Hristova and Terzieva *Advances in Difference Equations* (2016) 2016:322
DOI 10.1186/s13662-016-1045-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Lipschitz stability of differential equations with non-instantaneous impulses

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Abstract
Nonlinear differential equations with non-instantaneous impulses are studied. The impulses start abruptly at some points and their actions continue on given finite intervals. We pursue the study of Lipschitz stability using Lyapunov functions. Some sufficient conditions for Lipschitz stability, uniform Lipschitz stability, and uniform global Lipschitz stability are obtained. Examples are given to illustrate the results.

MSC: 34A37; 34D20

Keywords: non-instantaneous impulses; differential equations; Lipschitz stability

1 Introduction

The problems of stability of solutions of differential equations via Lyapunov functions have been successfully investigated in the past. One type of stability, very useful in real world problems, is the so-called Lipschitz stability. Dannan and Elaydi [1] introduced the notion of Lipschitz stability for ordinary differential equations. As is mentioned in [1] this type of stability is important only for nonlinear problems, since it coincides with uniform stability in linear systems.

There are a few different real life processes and phenomena that are characterized by rapid changes in their state. We will emphasize two main types of such kind of changes:

- The duration of these changes is relatively short compared to the overall duration of the whole process and the changes turn out to be irrelevant to the development of the studied process. The mathematical models in such cases can be adequately created with the help of impulsive equations (see, for example, [2–5], the monographs [6, 7] and the references therein).
- The duration of these changes is not negligible short, *i.e.* these changes start impulsively at arbitrary fixed points and remain active on finite initially time intervals. The model of this situation is the non-instantaneous impulsive differential equation. Hernandez and O'Regan [8] introduced this new class of differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions. We refer the reader for some recent results such as existence to [9, 10], to stability [11–16], to periodic boundary value problems [17, 18].

  Some examples of such processes can be found in physics, biology, population dynamics, ecology, pharmacokinetics, and others.

In this paper Lipschitz stability of solutions of nonlinear non-instantaneous impulsive differential equations is defined and studied. Several sufficient conditions for Lipschitz stability, uniform Lipschitz stability, and global uniform Lipschitz stability are obtained. Some examples illustrating the results are given. Note that non-instantaneous impulsive differential equations are natural generalizations of impulsive differential equations and some of the obtained sufficient conditions are a generalization of some results in [19]. Also, Lipschitz stability of impulsive functional-differential equations is studied in [20].

2 Preliminaries

In this paper we assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $0 < s_0 < t_i < s_i < t_{i+1} < s_{i+1}$, $i = 1, 2, \dots$, and $\lim_{k\to\infty} t_k = \infty$.

Let $t_0 \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1})$ be a given arbitrary point. Without loss of generality we will assume that $t_0 \in [0, s_0)$.

Consider the initial value problem (IVP) for the system of non-instantaneous impulsive differential equation (NIDE)

$$\begin{aligned} &x' = f(t,x) \quad \text{for } t \in \bigcup_{k=0}^{\infty}(t_k, s_k], \\ &x(t) = \Psi_k\bigl(t, x(t), x(s_k - 0)\bigr) \quad \text{for } t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots, \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x, x_0 \in \mathbb{R}^n, f : \bigcup_{k=0}^{\infty}[t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\Psi_k : [s_k, t_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ $(k = 1, 2, 3, \dots)$.

Remark 1 The functions Ψ_k are called impulsive functions and the intervals $(s_k, t_{k+1}]$, $k = 0, 1, 2, \dots$ are called intervals of non-instantaneous impulses.

Remark 2 In the partial case $s_k = t_{k+1}$, $k = 0, 1, 2, \dots$ each interval of non-instantaneous impulses is reduced to a point, and the problem (1) is reduced to an IVP for an impulsive differential equation with points of jump t_k and impulsive condition $x(t_k + 0) = I_k(x(t_k - 0)) \equiv \Psi_k(t_k, x(t_k - 0), x(t_k - 0))$.

