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Integro-differential fractional boundary value problem on an unbounded domain

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Abstract
This paper is concerned with the existence of solutions for nonlinear fractional differential equations of Volterra type with nonlocal fractional integro-differential boundary conditions on an infinite interval. The results are obtained by using the Altman fixed point theorem. An example is presented in order to illustrate the main results.

MSC: 26A33; 34B15; 34B40

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1 Introduction

The study of fractional calculus is gaining more and more attention. Compared with classical integer-order models, fractional-order models can describe reality more accurately, which has been shown recently in a variety of fields such as physics, chemistry, biology, economics, signal and image processing, control, porous media, aerodynamics, and so on [1–12].

In addition, scientists have found that many mathematics models can be reduced to the nonlocal problems with integral boundary conditions, such as the models on underground water flow, chemical engineering, plasma physics, and thermo-elasticity. For more information, see the excellent surveys by Corduneanu [13] and Agarwal and O'Regan [14] and some recent papers [15–28].

In the past decades, nonlocal boundary value problems of fractional differential equations on finite/infinite interval have been extensively investigated; see, for instance, [29–45]. However, to the best of our knowledge, very little is known regarding integro-differential fractional boundary value problem on an *infinite interval*.

Based on the reason mentioned, in this paper, we consider the following integro-differential fractional boundary value problem for nonlinear fractional differential equations of Volterra type on an *infinite interval*

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t),Tu(t))=0, \quad 3<\alpha\leq 4, \\ u(0)=u'(0)=u''(0)=0, \qquad D^{\alpha-1}u(\infty)=\xi I^{\beta}u(\eta), \quad \beta>0, \end{cases} \tag{1.1}$$

where $t \in J = [0, +\infty)$, $f \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $\xi \in \mathbb{R}$, $\eta \in J$, D^α is the Riemann-Liouville fractional derivative of order α, I^β is the Riemann-Liouville fractional integral of order β, and $(Tu)(t) = \int_0^t k(t,s)u(s)\,ds$ with $k(t,s) \in C[D, \mathbb{R}]$, $D = \{(t,s) \in \mathbb{R}^2 \mid 0 \le s \le t\}$.

Define the space

$$X = \left\{ u \in C(J, \mathbb{R}) : \sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}} < +\infty \right\}$$

equipped with the norm

$$\|u\|_X = \sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}}.$$

It is obvious that X is a Banach space.

2 Preliminaries

For the convenience of the reader, in this section, we first present some useful definitions and theorems.

Definition 2.1 ([4]) The Riemann-Liouville fractional derivative of order δ for a continuous function f is defined by

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\delta-1} f(s)\,ds, \quad n = [\delta] + 1,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([4]) The Riemann-Liouville fractional integral of order δ for a function f is defined as

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\,ds, \quad \delta > 0,$$

provided that such an integral exists.

Theorem 2.1 (Altman theorem [46]) *Let Ω be an open bounded subset of a Banach space E with $0 \in \Omega$, and $T : \overline{\Omega} \to E$ be a completely continuous operator. Then T has a fixed point in $\overline{\Omega}$, provided that*

$$\|Tx - x\|^2 \ge \|Tx\|^2 - \|x\|^2, \quad \forall x \in \partial\Omega.$$

Theorem 2.2 ([47]) *Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following conditions hold*:

(i) *for any $u(t) \in U$, $\frac{u(t)}{1+t^{\alpha-1}}$ is equicontinuous on any compact interval of J*;

(ii) *for any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that*

$$\left| \frac{u(t_1)}{1 + t_1^{\alpha-1}} - \frac{u(t_2)}{1 + t_2^{\alpha-1}} \right| < \varepsilon$$

for any $t_1, t_2 \ge T$ and $u \in U$.

Before proving our main result, we list the following assumptions:

(H_1) $\xi \geq 0$, $\Gamma(\alpha+\beta) > \xi\eta^{\alpha+\beta-1}$.

