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Existence of mild solutions for fractional nonlocal evolution equations with delay in partially ordered Banach spaces

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Abstract

This paper deals with the existence of mild solutions for the abstract fractional nonlocal evolution equations with noncompact semigroup in partially ordered Banach spaces. Under some mixed conditions, a group of sufficient conditions for the existence of abstract fractional nonlocal evolution equations are obtained by using a Krasnoselskii type fixed point theorem. The results we obtained are a generalization and continuation of the recent results on this issue. At the end, an example is given to illustrate the applicability of abstract result.

MSC: 35F25; 45N05

Keywords: partially ordered Banach space; fractional nonlocal evolution equation; positive C_0-semigroup; existence; Krasnoselskii type fixed point theorem

1 Introduction

Let $(E, \leq, \|\cdot\|)$ be a partially ordered Banach space. We consider the existence of mild solutions for the fractional nonlocal evolution equation

$$\begin{cases} D_t^q x(t) + Ax(t) = f(t, x_t) + h(t, x_t), & t \in J := [0, b], \\ x_0(t) = \varphi(t) + g(x), & t \in [-r, 0], \end{cases} \tag{1.1}$$

where D_t^q is the Caputo fractional derivative of order $q \in (0,1)$, $b > 0$ and $r > 0$ are constants, $-A : D(A) \subset E \to E$ generates a positive C_0-semigroup $T(t)$ $(t \geq 0)$ of uniformly bounded linear operator in E, f, h and g are given functions, $\varphi : [-r, 0] \to E$ is continuous, x_t is defined by $x_t(\tau) = x(t + \tau)$ for all $\tau \in [-r, 0]$ and $t \in J$.

It is well known that the differential equations with different type of boundary conditions have an extensive physical background and realistic applications, and the theory has been considerably developed in recent years [1–12]. Recently, in [1, 2, 7], the authors studied the existence of solutions for the initial value problem of the first order ordinary differential equation

$$x'(t) = f\big(t, x(t)\big) + g\big(t, x(t)\big), \quad t \in [t_0, t_0 + a], \qquad x(t_0) = x_0 \in \mathbb{R}, \tag{1.2}$$

where $t_0 \geq 0$, $a > 0$ are two constants. In [1], the existence theorems of the problem (1.2) are proved by using a hybrid Schaefer type fixed point theorem under strong Lipschitz and compactness type conditions. In [2, 7], under weaker partial continuity and partial compactness type conditions, some existence theorems are obtained by using a Krasnoselskii type fixed point theorem. These results are all for first order ordinary differential equations without delay, but as far as we know the existence of solutions for nonlinear fractional evolution equations with the nonlocal initial condition and delay in partially ordered Banach spaces has not been considered. This is the main motivation of the present paper. In this paper, we will prove the existence of mild solutions for the problem (1.1) in partially ordered Banach spaces by using the Krasnoselskii type fixed point theorem under some weaker partial continuity and partial compactness type conditions. The results we obtained are a generalization and continuation of the recent results on this issue. At the end, we give an example of the fractional parabolic equation to illustrate the applicability of abstract result.

2 Preliminaries

Let $(X, \leq, \|\cdot\|)$ be a partially ordered normed linear space. Two elements x, y in X are said to be comparable if either $x \leq y$ or $y \leq x$. A function $\Psi : X \to X$ is said to be upper semicontinuous (u.s.c.) if the set $\{x \in X : \Psi(x) \cap B \neq \varnothing\}$ is closed for any closed subset B in X. Definitions 1-8 have been introduced in [2, 3, 7] and are frequently used in the subsequent part of the article.

Definition 1 A partially ordered normed linear space X is said to be regular if either

(i) if a nondecreasing sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to x^*, then $x_n \leq x^*$ for all $n \in \mathbb{N}$, or

(ii) if a nonincreasing sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to x^*, then $x_n \geq x^*$ for all $n \in \mathbb{N}$.