The solution $x(t; t_0, x_0)$ of IVP for NIDE (1) is given by

$$x(t; t_0, x_0) = \begin{cases} X_k(t) & \text{for } t \in (t_k, s_k], k = 0, 1, 2, \dots, \\ \Psi_k(t, x(t; t_0, x_0), X_k(s_k - 0)) & \text{for } t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots, \end{cases} \tag{2}$$

where

- for any $k = 0, 1, 2, \dots$ the function $X_k(t)$, $t \in [t_k, s_k]$ is a solution of the initial value problem for ODE $x' = f(t,x)$, $x(t_k) = x(t_k; t_0, x_0)$, respectively;
- on any interval $(s_k, t_{k+1}]$, $k = 0, 1, 2, \dots$ the solution $x(t; t_0, x_0)$ satisfies the algebraic equation $x(t; t_0, x_0) = \Psi_k(t, x(t; t_0, x_0), X_k(t_1 - 0))$.

Let $J \subset \mathbb{R}^+$ be a given interval. Introduce the following classes of functions:

$$\begin{aligned} NPC(J) = \Biggl\{ u : J \to \mathbb{R}^n : u \in C\Biggl(J \Big/ \bigcup_{k=0}^{\infty}\{s_k\}, \mathbb{R}^n\Biggr) : \\ u(s_k) = u(s_k - 0) = \lim_{t\uparrow s_k} u(t) < \infty, u(s_k + 0) = \lim_{t\downarrow s_k} u(t) < \infty, k : s_k \in J \Biggr\}, \end{aligned}$$

$$NPC^1(J) = \left\{ u : J \to \mathbb{R}^n : u \in NPC(J), u \in C^1\left(J \Big/ \bigcup_{k=0}^{\infty}\{s_k\}, \mathbb{R}^n \right) : \right.$$
$$\left. u'(s_k) = u'(s_k - 0) = \lim_{t \uparrow s_k} u'(t) < \infty, k : s_k \in J \right\}.$$

Remark 3 According to the above description any solution of (1) might have a discontinuity at any point s_k, $k = 0, 1, 2, \ldots$.

Now we will illustrate the influence of the impulsive condition on the behavior of the solution.

Example 1 Consider the IVP for the NIDE

$$\begin{aligned} & x' = -x \quad \text{for } t \in \bigcup_{k=0}^{\infty}(2k, 2k+1], \\ & x(t) = \Psi_k\bigl(t, x(t), x(2k+1-0)\bigr) \quad \text{for } t \in (2k+1, 2k+2], k = 0, 1, 2, \ldots, \\ & x(0) = x_0. \end{aligned} \tag{3}$$

Case 1. Let $\Psi_k(t, x, y) = 2y$. Then the impulsive condition is $x(t) = 2x(2k + 1 - 0)$ and the solution of (3) is

$$x(t; 0, x_0) = \begin{cases} 2^k x_0 e^{-t+k} & \text{for } t \in (2k, 2k+1], k = 0, 1, 2, \ldots, \\ 2^k x_0 e^{-k} & \text{for } t \in (2k-1, 2k], k = 1, 2, \ldots. \end{cases} \tag{4}$$

The graph of the solutions of (3) on $[0, 8]$ with various initial values $x_0 = 0.5$, $x_0 = 1$, $x_0 = 1.5$ is given in Figure 1.

Case 2. Let $\Psi_k(t, x, y) = 2x$. Then the impulsive condition is $x(t) = 2x(t)$ for $t \in (2k-1, 2k]$, $k = 1, 2, \ldots$ with a solution $x(t) \equiv 0$, $t \in (2k-1, 2k]$, $k = 1, 2, \ldots$. The solution of (3) with the new impulsive condition is

$$x(t; 0, x_0) = \begin{cases} x_0 e^{-t} & \text{for } t \in (0, 1], \\ 0 & \text{for } t \geq 1. \end{cases} \tag{5}$$

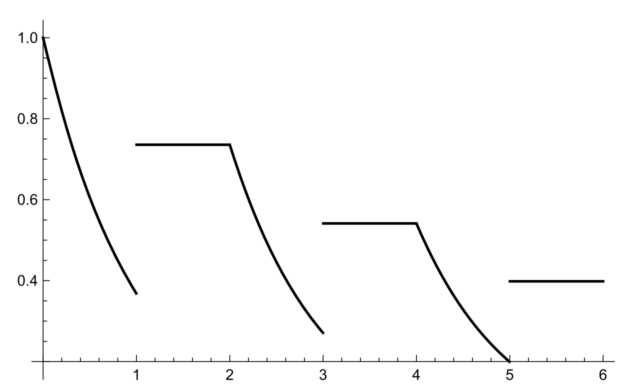

Figure 1 Graphs of solutions of (3) for various x_0.

