PC^1(J,\mathbb{R})$ with its Riemann-Liouville derivative of order α existing on J is a solution of (1.1) if it satisfies (1.1).

For convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the literature [2, 4, 6].

Definition 2.2 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (a,+\infty) \to \mathbb{R}$ is given by

$${}_aI_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\,ds, \quad a > 0,$$

provided that the right-hand side is point-wise defined on $(a,+\infty)$.

Definition 2.3 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (a, +\infty) \to \mathbb{R}$ is given by

$$ {}_aD_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds, $$

where $a > 0$, $n - 1 < \alpha \leq n$, provided that the right-hand side is point-wise defined on $(a, +\infty)$.

Lemma 2.1 *Assume that* $u \in C[a, b]$, $q \geq p \geq 0$, *then*

$$ {}_aD_t^p\, {}_aI_t^q u(t) = {}_aI_t^{q-p} u(t), \quad t \in [a, b]. \tag{2.2} $$

Lemma 2.2 (see [6], pp. 36-39) *Let* $\alpha > 0$, n *denotes the smallest integer greater than or equal to* α. *Then the following assertions hold.*

(i) *if* $\lambda > -1$, $\lambda \neq \alpha - i$, $i = 1, 2, \dots, n+1$, *then for* $t \in [a, b]$

$$ {}_aD_t^\alpha (t-a)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} (t-a)^{\lambda-\alpha}. \tag{2.3} $$

(ii) ${}_aD_t^\alpha (t-a)^{\alpha-i} = 0$, $i = 1, 2, \dots, n$.

(iii) ${}_aD_t^\alpha\, {}_aI_t^\alpha u(t) = u(t)$, *for all* $t \in [a, b]$.

(iv) ${}_aD_t^\alpha u(t) = 0$ *if and only if there exists* $c_i \in \mathbb{R}$ $(i = 1, 2, \dots, n)$ *such that*

$$ u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n}, \quad t \in [a, b]. \tag{2.4} $$

(v) *For all* $t \in [a, b]$, *then*

$$ {}_aI_t^\alpha\, {}_aD_t^\alpha u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n} + u(t). \tag{2.5} $$

Lemma 2.3 (Schauder fixed point theorem; see [46]) *If* U *is a closed bounded convex subset of a Banach space* X *and* $T : U \to U$ *is completely continuous, then* T *has at least one fixed point in* U.

Lemma 2.4 *For a given* $y \in C(J, \mathbb{R})$, *a function* $u \in PC^1(J, \mathbb{R})$ *is a solution of BVP* (2.6)

$$ \begin{cases} {}_{t_k}D_t^\alpha u(t) = y(t), & t \neq t_k, 2 < \alpha \leq 3, \\ \Delta D^{\alpha-1} u(t_k) = I_k(u(t_k)), & k = 1, \dots, m, \\ u(0) = u'(0) = 0, & u'(1) = \int_0^\eta g(s, u(s))\, ds, \end{cases} \tag{2.6} $$

if and only if $u \in PC^1(J, \mathbb{R})$ *is a solution of the impulsive fractional integral equation*

$$ \begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} y(s)\, ds \\ & - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{t \leq t_i} I_i\bigl(u(t_i)\bigr) + \frac{t^{\alpha-1}}{\alpha-1} \int_0^\eta g\bigl(s, u(s)\bigr)\, ds, \quad 0 \leq t \leq 1. \end{aligned} \tag{2.7} $$

Proof We denote the solution of (2.6) by $u(t) = u_k(t)$ in $[t_k, t_{k+1}]$ ($k = 1, 2, \ldots, m$). For $t \in J_0 = [0, t_1]$, by (2.5), we have

$$u_0(t) = {}_0I_t^\alpha y(t) + c_{01}t^{\alpha-1} + c_{02}t^{\alpha-2} + c_{03}t^{\alpha-3}.$$

$u(0) = u'(0) = 0$ implies that $c_{02} = c_{03} = 0$. Applying Lemma 2.2, we get

$$\begin{aligned} &u_0(t) = {}_0I_t^\alpha y(t) + c_{01}t^{\alpha-1}, \\ &D_t^{\alpha-1}u_0(t) = D_t^{\alpha-1}\left[{}_0I_t^\alpha y(t) + c_{01}t^{\alpha-1}\right] = \int_0^t y(s)\,ds + \Gamma(\alpha)c_{01}, \end{aligned}$$

and

$$\begin{aligned} D^{\alpha-1}u\left(t_1^+\right) &= D^{\alpha-1}u_0\left(t_1^+\right) = D^{\alpha-1}u_0(t_1) + I_1\left(u(t_1)\right) \\ &= \int_0^{t_1} y(s)\,ds + \Gamma(\alpha)c_{01} + I_1\left(u(t_1)\right). \end{aligned}$$

For $t \in J_1 = (t_1, t_2]$, by (2.5), we get

$$u_1(t) = {}_0I_t^\alpha y(t) + c_{11}t^{\alpha-1} + c_{12}t^{\alpha-2} + c_{13}t^{\alpha-3}$$

and

$$D_t^{\alpha-1}u_1(t) = \int_0^t y(s)\,ds + \Gamma(\alpha)c_{11}.$$

Noting that $u(0) = u'(0) = 0$ and $D^{\alpha-1}u_1(t_1) = D^{\alpha-1}u_0(t_1^+)$, we derive $c_{12} = c_{13} = 0$ and $c_{11} = c_{01} + \frac{I_1(u(t_1))}{\Gamma(\alpha)}$. So we can obtain

$$u_1(t) = {}_0I_t^\alpha y(t) + \left[c_{01} + \frac{I_1(u(t_1))}{\Gamma(\alpha)}\right]t^{\alpha-1}$$

and

$$D^{\alpha-1}u\left(t_2^+\right) = D^{\alpha-1}u_1\left(t_2^+\right) = D^{\alpha-1}u_1(t_2) + I_2\left(u(t_2)\right) = \int_0^{t_2} y(s)\,ds + \Gamma(\alpha)c_{01} + \sum_{i=1}^{2} I_i\left(u(t_i)\right).$$

By the recurrent method, for $t \in J_k = (t_k, t_{k+1}]$, $k = 2, 3, \ldots, m$, we get

$$\begin{aligned} u_k(t) &= {}_0I_t^\alpha y(t) + \left[c_{01} + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k} I_i\left(u(t_i)\right)\right]t^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + \left[c_{01} + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k} I_i\left(u(t_i)\right)\right]t^{\alpha-1} \end{aligned}$$

and

$$
\begin{aligned}
D^{\alpha-1}u\left(t_{k+1}^{+}\right) &= D^{\alpha-1}u_k\left(t_{k+1}^{+}\right) = D^{\alpha-1}u_k(t_{k+1}) + I_{k+1}\left(u(t_{k+1})\right) \\
&= \int_0^{t_{k+1}} y(s)\,ds + \Gamma(\alpha)c_{01} + \sum_{i=1}^{k+1} I_i\left(u(t_i)\right).
\end{aligned}
$$

So, for $t \in J_m = (t_m, t_{m+1}]$, we have

$$
u'(t) = u_m'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)\,ds + (\alpha-1)\left[c_{01} + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} I_i\left(u(t_i)\right)\right]t^{\alpha-2}.
$$

By $u'(1) = \int_0^\eta u(s)\psi(s)\,ds$, we have

$$
\int_0^\eta g\left(s,u(s)\right)ds = \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)\,ds + (\alpha-1)\left[c_{01} + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} I_i\left(u(t_i)\right)\right],
$$

which implies that

$$
c_{01} = \frac{1}{\alpha-1}\int_0^\eta g\left(s,u(s)\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}y(s)\,ds - \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} I_i\left(u(t_i)\right).
$$

Therefore, for $t \in J = [0,1]$, we have

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta g\left(s,u(s)\right)ds \\
&\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-2}y(s)\,ds + \sum_{i=1}^{m} I_i\left(u(t_i)\right) - \sum_{t_i<t} I_i\left(u(t_i)\right)\right] \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-2}y(s)\,ds + \sum_{t\le t_i} I_i\left(u(t_i)\right)\right] \\
&\quad + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta g\left(s,u(s)\right)ds,
\end{aligned}
$$

which indicates that u is a solution of (2.7). Conversely, noting that the above derivations are reversible, we assert that if u is a solution of the impulsive fractional integral equation (2.7), then u is also the solution of BVP (2.6). The proof is complete. □

3 Main results

According to Lemma 2.4, we obtain the following lemma.