Definition 2 The order relation $\leq$ and the norm $\|\cdot\|$ on partially ordered normed linear space $(X, \leq, \|\cdot\|)$ are compatible if for any monotone nondecreasing or monotone nonincreasing sequence $\{x_n\} \subset X$, the convergence of any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ to x^* implies that the whole sequence $\{x_n\}$ converges to x^*.

Definition 3 A mapping $Q : X \to X$ is nondecreasing if $x \leq y$ implies $Qx \leq Qy$ for all $x, y \in X$. Similarly, Q is nonincreasing if $x \leq y$ implies $Qx \geq Qy$ for all $x, y \in X$.

Definition 4 A mapping $Q : X \to X$ is partially continuous at a point $a \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|Qx - Qa\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. If Q is partially continuous at every point of X, then it is partially continuous on X.

Definition 5 A mapping $Q : X \to X$ is partially bounded if $Q(C)$ is bounded for every chain C in X. Q is bounded if $Q(X)$ is a bounded subset of X.

Remark 1 If Q is bounded in X, then it is partially bounded in X. However, the reverse description does not hold.

Definition 6 A mapping $Q : X \to X$ is partially compact if $Q(C)$ is relatively compact on X for all totally ordered sets or chains C of X.

Definition 7 A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a $\mathfrak{D}$-function if it is an upper semicontinuous and monotonic nondecreasing function satisfying $\psi(0) = 0$.

Remark 2 Examples of $\mathfrak{D}$-functions are $\psi(r) = kr$ for $k > 0$, $\psi(r) = \frac{r}{1+r}$ and $\psi(r) = \tan^{-1} r$ *etc.* If ϕ, ψ are two $\mathfrak{D}$-functions, then (i) $\phi + \psi$, (ii) $\lambda\phi, \lambda > 0$, and (iii) $\phi \circ \psi$ are all $\mathfrak{D}$-functions.

Definition 8 A mapping $Q : X \to X$ is called a partially nonlinear $\mathfrak{D}$-contraction if there exists a $\mathfrak{D}$-function ψ such that

$$\left\| Q(x) - Q(y) \right\| \leq \psi\left(\|x - y\|\right)$$

for all comparable elements $x, y \in X$ with $\psi(r) < r$ for $r > 0$.

Our result is based on the following Krasnoselskii type fixed point theorem, which can be found in [3].

Lemma 1 *Let $(X, \leq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\leq$ and the norm $\|\cdot\|$ in X are compatible. Let $A, B : X \to X$ be two nondecreasing operators such that*:

(a) *A is partially bounded and it is a partially nonlinear $\mathcal{D}$-contraction,*
(b) *B is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in X$ such that $x_0 \leq Ax_0 + Bx_0$.*

Then the operator equation $x = Ax + Bx$ has a solution $x^* \in X$ and the sequence $\{x_n\}$ defined by $x_{n+1} = Ax_n + Bx_n$, $n = 0, 1, 2, \ldots$, converges monotonically to x^*.

At the end of this section, we recall the definitions of fractional calculus. See [5, 12] for more details.

Definition 9 The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)\, ds, \quad t > 0,$$

where Γ is the gamma function.

The Caputo fractional derivative of order $n - 1 < \alpha < n$ with the lower limit zero for a function $f \in C^n[0, \infty)$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad t > 0, n \in \mathbb{N}.$$

Remark 3 By Definition 9, the Caputo derivative of a constant is equal to zero.