Case 3. Let $\Psi_k(t,x,y) = t - x$. Then the impulsive condition is $x(t) = t - x(t)$ for $t \in (2k-1, 2k]$, $k = 1,2,\dots$ with a solution $x(t) = 0.5t$, $t \in (2k-1,2k]$, $k = 1,2,\dots$. The solution of (3) with the new impulsive condition is

$$x(t;0,x_0) = \begin{cases} x_0 e^{-t} & \text{for } t \in (0,1], \\ 0.5t & \text{for } t \in (2k-1,2k], k = 1,2,\dots, \\ ke^{-t+2k} & \text{for } t \in (2k,2k+1], k = 1,2,\dots. \end{cases} \tag{6}$$

Therefore, if any impulsive function $\Psi_k(t,x,y)$ do not depend on y the solution does not depend on the initial value x_0 for $t > s_k$ (see Cases 2 and 3).

Remark 4 Note in some papers (see, for example [14]) the functions of non-instantaneous impulses are given in the form $g_k(t,x(t))$, *i.e.* they do not depend on the value of the solution before the jump $x(s_k - 0)$. Then the solution will depend on the initial value only on the interval $[t_0,s_0]$. Then the meaning of the stability as well dependence of the solution on the initial value is lost.

Introduce the following condition.

(H1) The function $f \in C(\bigcup_{k=0}^{\infty}[t_k,s_k] \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t,0) \equiv 0$.

(H2) For any $k = 0,1,2,\dots$ and any fixed $t \in [s_k,t_{k+1}]$ and $y \in \mathbb{R}^n$ the algebraic equation $x = \Psi_k(t,x,y)$ has unique solution $x = \phi_k(t,y)$ with $\phi_k \in C([s_k,t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$ and $\phi_k(t,0) \equiv 0$.

If condition (H2) is satisfied then IVP for NIDE (1) could be written in the form

$$\begin{aligned} & x' = f(t,x) \quad \text{for } t \in \bigcup_{k=0}^{\infty}(t_k,s_k], \\ & x(t) = \phi_k\big(t,x(t_k - 0)\big) \quad \text{for } t \in (s_k,t_{k+1}], k = 0,1,2,\dots, \\ & x(t_0) = x_0. \end{aligned} \tag{7}$$

Let $J \subset \mathbb{R}_+$, $0 \in J$, $\rho > 0$. Introduce the following sets:

$$\begin{aligned} M(J) = \big\{ & a \in C\big[J,\mathbb{R}^+\big] : a(0) = 0, a(r) \text{ is strictly increasing in } J, \text{and} \\ & a^{-1}(\alpha r) \le r q_a(\alpha) \text{ for some function } q_a : q_a(\alpha) \ge 1, \text{if } \alpha \ge 1\big\}, \\ K(J) = \big\{ & a \in C\big[J,\mathbb{R}^+\big] : a(0) = 0, a(r) \text{ is strictly increasing in } J, \text{and} \\ & a(r) \le K_a r \text{ for some constant } K_a > 0\big\}, \\ S_\rho = \big\{ & x \in \mathbb{R}^n : \|x\| \le \rho\big\}. \end{aligned}$$

Remark 5 The function $a(u) = K_1 u$, $K_1 \in (0,1]$ is from the class $K(\mathbb{R}_+)$ with $q(u) \equiv u$. The function $a(u) = K_1 u^2$, $K_1 > 0$ is from the class $M([0,1])$.

We will use the class Λ of Lyapunov-like functions, defined and used for impulsive differential equations in [7].