(H_2) There exists a constant k^* such that

$$k^* = \sup_{t\in J}\int_0^t \left|k(t,s)\right|\left(1+s^{\alpha-1}\right)ds < \infty.$$

(H_3) There exist nonnegative functions $a(t)$, $b(t)$, $c(t)$ defined on $[0,\infty)$ and constants $p, q \geq 0$ such that

$$\left|f(t,u,v)\right| \leq a(t) + b(t)|u|^p + c(t)|v|^q$$

and

$$\int_0^{+\infty} a(t)\,dt = a^* < +\infty,$$

$$\int_0^{+\infty} b(t)\left(1+t^{\alpha-1}\right)^p dt = b^* < +\infty,$$

$$\int_0^{+\infty} c(t)\,dt = c^* < +\infty.$$

3 Related lemmas

Firstly, we give an explicit expression of the Green's function related to the associated linear problem.

Lemma 3.1 *Let $h \in C([0,+\infty))$ with $\int_0^\infty h(s)\,ds < \infty$. If $\Gamma(\alpha+\beta) \neq \xi\eta^{\alpha+\beta-1}$, then the fractional integral boundary value problem*

$$\begin{cases} D^\alpha u(t) + h(t) = 0, \\ u(0) = u'(0) = u''(0) = 0, \qquad D^{\alpha-1}u(\infty) = \xi I^\beta u(\eta), \quad \beta > 0, \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t,s)h(s)\,ds,$$

where

$$G(t,s) = \frac{1}{\Delta}\begin{cases} [\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}]t^{\alpha-1} \\ \quad - [\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}](t-s)^{\alpha-1}, & s \leq t, s \leq \eta, \\ [\Gamma(\alpha+\beta) - \xi(\eta-s)^{\alpha+\beta-1}]t^{\alpha-1}, & 0 \leq t \leq s \leq \eta, \\ \Gamma(\alpha+\beta)[t^{\alpha-1} - (t-s)^{\alpha-1}] + \xi\eta^{\alpha+\beta-1}(t-s)^{\alpha-1}, & 0 \leq \eta \leq s \leq t, \\ \Gamma(\alpha+\beta)t^{\alpha-1}, & s \geq t, s \geq \eta, \end{cases} \tag{3.2}$$

and

$$\Delta = \Gamma(\alpha)\left[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}\right].$$

Proof By (3.2) we have

$$\begin{aligned} u(t) &= \int_0^{+\infty} G(t,s)h(s)\,ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}]}\biggl[\int_0^\infty h(s)\,ds \\ &\quad - \int_0^\eta \frac{\xi(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds\biggr]. \end{aligned} \tag{3.3}$$

Then, it is easy to get that $u(0) = u'(0) = u''(0) = 0$.

By (3.3) we have

$$\begin{aligned} D^{\alpha-1}u(t) &= D^{\alpha-1}\biggl(\int_0^{+\infty} G(t,s)h(s)\,ds\biggr) \\ &= D^{\alpha-1}\biggl(-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}]}\biggl[\int_0^\infty h(s)\,ds \\ &\quad - \int_0^\eta \frac{\xi(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds\biggr]\biggr) \\ &= -\int_0^t h(s)\,ds + \frac{\Gamma(\alpha+\beta)}{[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}]}\biggl[\int_0^\infty h(s)\,ds \\ &\quad - \int_0^\eta \frac{\xi(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds\biggr] \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} I^\beta u(t) &= I^\beta\biggl(\int_0^{+\infty} G(t,s)h(s)\,ds\biggr) \\ &= I^\beta\biggl(-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}]}\biggl[\int_0^\infty h(s)\,ds \\ &\quad - \int_0^\eta \frac{\xi(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds\biggr]\biggr) \\ &= -\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds + \frac{t^{\alpha+\beta-1}}{[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}]}\biggl[\int_0^\infty h(s)\,ds \\ &\quad - \int_0^\eta \frac{\xi(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds\biggr]. \end{aligned} \tag{3.5}$$

Thus, we can get the relation $D^{\alpha-1}u(\infty) = \xi I^\beta u(\eta)$.