3 Existence theorem

Let E be a partially ordered Banach space with the partial order $\leq$ and the norm $\|\cdot\|$, whose positive cone K is defined by $K = \{x \in E : x \geq 0\}$. It is well known that if cone K is normal, then the order relation $\leq$ and the norm $\|\cdot\|$ in E are compatible. Throughout this paper, we assume that $-A : D(A) \subset E \to E$ generates a positive C_0-semigroup $T(t)$ $(t \geq 0)$ of uniformly bounded linear operator in E. Namely, there exists a constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. A C_0-semigroup $T(t)$ $(t \geq 0)$ is said to be positive if the inequality $T(t)x \geq 0$ holds for all $x \geq 0$. Let $K \geq 0$ be a constant. Then $-(A + KI)$ generates a positive C_0-semigroup $S(t) = e^{-Kt}T(t)$ $(t \geq 0)$ in E. For more properties of operator semigroup and positive C_0-semigroup, please see [4, 8].

Denote by $C([-r,b],E)$ the Banach space of all continuous E-valued functions on the interval $[-r,b]$ with the norm $\|x\|_{[-r,b]} = \max_{t\in[-r,b]} \|x(t)\|$ for any $x \in C([-r,b],E)$. Define a positive cone K_C by

$$K_C = \{x \in C([-r,b],E) : x(t) \in K, t \in [-r,b]\}.$$

Then $C([-r,b],E)$ is a partially ordered Banach space with the order relation $\leq$ induced by K_C. K_C is normal if K is normal. Similarly, $(C([-r,0],E), \leq, \|x\|_{[-r,0]})$ is also a partially ordered Banach space.

In the rest of this paper, we consider the following assumptions:

(H1) $\|e^{-K\theta(t_2^q - t_1^q)}T(t_2^q\theta) - T(t_1^q\theta)\| \to 0$ as $t_2 - t_1 \to 0$ for every $\theta \in (0, +\infty)$.

(H2) The function $f(t,x) : J \times C([-r,0],E) \to E$ is continuous in x for all $t \in J$ and satisfies the following conditions:

(i) There exist a constant $0 < \sigma \leq \frac{\Gamma(q+1)}{Mb^q}$ and a $\mathcal{D}$-function ψ satisfying $\psi(r) < r$ for $r > 0$ such that

$$0 \leq [f(t,x_t) + Kx(t)] - [f(t,y_t) + Ky(t)] \leq \sigma\psi(x - y)$$

for all $t \in J$ and $x, y \in C([-r,b],E)$ with $x \geq y$, where $K > 0$ is a constant.

(ii) $f(t,x_t) + Kx(t)$ is bounded for all $t \in J$.

(H3) The function $h(t,x) : J \times C([-r,0],E) \to E$ is continuous, bounded and nondecreasing in x for all $t \in J$.

(H4) The function $g : C([-r,b],E) \to E$ is continuous, bounded and nondecreasing.

(H5) There exists a function $v \in C([-r,b],E)$ such that $D_t^q v(t) + Av(t) \leq f(t,v_t) + h(t,v_t)$ for $t \in J$ and $v_0(t) \leq \varphi(t) + g(v)$ for $t \in [-r,0]$.

By assumptions (H2)(ii), (H3), and (H4), there exist positive constants K_1, K_2 and K_3 such that

$$\|\hat{f}(t,x_t)\| \leq K_1, \qquad \|h(t,x_t)\| \leq K_2, \qquad \|g(x)\| \leq K_3, \tag{3.1}$$

where $\hat{f}(t,x_t) = f(t,x_t) + Kx(t)$.

Let $X = C([-r,b],E)$. Define an operator $A : X \to X$ by

$$(Ax)(t) = \begin{cases} 0, & t \in [-r,0], \\ \int_0^t (t-s)^{q-1}V(t-s)\hat{f}(s,x_s)\,ds, & t \in J, \end{cases} \tag{3.2}$$

where

$$V(t) = \int_0^\infty q\theta\eta_q(\theta)S(t^q\theta)\,d\theta, \quad 0 < q < 1,$$

where $S(t) = e^{-Kt}T(t)$ for all $t \geq 0$ and

$$\eta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q\big(\theta^{-\frac{1}{q}}\big),$$

$$\rho_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty).$$

Lemma 2 *Let the assumption* (H2) *hold. Then the operator A is nondecreasing, partially bounded, and it is a partially nonlinear $\mathcal{D}$-contraction in X.*