Definition 1 Let $J \subset \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$, belongs to the class $\Lambda(J,\Delta)$ if:

- the function $V(t,x)$ is a continuous on $J/\{t_k \in J\} \times \Delta$ and it is locally Lipschitz with respect to its second argument;
- for each $s_k \in J$ and $x \in \Delta$ there exist finite limits

$$V(s_k,x) = V(s_k - 0,x) = \lim_{t\uparrow s_k} V(t,x) \quad \text{and} \quad V(s_k + 0,x) = \lim_{t\downarrow s_k} V(t,x).$$

For any $t \in (t_k,s_k)$, $k = 0,1,2,\dots$, we define the Dini derivative of the function $V(t,x) \in \Lambda(J,\Delta)$ by

$$D_+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}\bigl\{V(t,x) - V\bigl(t-h, x-hf(t,x)\bigr)\bigr\},$$

where $x \in \Delta$, and for any $t \in (t_k,s_k)$ there exists $h_t > 0$: $t - h \in (t_k,s_k)$, $x - hf(t,x) \in \Delta$ for $0 < h < h_t$.

3 Main results

We define Lipschitz stability [1] of systems of differential equations with non-instantaneous impulses.

Definition 2 (Lipschitz stability) The zero solution of (7) is said to be:
- Lipschitz stable if there exists $M \geq 1$ and for every $t_0 \geq 0$ there exists $\delta = \delta(t_0) > 0$ such that, for any $x_0 \in \mathbb{R}^n$, the inequality $|x_0| < \delta$ implies $|x(t;t_0,x_0)| \leq M|x_0|$ for $t \geq t_0$;
- uniformly Lipschitz stable if there exist $M \geq 1$ and $\delta > 0$ such that for any $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ the inequality $|x_0| < \delta$ implies $|x(t;t_0,x_0)| \leq M|x_0|$ for $t \geq t_0$;
- globally uniformly Lipschitz stable if there exists $M \geq 1$ such that for any $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ the inequality $|x_0| < \infty$ implies $|x(t;t_0,x_0)| \leq M|x_0|$ for $t \geq t_0$.

Example 2 Let $t_0 \geq 0$ be an arbitrary point and without loss of generality we can assume $0 \leq t_0 < s_0$. Consider the IVP for the NIDE

$$\begin{aligned} &x' = \frac{x}{(1+t)^2} \quad \text{for } t \in \bigcup_{k=0}^{\infty}(t_k,s_k], \\ &x(t) = \Psi_k\bigl(t,x(t),x(s_k - 0)\bigr) \quad \text{for } t \in (s_{k-1},t_k], k = 1,2,\dots, \\ &x(0) = x_0. \end{aligned} \tag{8}$$

The solution of ODE $x' = \frac{x}{(1+t)^2}$, $x(\tau_0) = x_0$ is $x(t) = x_0 e^{\frac{1}{1+\tau_0} - \frac{1}{1+t}}$, $t \geq \tau_0$. Note, for any finite initial value x_0 the inequality $|x(t)| = x_0 e^{\frac{1}{1+\tau_0} - \frac{1}{1+t}} \leq M|x_0|$ for $t \geq \tau_0$ holds with $M = e$, *i.e.* the zero solution of ODE is globally Lipschitz stable but not asymptotically stable (see the graphs for $\tau_0 = 0$, $x_0 = 0.1, 0.5, 1, 1.5$ in Figure 2).

Case 1. Let $\Psi_k(t,x,y) = xy$, $k = 0,1,2,\dots$. Then the impulsive condition is $x(t) = x(t)x(s_k - 0)$ which unique solution is $x(t) = 0$ since $x(s_k - 0) = x_0 e^{\frac{1}{1+t_0} - \frac{1}{1+s_0}} \neq 0$ iff $x_0 \neq 0$ and $t_0 < s_0$. Then the solution of NIDE (8) will be

$$x(t;t_0,x_0) = \begin{cases} x_0 e^{\frac{1}{1+t_0} - \frac{1}{1+t}} & \text{for } t \in (t_0,s_0], \\ 0 & \text{for } t > s_0. \end{cases} \tag{9}$$

The zero solution is globally Lipschitz stable, since $x_0 e^{\frac{1}{1+t_0} - \frac{1}{1+t}} \leq M|x_0|$ for $t \geq t_0$ with $M = e^{\frac{1}{1+t_0} - \frac{1}{1+t}} = e > 1$. In this case the solution is also asymptotically stable.