Lemma 3.1 *A function $u \in PC^1(J,\mathbb{R})$ is a solution of BVP* (1.1) *if and only if $u \in PC^1(J,\mathbb{R})$ is a solution of the impulsive fractional integral equation*

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)ds \\
&\quad - \left[\int_0^1 (1-s)^{\alpha-2}f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)ds + \sum_{t\le t_i} I_i\left(u(t_i)\right)\right]\frac{t^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta g\left(s,u(s)\right)ds, \quad 0 \le t \le 1. \qquad (3.1)
\end{aligned}
$$

Define an operator $T : PC^1(J,\mathbb{R}) \to PC^1(J,\mathbb{R})$ as follows:

$$\begin{aligned}(Tu)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\,ds \\ &\quad - \left[\int_0^1 (1-s)^{\alpha-2} f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\,ds + \sum_{t\le t_i} I_i\bigl(u(t_i)\bigr)\right]\frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta g\bigl(s,u(s)\bigr)\,ds, \quad 0\le t\le 1. \end{aligned} \tag{3.2}$$

Then BVP (1.1) has a solution if and only if the operator T exists one fixed point.

Lemma 3.2 *Assume that* $f \in C(J\times\mathbb{R}^3,\mathbb{R})$, *and* $g \in C(J\times\mathbb{R},\mathbb{R})$. *Then* $T : PC^1(J,\mathbb{R}) \to PC^1(J,\mathbb{R})$ *defined by* (3.2) *is completely continuous.*

Proof Note that T is continuous in view of continuity of f, I_k, and g. Now we show that T is uniformly bounded. In fact, let $\Omega \subset PC^1(J,\mathbb{R})$ be bounded, then there exist some positive constants l_i ($i = 1,2,3$) such that $|f(t,u,u',D^{\alpha-1}u)| \le l_1$, $|g(t,u)| \le l_2$, $|I_k(u)| \le l_3$, for all $u \in \Omega$. Thus for $u \in \Omega$, we have

$$\begin{aligned}\bigl|(Tu)(t)\bigr| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds \\ &\quad + \left[\int_0^1 (1-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds + \sum_{t\le t_i}\bigl|I_i\bigl(u(t_i)\bigr)\bigr|\right]\frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta \bigl|g\bigl(s,u(s)\bigr)\bigr|\,ds \\ &\le \frac{2l_1+ml_3}{\Gamma(\alpha)} + \frac{l_2\eta}{\alpha-1} \triangleq M_1, \\ \bigl|(Tu)'(t)\bigr| &= \Biggl|\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\,ds \\ &\quad - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\,ds \\ &\quad - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{t\le t_i} I_i\bigl(u(t_i)\bigr) + t^{\alpha-2}\int_0^\eta g\bigl(s,u(s)\bigr)\,ds\Biggr| \\ &\le \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds \\ &\quad + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\sum_{t\le t_i}\bigl|I_i\bigl(u(t_i)\bigr)\bigr| + \int_0^\eta \bigl|g\bigl(s,u(s)\bigr)\bigr|\,ds \\ &\le \frac{2l_1+ml_3}{\Gamma(\alpha-1)} + l_2\eta \triangleq M_2 \end{aligned}$$

and

$$\begin{aligned}
&\left|D^{\alpha-1}(Tu)(t)\right| \\
&\quad= \left|\int_0^t f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\,ds - \int_0^1 (1-s)^{\alpha-2} f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\,ds \right.\\
&\qquad \left. - \sum_{t\le t_i} I_i\bigl(u(t_i)\bigr) + \Gamma(\alpha-1)\int_0^\eta g\bigl(s,u(s)\bigr)\,ds\right| \\
&\quad\le \int_0^t \left|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\right|ds + \int_0^1 (1-s)^{\alpha-2}\left|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\right|ds \\
&\qquad + \sum_{t\le t_i}\left|I_i\bigl(u(t_i)\bigr)\right| + \Gamma(\alpha-1)\int_0^\eta \left|g\bigl(s,u(s)\bigr)\right|ds \\
&\quad\le 2l_1 + ml_3 + l_2\eta\Gamma(\alpha-1) \triangleq M_3,
\end{aligned}$$

which means that $\|u\|_{PC^1} \le \max\{M_1, M_2, M_3\}$, that is, T is uniformly bounded.

Next, we should prove that T is equicontinuous on $J = [0,1]$. Indeed, for all $\bar{t}_1, \bar{t}_2 \in [0,1]$ with $\bar{t}_1 \le \bar{t}_2$, we have

$$\begin{aligned}
\left|(Tu)(\bar{t}_2) - (Tu)(\bar{t}_1)\right| &= \left|\int_{\bar{t}_1}^{\bar{t}_2} (Tu)'(s)\,ds\right| \le \int_{\bar{t}_1}^{\bar{t}_2}\left|(Tu)'(s)\right|ds \\
&\le M_2(\bar{t}_2 - \bar{t}_1) \to 0, \quad \text{as } \bar{t}_1 \to \bar{t}_2,
\end{aligned}$$

$$\begin{aligned}
&\left|(Tu)'(\bar{t}_2) - (Tu)'(\bar{t}_1)\right| \\
&\quad\le \frac{1}{\Gamma(\alpha-1)}\int_0^{\bar{t}_1}\left[(\bar{t}_2-s)^{\alpha-2} - (\bar{t}_1-s)^{\alpha-2}\right]\left|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\right|ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)}\int_{\bar{t}_1}^{\bar{t}_2}(\bar{t}_2-s)^{\alpha-2}\left|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\right|ds \\
&\qquad + \frac{\bar{t}_2^{\alpha-2} - \bar{t}_1^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\left|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\right|ds \\
&\qquad + \frac{\bar{t}_2^{\alpha-2} - \bar{t}_1^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{\bar{t}_2\le t_i}\left|I_i\bigl(u(t_i)\bigr)\right| + \frac{\bar{t}_1^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{\bar{t}_1\le t_i<\bar{t}_2}\left|I_i\bigl(u(t_i)\bigr)\right| \\
&\qquad + \bigl(\bar{t}_2^{\alpha-2} - \bar{t}_1^{\alpha-2}\bigr)\int_0^\eta\left|g\bigl(s,u(s)\bigr)\right|ds \\
&\quad\le \frac{l_1}{\Gamma(\alpha)}\bigl(\bar{t}_2^{\alpha-1} - \bar{t}_1^{\alpha-1}\bigr) + \frac{l_1}{\Gamma(\alpha)}(\bar{t}_2-\bar{t}_1)^{\alpha-1} + \frac{l_1}{\Gamma(\alpha)}\bigl(\bar{t}_2^{\alpha-2} - \bar{t}_1^{\alpha-2}\bigr) \\
&\qquad + \frac{ml_3}{\Gamma(\alpha-1)}\bigl(\bar{t}_2^{\alpha-2} - \bar{t}_1^{\alpha-2}\bigr) + \frac{l_3}{\Gamma(\alpha-1)}(\bar{t}_2-\bar{t}_1) + l_2\eta\bigl(\bar{t}_2^{\alpha-2} - \bar{t}_1^{\alpha-2}\bigr) \to 0, \\
&\qquad \text{as } \bar{t}_1 \to \bar{t}_2,
\end{aligned}$$

and

$$\begin{aligned}
\left|D^{\alpha-1}(Tu)(\bar{t}_2) - D^{\alpha-1}(Tu)(\bar{t}_1)\right| &= \left|\int_{\bar{t}_1}^{\bar{t}_2} f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\,ds + \sum_{\bar{t}_1\le t_i<\bar{t}_2} I_i\bigl(u(t_i)\bigr)\right| \\
&\le \int_{\bar{t}_1}^{\bar{t}_2}\left|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\right|ds + \sum_{\bar{t}_1\le t_i<\bar{t}_2}\left|I_i\bigl(u(t_i)\bigr)\right| \\
&\le l_1(\bar{t}_2-\bar{t}_1) + l_3(\bar{t}_2-\bar{t}_1) \to 0, \quad \text{as } \bar{t}_1 \to \bar{t}_2.
\end{aligned}$$