Proof Since $T(t)$ $(t \geq 0)$ is a positive C_0-semigroup, it follows that $\{V(t)\}_{t\geq 0}$ are positive operators. By the assumption (H2), it is easy to see that A is nondecreasing in X. By (3.1), for any $x \in X$, we have

$$\big\|(Ax)(t)\big\| \leq \int_0^t (t-s)^{q-1}\big\|V(t-s)\hat{f}(s,x_s)\big\|\,ds \leq \frac{Mb^qK_1}{\Gamma(q+1)}$$

for all $t \in J$. Taking the maximum over $t \in [-r,b]$ on both sides, we obtain

$$\|Ax\|_{[-r,b]} \leq \frac{Mb^qK_1}{\Gamma(q+1)}.$$

Therefore, A is bounded in X, which further implies that A is partially bounded in X.

We next prove that A is a partially nonlinear $\mathcal{D}$-contraction. For any comparable elements $x, y \in X$, by the assumption (H2), we have

$$\begin{aligned}\big\|(Ax)(t)-(Ay)(t)\big\| &\leq \int_0^t (t-s)^{q-1}\big\|V(t-s)\big[\hat{f}(s,x_s)-\hat{f}(s,y_s)\big]\big\|\,ds \\ &\leq \frac{M\sigma}{\Gamma(q)}\int_0^t (t-s)^{q-1}\psi\big(\|x_s-y_s\|\big)\,ds \\ &\leq \psi\big(\|x-y\|_{[-r,b]}\big)\end{aligned}$$

for all $t \in J$. Taking the maximum over $t \in [-r,b]$ on both sides, we obtain

$$\|Ax-Ay\|_{[-r,b]} \leq \psi\big(\|x-y\|_{[-r,b]}\big) \tag{3.3}$$

for all comparable elements $x, y \in X$. This implies that A is a partially nonlinear $\mathcal{D}$-contraction in X. This completes the proof. □

Define an operator $B : X \to X$ by

$$(Bx)(t) = \begin{cases}\varphi(t) + g(x), & t \in [-r,0],\\ U(t)(\varphi(0)+g(x)) + \int_0^t (t-s)^{q-1}V(t-s)h(s,x_s)\,ds, & t \in J,\end{cases} \tag{3.4}$$

where $V(t)$ is defined before and

$$U(t) = \int_0^\infty \eta_q(\theta)S\big(t^q\theta\big)\,d\theta, \quad 0 < q < 1,$$

where $S(\cdot)$, $\eta_q(\cdot)$, $\rho_q(\cdot)$ are defined before.

Lemma 3 *Let assumptions* (H1), (H3), *and* (H4) *hold. Then the operator B is nondecreasing, partially continuous and partially compact in X.*

Proof Since $T(t)$ $(t \geq 0)$ is a positive C_0-semigroup, it follows that $\{U(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\geq 0}$ are positive operators. Since the functions h and g are nondecreasing, it is easy to see that B is nondecreasing in X.

Take a sequence $\{x_n\}$ in a chain $C \subset X$ with $x_n \to x^*$ as $n \to +\infty$. Since the functions f and h are continuous in x for all $t \in J$, by the dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to+\infty}(Bx_n)(t) &= U(t)\Big(\varphi(0) + \lim_{n\to+\infty} g(x_n)\Big)\\ &\quad + \int_0^t (t-s)^{q-1} V(t-s) \lim_{n\to+\infty} h(s, x_{n,s})\,ds\\ &= \big(Bx^*\big)(t)\end{aligned}$$