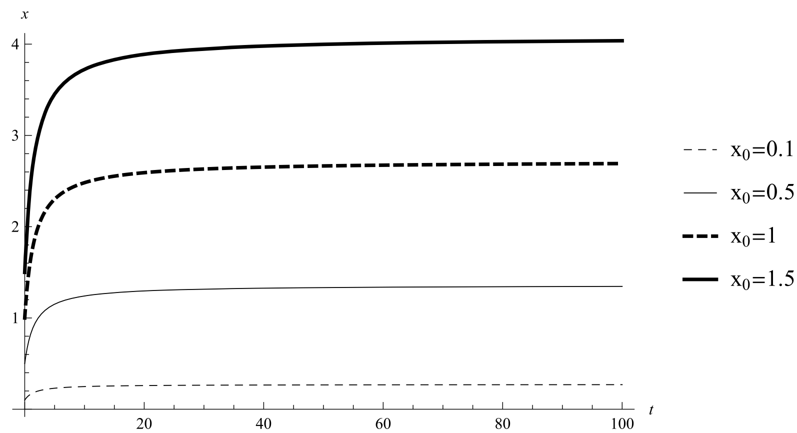

Figure 2 Graphs of solutions of (8), Case 1, for various x_0.

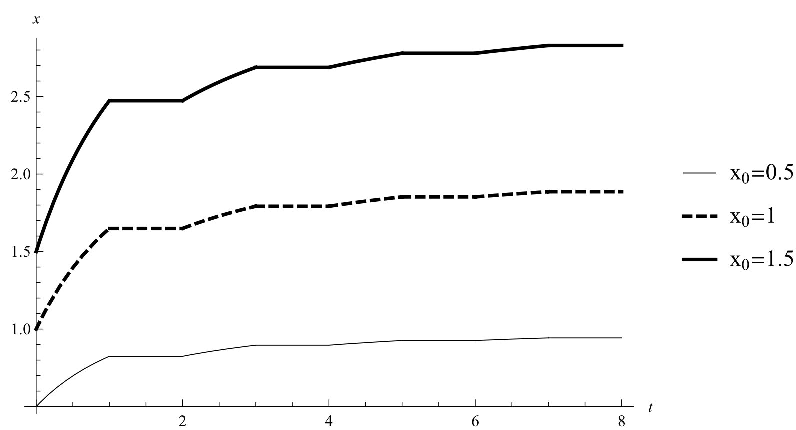

Figure 3 Graphs of solutions of (8), Case 2, for various x_0.

Case 2. Let $\Psi_k(t,x,y) = y$, $k = 0,1,2,\ldots$. Then the impulsive condition is $x(t) = x(s_k - 0)$ and the solution of (8) is

$$x(t;t_0,x_0) = \begin{cases} x_0(\prod_{i=0}^{k-1} e^{\frac{1}{1+t_i}-\frac{1}{1+s_i}})e^{\frac{1}{1+t_k}-\frac{1}{1+t}} & \text{for } t \in (t_k,s_k], k = 0,1,2,\ldots, \\ x_0\prod_{i=0}^{k-1} e^{\frac{1}{1+t_i}-\frac{1}{1+s_i}} & \text{for } t \in (s_{k-1},t_k], k = 1,2,\ldots. \end{cases} \tag{10}$$

The solution is a continuous function. The graphs of solutions for $t_0 = 0$, $s_k = 2k-1$, $t_k = 2k$, $k = 1,2,\ldots$ and various initial values x_0 are given in Figure 3. There exists $M = e > 1$ such that $x(t) < M|x_0|$, $t \geq t_0$ for any finite value of x_0. Therefore the zero solution of (8) is globally uniformly Lipschitz stable but not asymptotically stable.