Thus, for any $\varepsilon > 0$ (small enough), there exists $\delta = \delta(\varepsilon) > 0$ with independence of $\bar{t}_1$, $\bar{t}_2$ and u such that $\|(Tu)(\bar{t}_2) - (Tu)(\bar{t}_1)\|_{PC^1} < \varepsilon$, whenever $|\bar{t}_2 - \bar{t}_1| < \delta$. Therefore, T is equicontinuous on $J = [0,1]$. According to the Arzela-Ascoli theorem, it follows that $T : PC^1(J,\mathbb{R}) \to PC^1(J,\mathbb{R})$ is completely continuous. □

Theorem 3.1 *Assume that the conditions* (B_1)-(B_3) *hold.*

(B_1) $f \in C(J \times \mathbb{R}^3, \mathbb{R})$, *for all* $(t,u,v,w), (t,\bar{u},\bar{v},\bar{w}) \in J \times \mathbb{R}^3$, *there exist some functions* $\psi_i \in L([0,1])$ $(i = 1,2,3)$ *such that*

$$\begin{aligned}\bigl|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})\bigr| &\le \bigl|\psi_1(t)\bigr||u-\bar{u}| \\ &\quad + \bigl|\psi_2(t)\bigr||v-\bar{v}| + \bigl|\psi_3(t)\bigr||w-\bar{w}|.\end{aligned}$$

(B_2) $I_k \in C(\mathbb{R},\mathbb{R})$, *for all* $u, v \in \mathbb{R}$, *there exist some constants* $L_k > 0$ *such that*

$$\bigl|I_k(u) - I_k(v)\bigr| \le L_k|u-v|, \quad k = 1,2,\ldots,m.$$

(B_3) $g \in C(J,\mathbb{R})$, *for all* $(t,u), (t,v) \in J \times \mathbb{R}$, *there exists a function* $\psi \in L([0,1])$ *such that*

$$\bigl|g(t,u) - g(t,v)\bigr| \le \bigl|\psi(t)\bigr||u-v|.$$

If $\rho = 2\int_0^1 [|\psi_1(s)| + |\psi_2(s)| + |\psi_3(s)|]\,ds + \sum_{k=1}^m L_k + \Gamma(\alpha-1)\int_0^\eta |\psi(s)|\,ds < 1$, *then BVP* (1.1) *has a unique solution on* J.

Proof Let $M = \sup_{t\in J} |f(t,0,0,0)| + \sup_{t\in J} |g(t,0)|$ and $B_r = \{u \in PC(J,\mathbb{R}) : \|u\|_{PC^1} \le r\}$, where $r \ge \frac{1}{1-\rho}[(2 + \Gamma(\alpha-1)\eta)M + \sum_{k=1}^m |I_k(0)|]$. Define an operator $T : B_r \to PC(J,\mathbb{R})$ as (3.2). It is obvious that T is jointly continuous and maps bounded subsets of $J \times \mathbb{R}$ to bounded subsets of $\mathbb{R}$. We will prove Theorem 3.1 through the following two steps.

Step 1. We show that $T(B_r) \subset B_r$. In fact, noting that $u(0) = u'(0) = D^{\alpha-1}u(0) = 0$, we have, for $u \in B_r$, $t \in J = [0,1]$,

$$\begin{aligned}&\bigl|(Tu)(t)\bigr| \\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta \bigl|g\bigl(s,u(s)\bigr)\bigr|\,ds \\ &\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\biggl[\int_0^1 (1-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds + \sum_{t\le t_i}\bigl|I_i\bigl(u(t_i)\bigr)\bigr|\biggr] \\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl[\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr) - f(s,0,0,0)\bigr| + \bigl|f(s,0,0,0)\bigr|\bigr]\,ds \\ &\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}\bigl[\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr) - f(s,0,0,0)\bigr| + \bigl|f(s,0,0,0)\bigr|\bigr]\,ds \\ &\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{t\le t_i}\bigl[\bigl|I_i\bigl(u(t_i)\bigr) - I_i\bigl(u(0)\bigr)\bigr| + \bigl|I_i\bigl(u(0)\bigr)\bigr|\bigr] \\ &\qquad + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta \bigl[\bigl|g\bigl(s,u(s)\bigr) - g(s,0)\bigr| + \bigl|g(s,0)\bigr|\bigr]\,ds\end{aligned}$$

$$
\begin{aligned}
&\le \frac{r\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds+M}{\Gamma(\alpha)}+\frac{r\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds+M}{\Gamma(\alpha)}\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\bigl[L_k r+\bigl|I_k(0)\bigr|\bigr]+\frac{r}{\alpha-1}\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds+\frac{\eta M}{\alpha-1}\\
&=\biggl[\frac{2\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}L_k+\frac{1}{\alpha-1}\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]r\\
&\quad+\biggl[\frac{2}{\Gamma(\alpha)}+\frac{\eta}{\alpha-1}\biggr]M+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\bigl|I_k(0)\bigr|\\
&\le\biggl[2\int_0^1\bigl[\bigl|\psi_1(s)\bigr|+\bigl|\psi_2(s)\bigr|+\bigl|\psi_3(s)\bigr|\bigr]\,ds+\sum_{k=1}^{m}L_k+\Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]r\\
&\quad+\bigl[2+\Gamma(\alpha-1)\eta\bigr]M+\sum_{k=1}^{m}\bigl|I_k(0)\bigr|\\
&\le\rho r+(1-\rho)r=r,
\end{aligned}
$$