Finally, applying (3.3), by a simple deduction it follows

$$\begin{aligned} D^\alpha u(t) &= D^\alpha\biggl(-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}]}\biggl[\int_0^\infty h(s)\,ds \\ &\quad - \int_0^\eta \frac{\xi(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds\biggr]\biggr) \\ &= -h(t). \end{aligned} \tag{3.6}$$

Thus, the proof is complete. □

Through a careful computation, it is easy to obtain the following remark, so we omit its proof.

Remark 3.1 For $(s,t) \in J \times J$, if condition (H_1) holds, then we have

$$0 \leq \frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}]} := L. \tag{3.7}$$

Lemma 3.2 *If conditions* (H_2) *and* (H_3) *are satisfied, then we have*

$$\int_0^{+\infty} \bigl|f\bigl(s,u(s),Tu(s)\bigr)\bigr|\,ds \leq a^* + b^*\|u\|_X^p + c^*\bigl(k^*\bigr)^q\|u\|_X^q, \quad \forall u \in X. \tag{3.8}$$

Proof For all $u \in X$, by conditions (H_2) and (H_3) we have

$$\begin{aligned}
&\int_0^{+\infty} \bigl|f\bigl(s,u(s),Tu(s)\bigr)\bigr|\,ds \\
&\quad\leq \int_0^{+\infty} \bigl[a(s) + b(s)\bigl|u(s)\bigr|^p + c(s)\bigl|Tu(s)\bigr|^q\bigr]\,ds \\
&\quad\leq a^* + \int_0^{+\infty} b(s)\bigl(1+s^{\alpha-1}\bigr)^p \frac{|u(s)|^p}{(1+s^{\alpha-1})^p}\,ds + \int_0^{+\infty} c(s)\biggl[\int_0^s \bigl|K(s,r)u(r)\bigr|\,dr\biggr]^q\,ds \\
&\quad\leq a^* + b^*\|u\|_X^p + \int_0^{+\infty} c(s)\biggl[\int_0^s \bigl|K(s,r)\bigr|\bigl(1+r^{\alpha-1}\bigr)\frac{|u(r)|}{(1+r^{\alpha-1})}\,dr\biggr]^q\,ds \\
&\quad\leq a^* + b^*\|u\|_X^p + \int_0^{+\infty} c(s)\bigl(k^*\bigr)^q\|u\|_X^q\,ds \\
&\quad\leq a^* + b^*\|u\|_X^p + c^*\bigl(k^*\bigr)^q\|u\|_X^q.
\end{aligned} \tag{3.9}$$

□

4 Main results

Define the operator Q by

$$Qu(t) = \int_0^{+\infty} G(t,s)f\bigl(s,u(s),Tu(s)\bigr)\,ds. \tag{4.1}$$

Applying Lemma 3.1 with $h(t) = f(t,u(t),Tu(t))$, problem (1.1) reduces to a fixed point problem $u = Qu$, where Q is given by (4.1). Thus, problem (1.1) has a solution if and only if the operator Q has a fixed point.

Lemma 4.1 *Assume that conditions* (H_1)-(H_3) *are satisfied. Then* $Q: X \to X$ *is completely continuous.*

Proof Firstly, the operator $Q: X \to X$ is relatively compact.

(1) Let Ω be any bounded subset of X. Then there exists a constant $M > 0$ such that $\|u\|_X \leq M$. By Lemma 3.2 and Remark 3.1 we have

$$\begin{aligned}
\|Qu\|_X &= \sup_{t\in J}\int_0^{\infty} \frac{G(t,s)}{1+t^{\alpha-1}}\bigl|f\bigl(s,u(s),Tu(s)\bigr)\bigr|\,ds \\
&\leq L\int_0^{+\infty} \bigl|f\bigl(s,u(s),Tu(s)\bigr)\bigr|\,ds
\end{aligned}$$

$$\begin{aligned}&\leq L\left[a^* + b^*\|u\|_X^p + c^*\left(k^*\right)^q\|u\|_X^q\right]\\&\leq L\left[a^* + M^p b^* + M^q c^*\left(k^*\right)^q\right],\end{aligned} \tag{4.2}$$

which implies that $T\Omega$ is uniformly bounded.