Case 3. Let $\Psi_k(t,x,y) = y - x$, $k = 0,1,2,\ldots$. Then the impulsive condition is $x(t) = x(s_k - 0) - x(t)$ which unique solution is $x(t) = 0.5x(s_k - 0)$. The solution of (8) is

$$x(t;t_0,x_0) = \begin{cases} x_0 0.5^{k-1}(\prod_{i=0}^{k-1} e^{\frac{1}{1+t_i}-\frac{1}{1+s_i}})e^{\frac{1}{1+t_k}-\frac{1}{1+t}} & \text{for } t \in (t_k,s_k], k = 0,1,2,\ldots, \\ x_0 0.5^{k}\prod_{i=0}^{k} e^{\frac{1}{1+t_i}-\frac{1}{1+s_i}} & \text{for } t \in (s_{k-1},t_k], k = 1,2,\ldots. \end{cases} \tag{11}$$

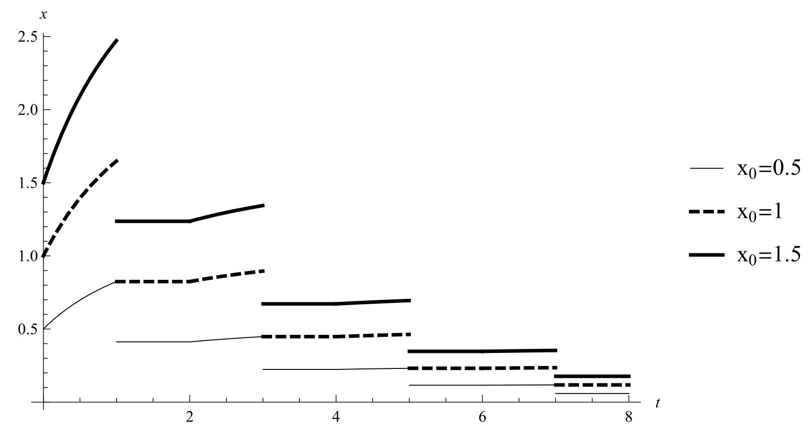

Figure 4 Graphs of solutions of (8), Case 3, for various x_0.

The graphs of solutions for $t_0 = 0$, $s_k = 2k - 1$, $t_k = 2k$, $k = 1, 2, \dots$, and various initial values x_0 are given in Figure 4. There exists $M = e > 1$ such that $x(t) < M|x_0|$, $t \geq t_0$ for any finite value of x_0. Therefore the zero solution of (8) is globally uniformly Lipschitz stable. Also it is asymptotically stable.

The above example shows the presence of non-instantaneous impulses and the type of impulsive functions that have influence on the behavior of the solution.

We study the Lipschitz stability using the following scalar comparison differential equation with non-instantaneous impulses:

$$\begin{aligned} &u' = g(t,u) \quad \text{for } t \in \bigcup_{k=0}^{\infty} (t_k, s_k], \\ &u(t) = \psi_k\bigl(t, u(s_k - 0)\bigr) \quad \text{for } t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots, \\ &u(t_0) = u_0, \end{aligned} \tag{12}$$

where $u, u_0 \in \mathbb{R}$, $g : \bigcup_{k=0}^{\infty} [t_k, s_k] \times \mathbb{R} \to \mathbb{R}$, $\psi_k : [s_k, t_{k+1}] \times \mathbb{R} \to \mathbb{R}$ $(k = 0, 1, 2, 3, \dots)$.

We introduce the following condition.

(H3) The function $g(t,u) \in C(\bigcup_{k=0}^{\infty} [t_k, s_k] \times \mathbb{R}_+, \mathbb{R})$, $g(t,0) = 0$, and for any $k = 0, 1, 2, \dots$ the functions $\psi_k : [s_k, t_{k+1}] \times \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing with respect to their second argument and $\psi_k(t, 0) = 0$.

In the main study we will use the following result.

Proposition 1 (Theorem 3.1.1 [21]) *Let the function $V \in C([t_0, T] \times \mathbb{R}^n, \mathbb{R}_+)$ and $V(t,x)$ be locally Lipschitz in x and $D_+V(t,x) \leq g(t, V(t,x))$ for $(t,x) \in [t_0, T] \times \mathbb{R}^n$, where $g \in C([t_0, T] \times \mathbb{R}_+, \mathbb{R})$. Let $\tilde{r}(t) = r(t; t_0, u_0)$ be the maximal solution of the scalar differential equation $u' = g(t,u)$ with initial condition $u(t_0) = u_0 \geq 0$, existing on $[t_0, T]$. If $x(t) = x(t; t_0, x_0)$ is any solution of the IVP for ODE $x' = f(t,x)$, $x(t_0) = x_0$ existing on $[t_0, T]$ such that $V(t_0, x_0) \leq u_0$, then the inequality $V(t, x(t)) \leq \tilde{r}(t)$ for $t \in [t_0, T]$ holds.*