$$
\begin{aligned}
&\bigl|(Tu)'(t)\bigr|\\
&\quad\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds\\
&\qquad+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)\bigr|\,ds\\
&\qquad+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{t\le t_i}\bigl|I_i\bigl(u(t_i)\bigr)\bigr|+t^{\alpha-2}\int_0^{\eta}\bigl|g\bigl(s,u(s)\bigr)\bigr|\,ds\\
&\quad\le\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl[\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)-f(s,0,0,0)\bigr|+\bigl|f(s,0,0,0)\bigr|\bigr]\,ds\\
&\qquad+\int_0^1\frac{[t(1-s)]^{\alpha-2}}{\Gamma(\alpha-1)}\bigl[\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)-f(s,0,0,0)\bigr|+\bigl|f(s,0,0,0)\bigr|\bigr]\,ds\\
&\qquad+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{t\le t_i}\bigl[\bigl|I_i\bigl(u(t_i)\bigr)-I_i\bigl(u(0)\bigr)\bigr|+\bigl|I_i\bigl(u(0)\bigr)\bigr|\bigr]\\
&\qquad+t^{\alpha-2}\int_0^{\eta}\bigl[\bigl|g\bigl(s,u(s)\bigr)-g(s,0)\bigr|+\bigl|g(s,0)\bigr|\bigr]\,ds\\
&\quad\le\frac{r\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds+M}{\Gamma(\alpha-1)}+\frac{r\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds+M}{\Gamma(\alpha-1)}\\
&\qquad+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\bigl[L_k r+\bigl|I_k(0)\bigr|\bigr]+r\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds+\eta M\\
&\quad=\biggl[\frac{2\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}L_k+\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]r\\
&\qquad+\biggl[\frac{2}{\Gamma(\alpha-1)}+\eta\biggr]M+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}\bigl|I_k(0)\bigr|\\
&\quad\le\biggl[2\int_0^1\bigl[\bigl|\psi_1(s)\bigr|+\bigl|\psi_2(s)\bigr|+\bigl|\psi_3(s)\bigr|\bigr]\,ds+\sum_{k=1}^{m}L_k+\Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]r
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[2 + \Gamma(\alpha - 1)\eta\right]M + \sum_{k=1}^{m}\left|I_k(0)\right| \\
&\leq \rho r + (1 - \rho)r = r
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|D^{\alpha-1}(Tu)(t)\right| \\
&\leq \int_0^t \left|f\left(s, u(s), u'(s), D^{\alpha-1}u(s)\right)\right| ds + \int_0^1 (1-s)^{\alpha-2}\left|f\left(s, u(s), u'(s), D^{\alpha-1}u(s)\right)\right| ds \\
&\quad + \sum_{t\leq t_i}\left|I_i\left(u(t_i)\right)\right| + \Gamma(\alpha - 1)\int_0^\eta \left|g\left(s, u(s)\right)\right| ds \\
&\leq \int_0^t \left[\left|f\left(s, u(s), u'(s), D^{\alpha-1}u(s)\right) - f(s, 0, 0, 0)\right| + \left|f(s, 0, 0, 0)\right|\right] ds \\
&\quad + \int_0^1 (1-s)^{\alpha-2}\left[\left|f\left(s, u(s), u'(s), D^{\alpha-1}u(s)\right) - f(s, 0, 0, 0)\right| + \left|f(s, 0, 0, 0)\right|\right] ds \\
&\quad + \Gamma(\alpha - 1)\int_0^\eta \left[\left|g\left(s, u(s)\right) - g(s, 0)\right| + \left|g(s, 0)\right|\right] ds \\
&\quad + \sum_{t\leq t_i}\left[\left|I_i\left(u(t_i)\right) - I_i\left(u(0)\right)\right| + \left|I_i\left(u(0)\right)\right|\right] \\
&\leq r\int_0^1 \left[\left|\psi_1(s)\right| + \left|\psi_2(s)\right| + \left|\psi_3(s)\right|\right] ds + M + r\int_0^1 \left[\left|\psi_1(s)\right| + \left|\psi_2(s)\right| + \left|\psi_3(s)\right|\right] ds + M \\
&\quad + \sum_{k=1}^{m}\left[L_k r + \left|I_k(0)\right|\right] + \Gamma(\alpha - 1)\left[r\int_0^\eta \left|\psi(s)\right| ds + \eta M\right] \\
&= \left[2\int_0^1 \left[\left|\psi_1(s)\right| + \left|\psi_2(s)\right| + \left|\psi_3(s)\right|\right] ds + \sum_{k=1}^{m} L_k + \Gamma(\alpha - 1)\int_0^\eta \left|\psi(s)\right| ds\right] r \\
&\quad + \left[2 + \Gamma(\alpha - 1)\eta\right]M + \sum_{k=1}^{m}\left|I_k(0)\right| \\
&\leq \rho r + (1 - \rho)r = r,
\end{aligned}
$$

which imply that $\|Tu\|_{PC^1} \leq r$, that is, $T(B_r) \subset B_r$.

Step 2. We show that T is a contraction mapping. Indeed, for all $u, v \in B_r$, for each $t \in J = [0,1]$, we obtain

$$
\begin{aligned}
&\left|(Tu)(t) - (Tv)(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, u(s), u'(s), D^{\alpha-1}u(s)\right) - f\left(s, v(s), v'(s), D^{\alpha-1}v(s)\right)\right| ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(s, u(s), u'(s), D^{\alpha-1}u(s)\right) - f\left(s, v(s), v'(s), D^{\alpha-1}v(s)\right)\right| ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{t\leq t_i}\left|I_i\left(u(t_i)\right) - I_i\left(v(t_i)\right)\right| + \frac{t^{\alpha-1}}{\alpha - 1}\int_0^\eta \left|g\left(s, u(s)\right) - g\left(s, v(s)\right)\right| ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \left[\frac{2\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}L_k\right.\\
&\qquad\left.+\frac{1}{\alpha-1}\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\right]\|u-v\|_{PC^1}\\
&\leq\left[2\int_0^1\bigl[\bigl|\psi_1(s)\bigr|+\bigl|\psi_2(s)\bigr|+\bigl|\psi_3(s)\bigr|\bigr]\,ds+\sum_{k=1}^{m}L_k+\Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\right]\|u-v\|_{PC^1}\\
&=\rho\|u-v\|_{PC^1},
\end{aligned}$$

$$\begin{aligned}
&\bigl|(Tu)'(t)-(Tv)'(t)\bigr|\\
&\quad\leq\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)-f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds\\
&\qquad+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)-f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds\\
&\qquad+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{t\leq t_i}\bigl|I_i\bigl(u(t_i)\bigr)-I_i\bigl(v(t_i)\bigr)\bigr|+t^{\alpha-2}\int_0^{\eta}\bigl|g\bigl(s,u(s)\bigr)-g\bigl(s,v(s)\bigr)\bigr|\,ds\\
&\quad\leq\left[\frac{2\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m}L_k+\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\right]\|u-v\|_{PC^1}\\
&\quad\leq\left[2\int_0^1\bigl[\bigl|\psi_1(s)\bigr|+\bigl|\psi_2(s)\bigr|+\bigl|\psi_3(s)\bigr|\bigr]\,ds+\sum_{k=1}^{m}L_k+\Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\right]\|u-v\|_{PC^1}\\
&\quad=\rho\|u-v\|_{PC^1},
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|D^{\alpha-1}(Tu)(t)-D^{\alpha-1}(Tv)(t)\bigr|\\
&\quad\leq\int_0^t\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)-f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds\\
&\qquad+\int_0^1(1-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr)-f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds\\
&\qquad+\sum_{t\leq t_i}\bigl|I_i\bigl(u(t_i)\bigr)-I_i\bigl(v(t_i)\bigr)\bigr|+\Gamma(\alpha-1)\int_0^{\eta}\bigl|g\bigl(s,u(s)\bigr)-g\bigl(s,v(s)\bigr)\bigr|\,ds\\
&\quad\leq\left[2\int_0^1\bigl[\bigl|\psi_1(s)\bigr|+\bigl|\psi_2(s)\bigr|+\bigl|\psi_3(s)\bigr|\bigr]\,ds+\sum_{k=1}^{m}L_k+\Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\right]\|u-v\|_{PC^1}\\
&\quad=\rho\|u-v\|_{PC^1},
\end{aligned}$$

which indicates $\|Tu-Tv\|_{PC^1}\leq\rho\|u-v\|_{PC^1}$, where $\rho=2\int_0^1[|\psi_1(s)|+|\psi_2(s)|+|\psi_3(s)|]\,ds+\sum_{k=1}^{m}L_k+\Gamma(\alpha-1)\int_0^{\eta}|\psi(s)|\,ds<1$. Therefore T is a contraction mapping on $PC^1(J,\mathbb{R})$. According to the contraction mapping principle, we conclude that T has a unique fixed point $u(t)\in PC^1(J,\mathbb{R})$, which is the unique solution of BVP (1.1). The proof is complete. □

Now we give a simple and easily verifiable result as follows.