(2) We prove that Q is equicontinuous.

(I) Let $I \subset J$ be any compact interval. Let Ω be any bounded subset of X. For all $t_1, t_2 \in I$, $t_2 > t_1$, and $u \in \Omega$, we have

$$\begin{aligned}\left|\frac{Qu(t_2)}{1+t_2^{\alpha-1}} - \frac{Qu(t_1)}{1+t_1^{\alpha-1}}\right| &= \left|\int_0^\infty \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right) f\left(s,u(s),Tu(s)\right)ds\right|\\&\leq \int_0^\infty \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| \left|f\left(s,u(s),Tu(s)\right)\right|ds.\end{aligned} \tag{4.3}$$

Since $G(t,s) \in C(J \times J)$, $\frac{G(t,s)}{1+t^{\alpha-1}}$ is uniformly continuous on any compact set $I \times I$. Note that this function only depends on t for $s \geq t$, so it is uniformly continuous on $I \times (J \backslash I)$. Thus, for all $s \in J$ and $t_1, t_2 \in I$, we have

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that if } |t_1 - t_2| < \delta, \text{then } \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| < \varepsilon. \tag{4.4}$$

By Lemma 3.2, for all $u \in \Omega$, we have

$$\int_0^\infty \left|f\left(s,u(s),Tu(s)\right)\right|ds < \infty, \quad \forall u \in \Omega. \tag{4.5}$$

This, together with (4.3) and (4.4), implies that $Q\Omega$ is equicontinuous on I.

(II) We have

$$\lim_{t\to\infty} \frac{G(t,s)}{1+t^{\alpha-1}} = \frac{1}{\Gamma(\alpha)[\Gamma(\alpha+\beta) - \xi\eta^{\alpha+\beta-1}]} \begin{cases} \xi\eta^{\alpha+\beta-1} - \xi(\eta-s)^{\alpha+\beta-1}, & 0 \leq s \leq \eta, \\ \xi\eta^{\alpha+\beta-1}, & \eta \leq s. \end{cases} \tag{4.6}$$

From this it is easy to verify that, for any $\varepsilon > 0$, there exists a constant $T' = T'(\varepsilon) > 0$ such that

$$\left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| < \varepsilon$$

for any $t_1, t_2 \geq T'$ and $s \in J$. Combining this with Lemma 3.2 and (4.3), we get that the same property holds for $Q\Omega$, uniformly on $u \in \Omega$. Hence, Q is equiconvergent at ∞.

Therefore, by Theorem 2.2 we know that Q is relatively compact on J.

Next, we show that $Q : X \to X$ is continuous.

Let $u_n, u \in X$ be such that $u_n \to u$ $(n \to \infty)$. Then, $\|u_n\|_X < \infty$ and $\|u\|_X < \infty$. By Lemma 3.2 we have

$$\begin{aligned}\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f\left(s,u_n(s),Tu_n(s)\right)ds &\leq L\int_0^\infty \left|f\left(s,u_n(s),Tu_n(s)\right)\right|ds\\&\leq L\left[a^* + b^*\|u_n\|_X^p + c^*\left(k^*\right)^q\|u_n\|_X^q\right] < \infty,\end{aligned} \tag{4.7}$$

where L is defined in (3.7).

By the Lebesgue dominated convergence theorem and continuity of f we get

$$\lim_{n\to\infty}\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f\big(s,u_n(s),Tu_n(s)\big)\,ds = \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} f\big(s,u(s),Tu(s)\big)\,ds.$$

Hence, we have

$$\begin{aligned}\|Qu_n - Qu\|_X &= \sup_{t\in J}\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}\big|f\big(s,u_n(s),Tu_n(s)\big) - f\big(s,u(s),Tu(s)\big)\big|\,ds\\ &\to 0 \quad (n\to\infty),\end{aligned} \tag{4.8}$$

which shows that Q is continuous. Therefore, $Q: X \to X$ is completely continuous. This completes the proof. □

Next, we give several existence results for integro-differential fractional boundary value problem (1.1).