for any $t \in [-r, b]$, where $x_{n,s} = (x_n)_s$. Therefore, we deduce that Bx_n converges to Bx^* pointwise on $[-r, b]$. On the other hand, for any $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}&\big\|(Bx_n)(t_2) - (Bx_n)(t_1)\big\|\\ &\quad\leq \big\|U(t_2)\big(\varphi(0) + g(x_n)\big) - U(t_1)\big(\varphi(0) + g(x_n)\big)\big\|\\ &\qquad + \left\|\int_0^{t_1} (t_2-s)^{q-1}\big[V(t_2-s) - V(t_1-s)\big]h(s, x_{n,s})\,ds\right\|\\ &\qquad + \left\|\int_0^{t_1} \big[(t_2-s)^{q-1} - (t_1-s)^{q-1}\big]V(t_1-s)h(s, x_{n,s})\,ds\right\|\\ &\qquad + \left\|\int_{t_1}^{t_2} (t_2-s)^{q-1} V(t_2-s)h(s, x_{n,s})\,ds\right\|\\ &\quad\leq \int_0^\infty \eta_q(\theta)e^{-Kt_1^q\theta}\big\|e^{-K\theta(t_2^q-t_1^q)}T\big(t_2^q\theta\big) - T\big(t_1^q\theta\big)\big\|\,d\theta\big(\|\varphi\|_{[-r,0]} + K_3\big)\\ &\qquad + qK_2\int_0^{t_1}(t_2-s)^{q-1}\\ &\qquad \times \int_0^\infty \theta\eta_q(\theta)\big\|e^{-K(t_2-s)^q\theta}T\big((t_2-s)^q\theta\big) - e^{-K(t_1-s)^q\theta}T\big((t_1-s)^q\theta\big)\big\|\,d\theta\,ds\\ &\qquad + \frac{MK_2}{\Gamma(q)}\int_0^{t_1}\big|(t_2-s)^{q-1} - (t_1-s)^{q-1}\big|\,ds\\ &\qquad + \frac{MK_2(t_2-t_1)^q}{\Gamma(q+1)}\\ &\quad\to 0\end{aligned}$$

as $t_2 - t_1 \to 0$ uniformly for all $n \in \mathbb{N}$. This implies that $Bx_n \to Bx^*$ uniformly. Hence the operator B is partially continuous in X.

We next prove that B is partially compact in X. We will prove that $B(C)$ is uniformly bounded and equi-continuous for any chain $C \subset X$. Using a similar method as above we can prove that, for any $t_1, t_2 \in J$ with $t_1 < t_2$, $\|y(t_2) - y(t_1)\| \to 0$ as $t_2 - t_1 \to 0$ uniformly for all $y \in B(C)$. This implies that $B(C)$ is equi-continuous for any chain $C \subset X$. For any

$y \in B(C)$, by assumptions (H3) and (H4), if $t \in [-r, 0]$, we have

$$\|y(t)\| \le \|\varphi\|_{[-r,0]} + K_3.$$

If $t \in J$, we have

$$\|y(t)\| \le M(\|\varphi\|_{[-r,0]} + K_3) + \frac{Mb^q K_2}{\Gamma(q+1)}.$$

Therefore,

$$\|y\|_{[-r,b]} \le (M+1)(\|\varphi\|_{[-r,0]} + K_3) + \frac{Mb^q K_2}{\Gamma(q+1)},$$

which implies that $B(C)$ is uniformly bounded in X. Therefore, B is partially compact in X. This completes the proof. □

Theorem 1 *Let* $(E, \le, \|\cdot\|)$ *be a partially ordered Banach space, whose positive cone* P *is normal, and let* $-A$ *generate a positive* C_0*-semigroup* $T(t)$ $(t \ge 0)$ *in* E. *Assume that the conditions* (H1)-(H5) *hold. Then the fractional nonlocal evolution equation* (1.1) *has a solution* $x^* \in C([-r, b], E)$ *and the sequence* $\{x_n\}$ *defined by* $x_0 = v$, $x_{n+1} = Ax_n + Bx_n$, $n = 0, 1, 2, \dots$, *converges monotonically to* x^*.