Corollary 3.1 *Assume that the conditions* (B_2), (B_3), *and* (B_1') *hold.*

(B_1') $f \in C(J \times \mathbb{R}^3, \mathbb{R})$, *for all* $(t,u,v,w), (t,\bar{u},\bar{v},\bar{w}) \in J \times \mathbb{R}^3$, *there exist some constants* $N_i > 0$ $(i = 1,2,3)$ *such that*

$$\left|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})\right| \le N_1|u-\bar{u}| + N_2|v-\bar{v}| + N_3|w-\bar{w}|.$$

If $\rho = \frac{\alpha(N_1+N_2+N_3)}{\alpha-1} + \sum_{k=1}^m L_k + \Gamma(\alpha-1)\int_0^\eta |\psi(s)|\,ds < 1$, *then BVP* (1.1) *has a unique solution on* J.

Proof Let $M = \sup_{t\in J}|f(t,0,0,0)| + \sup_{t\in J}|g(t,0)|$ and $B_r = \{u \in PC(J,\mathbb{R}) : \|u\|_{PC^1} \le r\}$, where $r \ge \frac{1}{1-\rho}[(\frac{\alpha}{\alpha-1} + \Gamma(\alpha-1)\eta)M + \sum_{k=1}^m |I_k(0)|]$. Define an operator $T : B_r \to PC(J,\mathbb{R})$ as (3.2). It is obvious that T is jointly continuous and maps bounded subsets of $J \times \mathbb{R}$ to bounded subsets of $\mathbb{R}$. Similarly, we will prove Corollary 3.1 through the following two steps.

Step 1. We show that $T(B_r) \subset B_r$. In fact, noting that $u(0) = u'(0) = D^{\alpha-1}u(0) = 0$, we have, for $u \in B_r$, $t \in J = [0,1]$,

$$\begin{aligned}
&\left|(Tu)(t)\right| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right|ds + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta \left|g\left(s,u(s)\right)\right|ds \\
&\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right|ds + \sum_{t\le t_i}\left|I_i\left(u(t_i)\right)\right|\right] \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right) - f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right]ds \\
&\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}\left[\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right) - f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right]ds \\
&\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{t\le t_i}\left[\left|I_i\left(u(t_i)\right) - I_i\left(u(0)\right)\right| + \left|I_i\left(u(0)\right)\right|\right] \\
&\qquad + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta \left[\left|g\left(s,u(s)\right) - g(s,0)\right| + \left|g(s,0)\right|\right]ds \\
&\quad\le \frac{2[(N_1+N_2+N_3)r + M]}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^m\left[L_k r + \left|I_k(0)\right|\right] + \frac{r}{\alpha-1}\int_0^\eta \left|\psi(s)\right|ds + \frac{\eta M}{\alpha-1} \\
&\quad= \left[\frac{2(N_1+N_2+N_3)}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^m L_k + \frac{1}{\alpha-1}\int_0^\eta \left|\psi(s)\right|ds\right]r \\
&\qquad + \left[\frac{2}{\Gamma(\alpha+1)} + \frac{\eta}{\alpha-1}\right]M + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^m \left|I_k(0)\right| \\
&\quad\le \left[\frac{\alpha(N_1+N_2+N_3)}{\alpha-1} + \sum_{k=1}^m L_k + \Gamma(\alpha-1)\int_0^\eta \left|\psi(s)\right|ds\right]r \\
&\qquad + \left[\frac{\alpha}{\alpha-1} + \Gamma(\alpha-1)\eta\right]M + \sum_{k=1}^m \left|I_k(0)\right| \\
&\quad\le \rho r + (1-\rho)r = r,
\end{aligned}$$

$$
\begin{aligned}
&\left|(Tu)'(t)\right| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right|ds + t^{\alpha-2}\int_0^\eta \left|g\left(s,u(s)\right)\right|ds \\
&\qquad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\left[\int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right|ds + \sum_{t\le t_i}\left|I_i\left(u(t_i)\right)\right|\right] \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left[\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)-f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right]ds \\
&\qquad + \int_0^1 \frac{[t(1-s)]^{\alpha-2}}{\Gamma(\alpha-1)}\left[\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)-f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right]ds \\
&\qquad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{t\le t_i}\left[\left|I_i\left(u(t_i)\right)-I_i\left(u(0)\right)\right| + \left|I_i\left(u(0)\right)\right|\right] \\
&\qquad + t^{\alpha-2}\int_0^\eta \left[\left|g\left(s,u(s)\right)-g(s,0)\right| + \left|g(s,0)\right|\right]ds \\
&\quad\le \frac{2[(N_1+N_2+N_3)r+M]}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^m\left[L_k r + \left|I_k(0)\right|\right] + r\int_0^\eta \left|\psi(s)\right|ds + \eta M \\
&\quad= \left[\frac{2(N_1+N_2+N_3)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^m L_k + \int_0^\eta\left|\psi(s)\right|ds\right]r \\
&\qquad + \left[\frac{2}{\Gamma(\alpha)} + \eta\right]M + \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^m\left|I_k(0)\right| \\
&\quad\le \left[\frac{\alpha(N_1+N_2+N_3)}{\alpha-1} + \Gamma(\alpha-1)\int_0^\eta\left|\psi(s)\right|ds + \sum_{k=1}^m L_k\right]r \\
&\qquad + \left[\frac{\alpha}{\alpha-1} + \Gamma(\alpha-1)\eta\right]M + \sum_{k=1}^m\left|I_k(0)\right| \\
&\quad\le \rho r + (1-\rho)r = r,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|D^{\alpha-1}(Tu)(t)\right| \\
&\quad\le \int_0^t \left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right|ds + \int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right|ds \\
&\qquad + \sum_{t\le t_i}\left|I_i\left(u(t_i)\right)\right| + \Gamma(\alpha-1)\int_0^\eta\left|g\left(s,u(s)\right)\right|ds \\
&\quad\le \int_0^t \left[\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)-f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right]ds \\
&\qquad + \int_0^1 (1-s)^{\alpha-2}\left[\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)-f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right]ds \\
&\qquad + \sum_{t\le t_i}\left[\left|I_i\left(u(t_i)\right)-I_i\left(u(0)\right)\right| + \left|I_i\left(u(0)\right)\right|\right] \\
&\qquad + \Gamma(\alpha-1)\int_0^\eta\left[\left|g\left(s,u(s)\right)-g(s,0)\right| + \left|g(s,0)\right|\right]ds
\end{aligned}
$$

$$\begin{aligned}
&\leq r\sum_{i=0}^{3} N_i + M + \frac{r\sum_{i=0}^{3} N_i + M}{\alpha-1} + \sum_{k=1}^{m}\bigl[L_k r + \bigl|I_k(0)\bigr|\bigr] \\
&\quad + \Gamma(\alpha-1)\biggl[r\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds + \eta M\biggr] \\
&= \biggl[\frac{\alpha(N_1+N_2+N_3)}{\alpha-1} + \sum_{k=1}^{m} L_k + \Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr] r \\
&\quad + \biggl[\frac{\alpha}{\alpha-1} + \Gamma(\alpha-1)\eta\biggr] M + \sum_{k=1}^{m}\bigl|I_k(0)\bigr| \\
&\leq \rho r + (1-\rho) r = r,
\end{aligned}$$

which imply that $\|Tu\|_{PC^1} \leq r$, that is, $T(B_r) \subset B_r$.