According to the range of p and q, we have the following theorems.

Theorem 4.1 *Assume that conditions* (H_1)-(H_3) *are satisfied. If* $0 \le p, q < 1$, *then problem* (1.1) *has at least one solution.*

Proof Let us choose

$$R \ge \max\Big\{3La^*, \big(3Lb^*\big)^{\frac{1}{1-p}}, \big(3Lc^*\big(k^*\big)^q\big)^{\frac{1}{1-q}}\Big\}$$

and define $U = \{u \in X, \|u\|_X < R\}$. In view of Theorem 2.1, we just need to show that

$$\|Qu\|_X \le \|u\|_X, \quad \forall u \in \partial U. \tag{4.9}$$

For any $u \in \partial U$, by Lemma 3.2 and Remark 3.1 we have

$$\begin{aligned}\|Qu\|_X &= \sup_{t\in J}\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}\big|f\big(s,u(s),Tu(s)\big)\big|\,ds\\ &\le L\int_0^{+\infty}\big|f\big(s,u(s),Tu(s)\big)\big|\,ds\\ &\le L\big[a^* + b^*\|u\|_X^p + c^*\big(k^*\big)^q\|u\|_X^q\big]\\ &\le L\big[a^* + R^p b^* + R^q c^*\big(k^*\big)^q\big]\\ &\le L\left(\frac{R}{3L}+\frac{R}{3L}+\frac{R}{3L}\right)\\ &= R.\end{aligned} \tag{4.10}$$

Thus, $QU \subset U$ and $\|Qu\|_X \le \|u\|_X$ for all $u \in \partial U$, which completes the proof. □

Theorem 4.2 *Assume that conditions* (H_1)-(H_3) *are satisfied. Then problem* (1.1) *has at least one solution, provided that one of the following eight conditions holds*:

Case 1. $p = q = 1$, $L(b^* + c^*k^*) < 1$;

Case 2. $0 \le p < 1$, $q = 1$, $2Lc^*k^* < 1$;
Case 3. $p > 1$, $q = 1$, $2La^*(1 - 2Lc^*k^*)^{-1} \le (2Lb^*)^{\frac{1}{1-p}}$, $2Lc^*k^* < 1$;
Case 4. $p = 1$, $0 \le q < 1$, $2Lb^* < 1$;
Case 5. $p = 1$, $q > 1$, $2La^*(1 - 2Lb^*)^{-1} \le (2Lc^*(k^*)^q)^{\frac{1}{1-q}}$, $2Lb^* < 1$;
Case 6. $p, q > 1$, $3La^* \le \min\{(3Lb^*)^{\frac{1}{1-p}}, (3Lc^*(k^*)^q)^{\frac{1}{1-q}}\}$;
Case 7. $0 \le p < 1$, $q > 1$, $\max\{3La^*, (3Lb^*)^{\frac{1}{1-p}}\} \le (3Lc^*(k^*)^q)^{\frac{1}{1-q}}$;
Case 8. $p > 1$, $0 \le q < 1$, $\max\{3La^*, (3Lc^*(k^*)^q)^{\frac{1}{1-q}}\} \le (3Lb^*)^{\frac{1}{1-p}}$;

here L is defined in (3.7).

Proof The proofs of Cases 1-5 are similar, so we only give the proof of Case 1.

For $p = q = 1$, let us take

$$R \ge \frac{La^*}{1 - L(b^* + c^*k^*)}$$

and define $U = \{u \in X, \|u\|_X < R\}$.