Proof We consider an auxiliary fractional evolution equation

$$\begin{cases} D_t^q x(t) + (A + KI)x(t) = \hat{f}(t, x_t) + h(t, x_t), & t \in J, \\ x_0(t) = \varphi(t) + g(x), & t \in [-r, 0], \end{cases} \tag{3.5}$$

where $\hat{f}(t, x_t) = f(t, x_t) + Kx(t)$. Clearly, the mild solution of fractional nonlocal evolution equation (1.1) is equivalent to the mild solution of fractional nonlocal evolution equation (3.5). We will use Lemma 1 to prove that the fractional nonlocal evolution equation (3.5) has a mild solutions in X. For this purpose, we define two operators $A, B : X \to X$ as in (3.2) and (3.4). Then by Lemmas 2 and 3, we deduce that A is nondecreasing, partially bounded, and it is a partially nonlinear $\mathcal{D}$-contraction, and B is nondecreasing, partially continuous, and partially compact. It remains to prove that v satisfies the inequality $v \le Av + Bv$.

By assumption (H5), we have

$$\begin{cases} D_t^q v(t) + (A + KI)v(t) \le \hat{f}(t, v_t) + h(t, v_t), & t \in J, \\ v_0(t) \le \varphi(t) + g(v), & t \in [-r, 0]. \end{cases}$$

Let $D^q v(t) + (A + KI)v(t) = G(t)$ for all $t \in J$. Then

$$\begin{aligned} v(t) &= U(t)v_0(0) + \int_0^t (t-s)^{q-1} V(t-s) G(s)\, ds \\ &\le U(t)(\varphi(0) + g(v)) + \int_0^t (t-s)^{q-1} V(t-s) [\hat{f}(t, v_t) + h(t, v_t)]\, ds \\ &= Av(t) + Bv(t) \end{aligned}$$

for all $t \in J$. For $t \in [-r, 0]$, $v_0(t) \le \varphi(t) + g(v) = (Bv)(t) = (Av)(t) + (Bv)(t)$. This implies that $v \le Av + Bv$. Hence, all conditions of the Lemma 1 are satisfied. By Lemma 1, the fractional nonlocal evolution equation (3.5) has a mild solution $x^* \in C([-r, b], E)$, which is also a mild solution of the fractional nonlocal evolution equation (1.1), and the sequence $\{x_n\}$ defined by $x_0 = v$, $x_{n+1} = Ax_n + Bx_n$, $n = 0, 1, 2, \ldots$, converges monotonically to x^*. This completes the proof. □

Remark 4 If $g \equiv 0$, $A \equiv 0$ and without delay in equation (1.1), Theorem 1 still extends the main result in [2], because the condition (H2) in this paper is much weaker than the condition (A_1) in [2] and the condition (H1) is not needed if $A \equiv 0$ in equation (1.1). On the other hand, in the case of $A \not\equiv 0$, we do not use the assumptions of the compactness and equi-continuity of the C_0-semigroup in this paper, just assume that the C_0-semigroup $T(t)$ $(t \ge 0)$ is positive and satisfies the condition (H1). Hence, our result also extends some existing results of evolution equation.

Remark 5 In this paper, the order relation in the partially ordered Banach space is induced by positive cone. By the closed property of positive cone, we obtain the regularity (see Definition 1) of the partially ordered Banach space. And by the normal property of positive cone, we see that the order relation and the norm in the partially ordered Banach space are compatible (see Definition 2). Hence, it is different from [1–3, 7].