Step 2. We show that T is a contraction mapping. Indeed, for all $u, v \in B_r$, for each $t \in J = [0,1]$, we obtain

$$\begin{aligned}
&\bigl|(Tu)(t) - (Tv)(t)\bigr| \\
&\quad\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr) - f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr) - f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{t\leq t_i}\bigl|I_i\bigl(u(t_i)\bigr) - I_i\bigl(v(t_i)\bigr)\bigr| + \frac{t^{\alpha-1}}{\alpha-1}\int_0^{\eta}\bigl|g\bigl(s,u(s)\bigr) - g\bigl(s,v(s)\bigr)\bigr|\,ds \\
&\quad\leq \biggl[\frac{N_1+N_2+N_3}{\Gamma(\alpha+1)} + \frac{N_1+N_2+N_3}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m} L_k + \frac{1}{\alpha-1}\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]\|u-v\|_{PC^1} \\
&\quad\leq \biggl[\frac{\alpha(N_1+N_2+N_3)}{\alpha-1} + \sum_{k=1}^{m} L_k + \Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]\|u-v\|_{PC^1} = \rho\|u-v\|_{PC^1}, \\
&\bigl|(Tu)'(t) - (Tv)'(t)\bigr| \\
&\quad\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr) - f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\bigl|f\bigl(s,u(s),u'(s),D^{\alpha-1}u(s)\bigr) - f\bigl(s,v(s),v'(s),D^{\alpha-1}v(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\sum_{t\leq t_i}\bigl|I_i\bigl(u(t_i)\bigr) - I_i\bigl(v(t_i)\bigr)\bigr| + t^{\alpha-2}\int_0^{\eta}\bigl|g\bigl(s,u(s)\bigr) - g\bigl(s,v(s)\bigr)\bigr|\,ds \\
&\quad\leq \biggl[\frac{2(N_1+N_2+N_3)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\sum_{k=1}^{m} L_k + \int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]\|u-v\|_{PC^1} \\
&\quad\leq \biggl[\frac{\alpha(N_1+N_2+N_3)}{\alpha-1} + \sum_{k=1}^{m} L_k + \Gamma(\alpha-1)\int_0^{\eta}\bigl|\psi(s)\bigr|\,ds\biggr]\|u-v\|_{PC^1} = \rho\|u-v\|_{PC^1},
\end{aligned}$$

and

$$
\begin{aligned}
&\left|D^{\alpha-1}(Tu)(t)-D^{\alpha-1}(Tv)(t)\right| \\
&\quad\le \int_0^t \left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)-f\left(s,v(s),v'(s),D^{\alpha-1}v(s)\right)\right| ds \\
&\qquad+ \int_0^1 (1-s)^{\alpha-2}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)-f\left(s,v(s),v'(s),D^{\alpha-1}v(s)\right)\right| ds \\
&\qquad+ \sum_{t\le t_i}\left|I_i\left(u(t_i)\right)-I_i\left(v(t_i)\right)\right| + \Gamma(\alpha-1)\int_0^\eta \left|g\left(s,u(s)\right)-g\left(s,v(s)\right)\right| ds \\
&\quad\le \left[N_1+N_2+N_3+\frac{N_1+N_2+N_3}{\alpha-1}+\sum_{k=1}^m L_k+\Gamma(\alpha-1)\int_0^\eta\left|\psi(s)\right| ds\right]\|u-v\|_{PC^1} \\
&\quad= \left[\frac{\alpha(N_1+N_2+N_3)}{\alpha-1}+\sum_{k=1}^m L_k+\Gamma(\alpha-1)\int_0^\eta\left|\psi(s)\right| ds\right]\|u-v\|_{PC^1} = \rho\|u-v\|_{PC^1},
\end{aligned}
$$

which indicates $\|Tu-Tv\|_{PC^1}\le\rho\|u-v\|_{PC^1}$, where $\rho=\frac{\alpha(N_1+N_2+N_3)}{\alpha-1}+\sum_{k=1}^m L_k+\Gamma(\alpha-1)\int_0^\eta|\psi(s)|\,ds<1$. Therefore T is a contraction mapping on $PC^1(J,\mathbb{R})$. According to the contraction mapping principle, we conclude that T has a unique fixed point $u(t)\in PC^1(J,\mathbb{R})$, which is the unique solution of BVP (1.1). The proof is complete. □

For some fixed $r>0$, considering BVP (1.1) on the cylinder $\mathcal{R}=[0,1]\times B(0,r)$, we obtain the following theorem.

Theorem 3.2 *Assume that conditions* (B_4)-(B_6) *hold. Then BVP* (1.1) *has at least one solution in J, provided that* $\varrho=\sum_{k=1}^m L_k+\Gamma(\alpha-1)\int_0^\eta|\psi(s)|\,ds<1$.

(B_4) $f\in C(J\times\mathbb{R}^3,\mathbb{R})$, *for all* $(t,u,u',D^{\alpha-1}u)\in J\times\mathbb{R}^3$, *there exist* $p\in(0,1)$, $h\in L_{1/p}([0,1],\mathbb{R}^+)$ *such that* $|f(t,u,u',D^{\alpha-1}u)|\le h(t)$, *where* $L_{1/p}([0,1],\mathbb{R}^+)$ *denotes space* $1/p$-*Lebesgue measurable functions from* $[0,1]$ *to* $\mathbb{R}^+$ *with the norm* $\|v\|_{1/p}=(\int_0^1|v(s)|^{\frac{1}{p}}\,ds)^p$, *for* $v\in L_{1/p}([0,1],\mathbb{R}^+)$.

(B_5) $I_k\in C(\mathbb{R},\mathbb{R})$, *for all* $u\in\mathbb{R}$, *there exist some constants* $L_k>0$ *such that* $|I_k(u)|<L_k|u|$, $k=1,2,\ldots,m$.

(B_6) $g\in C(J,\mathbb{R})$, *for all* $(t,u)\in(J,\mathbb{R})$, *there exists* $\psi\in L[0,1]$ *such that* $|g(t,u)|\le|\psi(t)||u|$.

Proof Let B_λ be a closed bounded convex subset of $PC^1([0,1],\mathbb{R})$ defined by $B_\lambda=\{u:\|u\|\le\lambda\}$, $\lambda\ge\frac{A}{1-\varrho}$, $A=[1+(\frac{1-p}{\alpha-p-1})^{1-p}]\|h\|_{1/p}$.

Define the operator $T:B_\lambda\to PC^1([0,1],\mathbb{R})$ as (3.2). For $u\in\partial B_\lambda$, we have

$$
\begin{aligned}
&\left|(Tu)(t)\right| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right| ds + \frac{t^{\alpha-1}}{\alpha-1}\int_0^\eta\left|g\left(s,u(s)\right)\right| ds \\
&\qquad+ \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-2}\left|f\left(s,u(s),u'(s),D^{\alpha-1}u(s)\right)\right| ds+\sum_{t\le t_i}\left|I_i\left(u(t_i)\right)\right|\right] \\
&\quad\le \int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds+\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}h(s)\,ds \\
&\qquad+ \frac{1}{\Gamma(\alpha)}\sum_{t\le t_i}L_i\left|u(t_i)\right|+\frac{1}{\alpha-1}\int_0^\eta\left|\psi(s)\right|\left|u(s)\right| ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\frac{\alpha-1}{1-p}}\,ds\right)^{1-p}\left(\int_0^1 \big(h(s)\big)^{\frac{1}{p}}\,ds\right)^{p} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\frac{\alpha-2}{1-p}}\,ds\right)^{1-p}\left(\int_0^1 \big(h(s)\big)^{\frac{1}{p}}\,ds\right)^{p} \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m} L_k\|u\|_{PC^1} + \frac{\|u\|_{PC^1}}{\alpha-1}\int_0^\eta \big|\psi(s)\big|\,ds \\
&\le \left[\left(\frac{1-p}{\alpha-p}\right)^{1-p} + \left(\frac{1-p}{\alpha-p-1}\right)^{1-p}\right]\frac{\|h\|_{1/p}}{\Gamma(\alpha)} + \lambda\left[\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m} L_k + \frac{1}{\alpha-1}\int_0^\eta \big|\psi(s)\big|\,ds\right] \\
&\le \left[1 + \left(\frac{1-p}{\alpha-p-1}\right)^{1-p}\right]\|h\|_{1/p} + \lambda\left[\sum_{k=1}^{m} L_k + \Gamma(\alpha-1)\int_0^\eta \big|\psi(s)\big|\,ds\right] = A + \lambda\varrho \le \lambda,
\end{aligned}
$$