For any $u \in \partial U$, by Lemma 3.2 and Remark 3.1 we have

$$\begin{aligned} \|Qu\|_X &= \sup_{t\in J} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} \bigl|f\bigl(s,u(s),Tu(s)\bigr)\bigr|\,ds \\ &\le L \int_0^{+\infty} \bigl|f\bigl(s,u(s),Tu(s)\bigr)\bigr|\,ds \\ &\le L\bigl[a^* + b^*\|u\|_X + c^*k^*\|u\|_X\bigr] \\ &\le L\bigl[a^* + \bigl(b^* + c^*k^*\bigr)R\bigr] \\ &\le R. \end{aligned} \tag{4.11}$$

Thus, $QU \subset U$ and $\|Qu\|_X \le \|u\|_X$ for all $u \in \partial U$. In view of Theorem 2.1, we get that problem (1.1) has at least one solution $u(t)$ satisfying

$$0 \le \frac{|u(t)|}{1+t^{\alpha-1}} \le R \quad \text{for } t \in J.$$

The proofs of Cases 6-8 are similar to that of Theorem 4.1, so we omit it. This completes the proof. □

5 Example

Example 5.1 Take $\alpha = 3.5$ and $\beta = 1.5$. We consider the following integro-differential fractional boundary value problem for nonlinear fractional differential equations of Volterra type on an unbounded domain:

$$\begin{cases} D^{3.5}u(t) + \frac{\ln(1+t)}{(1+t^2)^2} + \frac{e^{-t}|u(t)|^p}{(1+t^{2.5})^p} + \frac{1}{(3+t)^2}|\int_0^t \frac{e^{-t}\cos(t^2-s)}{(1+s^{2.5})}u(s)\,ds|^q = 0, \quad t \in [0,+\infty), \\ u(0) = u'(0) = u''(0) = 0, \qquad D^{2.5}u(\infty) = \xi I^{1.5}u(\eta), \end{cases} \tag{5.1}$$

where $f(t,u,v) = \frac{\ln(1+t)}{(1+t^2)^2} + \frac{e^{-t}|u|^p}{(1+t^{2.5})^p} + \frac{1}{(3+t)^2}|v|^q$, $0 \le p, q \le 1$, and ξ, η satisfy $0 \le \xi\eta^4 < 24$. For example, we take $\xi = 8$, $\eta = 1$.

Firstly, it is obvious that $\Gamma(\alpha + \beta) = 24$ and $\xi\eta^{\alpha+\beta-1} = \xi\eta^4 = 8$. Then (H_1) holds.

Secondly, we have

$$\begin{aligned}\big|f(t,u,v)\big| &= \frac{\ln(1+t)}{(1+t^2)^2} + \frac{e^{-t}|u|^p}{(1+t^{2.5})^p} + \frac{1}{(3+t)^2}|v|^q \\ &\le \frac{t}{(1+t^2)^2} + \frac{e^{-t}|u|^p}{(1+t^{2.5})^p} + \frac{1}{(3+t)^2}|v|^q.\end{aligned}$$

Take $a(t) = \frac{t}{(1+t^2)^2}$, $b(t) = \frac{e^{-t}}{(1+t^{2.5})^p}$, $c(t) = \frac{1}{(3+t)^2}$. By a direct computation we can obtain

$$\begin{aligned}a^* &= \int_0^{+\infty} a(t)\,dt = \frac{1}{2} < +\infty, \\ b^* &= \int_0^{+\infty} b(t)\big(1+t^{\alpha-1}\big)^p\,dt = 1 < +\infty, \\ c^* &= \int_0^{+\infty} c(t)\,dt = 1 < +\infty,\end{aligned}$$

which implies that (H_3) holds.

Noting that $k(t,s) = \frac{e^{-t}\cos(t^2-s)}{(1+s^{2.5})}$, we have

$$k^* = \sup_{t\in J}\int_0^t \big|k(t,s)\big|\big(1+s^{\alpha-1}\big)\,ds \le 1.$$

Thus, conditions (H_1)-(H_3) hold.

Therefore, for the case $0 \le p,q < 1$, by Theorem 4.1 the nonlinear fractional differential equation (5.1) has at least one solution.

In addition, since $L = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\eta^{\alpha+\beta-1}]} = \frac{4}{5\sqrt{\pi}}$, Cases 1, 2, and 4 of Theorem 4.2 hold, Thus, all conditions of Theorem 4.2 are satisfied.

To sum up our arguments, for $0 \le p,q \le 1$, the integro-differential fractional boundary value problem (5.1) has at least one solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors have equal contributions. Both authors read and approved the final manuscript.

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