4 Application

At the end of this paper, we give an example of the fractional parabolic equation to illustrate the applicability of the abstract result.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Consider the following nonlocal problem of the fractional parabolic equation:

$$\begin{cases} D_t^q x(t,y) - \Delta x(t,y) \\ \quad = f(t,y,x_t(\tau,y)) + h(t,y,x_t(\tau,y)), \quad t \in [0,b], \tau \in [-r,0], y \in \Omega, \\ x|_{\partial\Omega} = 0, \\ x_0(\tau,y) = \varphi(\tau) + g(x)(y), \quad \tau \in [-r,0], y \in \Omega, \end{cases} \tag{4.1}$$

where D_t^q denotes the Caputo fractional derivative of order $q \in (0,1)$, Δ is the Laplace operator, $b > 0$ and $r > 0$ are two constants. $x_t(\tau,y)$ is defined by $x_t(\tau,y) = (t+\tau, y)$ for all $t \in [0,b]$, $\tau \in [-r,0]$ and $y \in \Omega$.

Let $E = L^2(\Omega, \mathbb{R})$. Define an operator $A : D(A) \subset E \to E$ by

$$D(A) = \{x \in E : \Delta x \in E, x|_{\partial\Omega} = 0\}, \qquad Ax = -\Delta x.$$

Then, by [9], $-A$ generates an analysis semigroup $T(t)$ $(t \ge 0)$ in E and there exists $M > 0$ such that $\|T(t)\| \le M$. By the maximum principle of the parabolic type equation, $T(t)$ $(t \ge 0)$ is a positive C_0-semigroup in E. Hence we see that the condition (H1) holds.

Let $f(t,y,x_t(\tau,y)) = \frac{\sigma|x_t(\tau,y)|}{1+|x_t(\tau,y)|} - x(t,y)$, where $\sigma \in (0, \frac{\Gamma(q+1)}{Mb^q})$ is a constant and $\Gamma(\cdot)$ is the Gamma function. Then f is continuous, $\hat{f}(t,y,x_t(\tau,y)) = \frac{\sigma|x_t(\tau,y)|}{1+|x_t(\tau,y)|} \le \frac{\Gamma(q+1)}{Mb^q}$ and

$$0 \le \hat{f}(t,y,w) - \hat{f}(t,y,u) \le \sigma\left(\frac{|w-u|}{1+|w-u|}\right) = \sigma\psi\big(|w-u|\big) = \sigma\psi(w-u)$$

for all $t \in [0,b]$, $y \in \Omega$ and $w \geq u$, where $\psi(r) = \frac{r}{1+r}$. This implies that the condition (H2) holds.

Let

$$h\bigl(t,y,x_t(\tau,y)\bigr) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 + 2x_t(\tau,y), & \text{if } 0 < x < 2, \\ 5, & \text{if } x \geq 2, \end{cases} \qquad g(x)(y) = \begin{cases} 1, & \text{if } x \leq 0, \\ 1 + \frac{x(t,y)}{1+x(t,y)}, & \text{if } x > 0. \end{cases}$$

Then the conditions (H3) and (H4) hold with $K_2 = 5$, $K_3 = 2$.

Let λ_1 be an eigenvalue of the Laplace operator $-\Delta$ under the Dirichlet boundary condition $x|_{\partial\Omega} = 0$ and $e_1 \in E$ the corresponding eigenvector. Assume that $\lambda_1 e_1(y) \leq f(t,y,e_1(y)) + h(t,y,e_1(y))$, $t \in [0,b]$, $y \in \Omega$ and $e_1(y) \leq \varphi(t) + g(e_1)(y)$, $t \in [-r,0]$, $y \in \Omega$. Then the condition (H5) holds with $v = e_1(y)$ for all $y \in \Omega$.

Let

$$\begin{aligned} &x(t)(\cdot) = x(t,\cdot), \\ &f(t,x_t)(\cdot) = f\bigl(t,\cdot,x_t(\tau,\cdot)\bigr), \\ &h(t,x_t)(\cdot) = h\bigl(t,\cdot,x_t(\tau,\cdot)\bigr), \\ &g(x) = g(x)(\cdot). \end{aligned}$$

Then the fractional parabolic equation (4.1) can be rewritten into the abstract fractional nonlocal evolution equation (1.1). Hence by Theorem 1, the nonlocal problem of the fractional parabolic equation (4.1) has a solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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