$$
\begin{aligned}
&\big|(Tu)'(t)\big| \\
&\le \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\big|f\big(s,u(s),u'(s),D^{\alpha-1}u(s)\big)\big|\,ds + t^{\alpha-2}\int_0^\eta \big|g\big(s,u(s)\big)\big|\,ds \\
&\quad + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\left[\int_0^1 (1-s)^{\alpha-2}\big|f\big(s,u(s),u'(s),D^{\alpha-1}u(s)\big)\big|\,ds + \sum_{t\le t_i}\big|I_i\big(u(t_i)\big)\big|\right] \\
&\le \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}h(s)\,ds + \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}h(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\sum_{t\le t_i} L_i\big|u(t_i)\big| + \int_0^\eta \big|\psi(s)\big|\big|u(s)\big|\,ds \\
&\le \frac{\alpha-1}{\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\frac{\alpha-2}{1-p}}\,ds\right)^{1-p}\left(\int_0^1 \big(h(s)\big)^{\frac{1}{p}}\,ds\right)^{p} \\
&\quad + \frac{\alpha-1}{\Gamma(\alpha)}\left(\int_0^1 (1-s)^{\frac{\alpha-2}{1-p}}\,ds\right)^{1-p}\left(\int_0^1 \big(h(s)\big)^{\frac{1}{p}}\,ds\right)^{p} \\
&\quad + \frac{\alpha-1}{\Gamma(\alpha)}\sum_{k=1}^{m} L_k\|u\|_{PC^1} + \|u\|_{PC^1}\int_0^\eta \big|\psi(s)\big|\,ds \\
&\le 2(\alpha-1)\left(\frac{1-p}{\alpha-p-1}\right)^{1-p}\frac{\|h\|_{1/p}}{\Gamma(\alpha)} + \lambda\left[\frac{\alpha-1}{\Gamma(\alpha)}\sum_{k=1}^{m} L_k + \int_0^\eta \big|\psi(s)\big|\,ds\right] \\
&\le \left[1 + \left(\frac{1-p}{\alpha-p-1}\right)^{1-p}\right]\|h\|_{1/p} + \lambda\left[\sum_{k=1}^{m} L_k + \Gamma(\alpha-1)\int_0^\eta \big|\psi(s)\big|\,ds\right] = A + \lambda\varrho \le \lambda,
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|D^{\alpha-1}(Tu)(t)\big| \\
&\le \int_0^t \big|f\big(s,u(s),u'(s),D^{\alpha-1}u(s)\big)\big|\,ds + \int_0^1 (1-s)^{\alpha-2}\big|f\big(s,u(s),u'(s),D^{\alpha-1}u(s)\big)\big|\,ds \\
&\quad + \sum_{t\le t_i}\big|I_i\big(u(t_i)\big)\big| + \Gamma(\alpha-1)\int_0^\eta \big|g\big(s,u(s)\big)\big|\,ds \\
&\le \int_0^1 h(s)\,ds + \int_0^1 (1-s)^{\alpha-2}h(s)\,ds + \sum_{t\le t_i} L_i\big|u(t_i)\big| + \Gamma(\alpha-1)\int_0^\eta \big|\psi(s)\big|\big|u(s)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \left(\int_0^1 \bigl(h(s)\bigr)^{\frac{1}{p}}\,ds\right)^p + \left(\int_0^1 (1-s)^{\frac{\alpha-2}{1-p}}\,ds\right)^{1-p}\left(\int_0^1 \bigl(h(s)\bigr)^{\frac{1}{p}}\,ds\right)^p + \sum_{k=1}^m L_k\|u\|_{PC^1} \\
&\quad + \Gamma(\alpha-1)\|u\|_{PC^1}\int_0^\eta \bigl|\psi(s)\bigr|\,ds \\
&= \left[1+\left(\frac{1-p}{\alpha-p-1}\right)^{1-p}\right]\|h\|_{1/p} + \lambda\left[\sum_{k=1}^m L_k + \Gamma(\alpha-1)\int_0^\eta \bigl|\psi(s)\bigr|\,ds\right] = A + \lambda\varrho \le \lambda.
\end{aligned}
$$

Therefore, $T(B_\lambda) \subset B_\lambda$. By Lemma 3.2, we see that $T: B_\lambda \to B_\lambda$ is completely continuous. Thus BVP (1.1) has at least one solution by Lemma 2.3. The proof is complete. □

4 Illustrative example

As an application of the main results, we consider the following impulsive fractional differential equation with integral boundary conditions:

$$
\begin{cases}
{}_{t_k}D_t^{\frac{5}{2}}u(t) = f(t,u,u',D^{\frac{3}{2}}u), & t \ne t_k, \\
\Delta D^{\frac{3}{2}}u(t_k) = I_k(u(t_k)), & k = 1,\dots,m, \\
u(0) = u'(0) = 0, & u'(1) = \int_0^\eta g(s,u(s))\,ds,
\end{cases}
\tag{4.1}
$$

here $t \in J = [0,1]$, $t_k = 1 - \frac{1}{2^k}$ $(k = 1,2,\dots,m)$, $\alpha = \frac{5}{2}$, $\eta = \frac{1}{2}$.

Case 1 Let

$$
\begin{aligned}
f\bigl(t,u,u',D^{\frac{3}{2}}u\bigr) &= \frac{t^4u(t)}{10 + h_1(u(t),u'(t),D^{\frac{3}{2}}u(t))} + \frac{e^{-t}D^{\frac{3}{2}}u(t)}{20 + h_3(u(t),u'(t),D^{\frac{3}{2}}u(t))} \\
&\quad + \frac{u'(t)}{(1+t)^3(15 + h_2(u(t),u'(t),D^{\frac{3}{2}}u(t))}, \\
I_k\bigl(u(t_k)\bigr) &= \frac{|u(t_k)|}{5^k + |u(t_k)|} + 1, \qquad g(t,u) = \frac{t^2u^2(t)}{1+u^2(t)},
\end{aligned}
$$

where $h_i(u,u',D^{\frac{3}{2}}u) \ge 0$ $(i = 1,2,3)$. By simple computation, we have $\psi_1(s) = \frac{s^4}{10}$, $\psi_2(s) = \frac{1}{15(1+s)^3}$, $\psi_3(s) = \frac{1}{20e^t}$, $L_k = \frac{1}{5^k}$, $\psi(s) = 2s^2$,

$$
\begin{aligned}
\rho &= 2\int_0^1 \bigl[\bigl|\psi_1(s)\bigr| + \bigl|\psi_2(s)\bigr| + \bigl|\psi_3(s)\bigr|\bigr]\,ds + \sum_{k=1}^m L_k + \Gamma(\alpha-1)\int_0^\eta \bigl|\psi(s)\bigr|\,ds \\
&= \frac{1}{25} + \frac{1}{20} + \frac{1-e^{-1}}{10} + \frac{1}{4}\left(1 - \frac{1}{5^{m+1}}\right) + \frac{\Gamma(\frac{3}{2})}{12} \\
&< \frac{264 + 25\sqrt{\pi}}{600} < 1.
\end{aligned}
$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence BVP (4.1) has a unique solution on $J = [0,1]$.

Case 2 Take

$$
f\bigl(t,u,u',D^{\frac{3}{2}}u\bigr) = \frac{t^2(u+u+D^{\frac{3}{2}}u)^2}{1 + (u+u+D^{\frac{3}{2}}u)^2}.
$$

$I_k(u(t_k))$ and $g(t,u)$ are the same as Case 1. It is clear that $|f(t,u,u',D^{\frac{3}{2}}u)| \leq t^2 \triangleq h(t)$ and $\varrho = \sum_{k=1}^{m} L_k + \Gamma(\alpha-1)\int_0^\eta |\psi(s)|\,ds < \frac{6+\sqrt{\pi}}{24} < 1$. Thus, BVP (4.1) has at least one solution in $J = [0,1]$ by Theorem 3.2.

5 Conclusions

Compared with previous papers involving impulsive fractional order differential equations, the impulse of our boundary value problem (1.1) is related to the fractional order derivative, namely, $\Delta D^{\alpha-1}u(t_k) = I_k(u(t_k))$. It is difficult and challenging to find the Green function of (1.1). Our results are new and interesting. Our methods can be used to study the existence of positive solutions for the high order or multiple-point boundary value problems of nonlinear fractional differential equation with the impulses involving the fractional order derivative. However, there exist some difficulties and complexities to address the structure of the Green function for these boundary value problems.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the anonymous referees for their useful and valuable suggestions. This work is supported by the National Natural Sciences Foundation of Peoples Republic of China under Grant (No. 11161025; No. 11661047), and the Yunnan Province natural scientific research fund project (No. 2011FZ058).

Received: 10 November 2016 Accepted: 30 January 2017 Published online: 09 February 2017

References

1. Kilbas, AA, Srivastava, H, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
2. Podlubny, I: Fractional Differential Equation. Academic Press, San Diego (1999)
3. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993)
4. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
5. Tarasov, VE: Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Berlin (2010)
6. Diethelm, K: The Analysis of Fractional Differential Equations. Springer, Berlin (2010)
7. Feng, M, Ge, W: Existence results for a class of *n*th order *m*-point boundary value problems in Banach spaces. Appl. Math. Lett. **22**, 1303-1308 (2009)
8. Chang, Y, Nieto, JJ: Some new existence results for fractional differential inclusions with boundary conditions. Math. Comput. Model. **49**, 605-609 (2009)
9. Goodrich, C: Existence of a positive solution to a class of fractional differential equations. Comput. Math. Appl. **59**, 3489-3499 (2010)
10. Bai, Z: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. **72**, 916-924 (2010)
11. Zhang, X, Liu, L, Wu, Y: Multiple positive solutions of a singular fractional differential equation with negatively perturbed term. Math. Comput. Model. **55**(3), 1263-1274 (2012)
12. Li, C, Luo, X, Zhou, Y: Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations. Comput. Math. Appl. **59**, 1363-1375 (2010)
13. Zhang, S: Positive solutions to singular boundary value problem for nonlinear fractional differential equation. Comput. Math. Appl. **59**, 1300-1309 (2010)
14. Wang, Y, Liu, Y, Wu, Y: Positive solutions for a class of fractional boundary value problem with changing sign nonlinearity. Nonlinear Anal., Theory Methods Appl. **74**(17), 6434-6441 (2011)
15. Goodrich, C: Existence of a positive solution to a class of fractional differential equations. Appl. Math. Lett. **23**, 1050-1055 (2010)
16. Salem, H: On the existence of continuous solutions for a singular system of nonlinear fractional differential equations. Appl. Math. Comput. **198**, 445-452 (2008)
17. Jafari, H, Gejji, V: Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method. Appl. Math. Comput. **180**, 700-706 (2006)
18. Jiang, D, Yuan, C: The positive properties of the green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application. Nonlinear Anal. TMA **72**, 710-719 (2010)
19. Zhao, KH, Gong, P: Existence of positive solutions for a class of higher-order Caputo fractional differential equation. Qual. Theory Dyn. Syst. **14**(1), 157-171 (2015)
20. Zhao, KH, Gong, P: Positive solutions of Riemann-Stieltjes integral boundary problems for the nonlinear coupling system involving fractional-order differential. Adv. Differ. Equ. **2014**, 254 (2014)

21. Zhao, KH, Gong, P: Positive solutions of m-point multi-term fractional integral BVP involving time-delay for fractional differential equations. Bound. Value Probl. **2015**, 19 (2015)
22. Zhao, KH, Gong, P: Existence and nonexistence of positive solutions for a singular higher-order nonlinear fractional differential equation. WSEAS Trans. Math. **14**, 87-96 (2015)
23. Zhao, KH, Wang, K: Existence of solutions for the delayed nonlinear fractional functional differential equations with three-point integral boundary value conditions. Adv. Differ. Equ. **2016**, 284 (2016)
24. Gao, Y, Chen, P: Existence of solutions for a class of nonlinear higher-order fractional differential equation with fractional nonlocal boundary condition. Adv. Differ. Equ. **2016**, 314 (2016)
25. Rong, J, Bai, C: Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions. Adv. Differ. Equ. **2015**, 82 (2015)
26. Ahmad, B, Alsaedi, A: Nonlinear fractional differential equations with nonlocal fractional integro-differential boundary conditions. Bound. Value Probl. **2012**, 124 (2012)
27. Li, RG: Existence of solutions for nonlinear singular fractional differential equations with fractional derivative condition. Adv. Differ. Equ. **2014**, 292 (2014)
28. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions. Hindawi Publ. Corp., New York (2006)
29. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
30. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
31. Bai, C: Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative. J. Math. Anal. Appl. **384**, 211-231 (2011)
32. Anguraj, A, Karthikeyan, P, Rivero, M, Trujillo, JJ: On new existence results for fractional integro-differential equations with impulsive and integral conditions. Comput. Math. Appl. **66**(12), 2587-2594 (2014)
33. Cao, J, Chen, H: Impulsive fractional differential equations with nonlinear boundary conditions. Math. Comput. Model. **55**(3), 303-311 (2012)
34. Wang, G, Ahmad, B, Zhang, L: Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order. Nonlinear Anal. **74**, 792-804 (2011)
35. Zhou, Y, Wang, JY: On the concept and existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**(7), 3050-3060 (2012)
36. Mophou, GN: Existence and uniqueness of mild solutions to impulsive fractional differential equations. Nonlinear Anal. TMA **72**(3), 1604-1615 (2010)
37. Ahmad, B, Sivasundaram, S: Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations. Nonlinear Anal. Hybrid Syst. **3**, 251-258 (2009)
38. Zhao, KH, Gong, P: Positive solutions for impulsive fractional differential equations with generalized periodic boundary value conditions. Adv. Differ. Equ. **2014**, 255 (2014)
39. Zhao, KH: Multiple positive solutions of integral BVPs for high-order nonlinear fractional differential equations with impulses and distributed delays. Dyn. Syst. **30**(2), 208-223 (2015)
40. Zhao, KH: Impulsive boundary value problems for two classes of fractional differential equation with two different Caputo fractional derivatives. Mediterr. J. Math. **13**, 1033-1050 (2016)
41. Fu, X, Bao, XM: Some existence results for nonlinear fractional differential equations with impulsive and fractional integral boundary conditions. Adv. Differ. Equ. **2014**, 129 (2014)
42. Zhou, J, Feng, MQ: Green's function for Sturm-Liouville-type boundary value problems of fractional order impulsive differential equations and its application. Bound. Value Probl. **2014**, 69 (2014)
43. Zhou, WX, Liu, X, Zhang, JG: Some new existence and uniqueness results of solutions to semilinear impulsive fractional integro-differential equations. Adv. Differ. Equ. **2015**, 38 (2015)
44. Asawasamrit, S, Ntouyas, SK, Thiramanus, P, Tariboon, J: Periodic boundary value problems for impulsive conformable fractional integro-differential equations. Bound. Value Probl. **2016**, 122 (2016)
45. Mahmudov, N, Unul, S: On existence of BVP's for impulsive fractional differential equations. Adv. Differ. Equ. **2017**, 15 (2017)
46. Hale, JK: Theory of Functional Differential Equations. Springer, New York (1977)