Lemma 1 *Assume the following conditions are satisfied*:

1. *Conditions* (H1), (H2), *and* (H3) *are satisfied.*

2. *The function $x^*(t) = x(t;t_0,x_0) \in NPC^1([t_0,T],\Delta)$ is a solution of (7), where $T \geq t_0$ is a given constants, $\Delta \subset \mathbb{R}^n$.*
3. *The function $V \in \Lambda([t_0,T],\Delta)$ is such that:*
   (i) *the inequality $D_+V(t,x^*(t)) \leq g(t,V(t,x^*(t)))$ for $t \in [t_0,T] \cap (\bigcup_{k=0}^{\infty}(t_k,s_k))$ holds;*
   (ii) *for all $k = 0,1,2,3,\ldots$ the inequality*

$$V\bigl(t,\phi_k\bigl(t,x^*(s_k-0)\bigr)\bigr) \leq \psi_k\bigl(t,V\bigl(s_k-0,x^*(s_k-0)\bigr)\bigr) \quad \text{for } t \in [t_0,T] \cap (s_k,t_{k+1}]$$

holds.

If $V(t_0,x_0) \leq u_0$, then the inequality $V(t,x^*(t)) \leq r(t)$ for $t \in [t_0,T]$ holds, where $r(t) = r(t;t_0,u_0)$ is the maximal solution of (12) with $u_0 \geq 0$.

Proof We use induction to prove Lemma 1.

The function $x^*(t) \in C^1([t_0,s_0] \cap [t_0,T],\Delta)$. According to condition 3(i) and Proposition 1 applied to the interval $[t_0,s_0] \cap [t_0,T]$ the inequality

$$V\bigl(t,x^*(t)\bigr) \leq r(t;t_0,u_0) \tag{13}$$

holds.

Let $T > s_0$ and $t \in (s_0,t_1] \cap [t_0,T]$. From condition 3(ii)

$$V\bigl(t,\phi_0\bigl(t,x^*(s_0-0)\bigr)\bigr) \leq \psi_0\bigl(t,V\bigl(s_0-0,x^*(s_0-0)\bigr)\bigr).$$

From the inequality (13) we get $V(s_0-0,x^*(s_0-0)) \leq r(s_0-0;t_0,u_0)$ and the monotonicity of ψ_0 we get

$$\psi_0\bigl(t,V\bigl(s_0-0,x^*(s_0-0)\bigr)\bigr) \leq \psi_0\bigl(t,r(s_0-0;t_0,u_0)\bigr) = r(t;t_0,u_0),$$

i.e. $V(t,x^*(t)) \leq r(t;t_0,u_0)$ for $t \in (s_0,t_1] \cap [t_0,T]$.

Let $T > t_1$ and $t \in (t_1,s_1] \cap [t_0,T]$. Consider the function $\overline{x}(t) = x^*(t)$ for $t \in (t_1,s_1]$ and $\overline{x}(t_1) = x^*(t_1) = \phi_0(t_1,x^*(s_0-0))$. Since $\lim_{t\to t_1+0}\overline{x}(t) = \overline{x}(t_1+0) = \phi_0(t_1,\overline{x}(s_1-0)) = \overline{x}(t_1+0)$, $\overline{x}(t) \in C^1([t_1,s_1],\Delta)$. From condition 3(*ii*) for the interval $[t_1,s_1] \cap [t_0,T]$ and the proof above we obtain

$$\begin{aligned} V\bigl(t_1,\overline{x}(t_1)\bigr) &= V\bigl(t_1,\phi_0\bigl(t_1,x^*(s_0-0)\bigr)\bigr) \\ &\leq \psi_0\bigl(t_1,V\bigl(s_0-0,x^*(s_0-0)\bigr)\bigr) \\ &\leq \psi_0\bigl(t_1,r(s_0-0;t_0,u_0)\bigr) \\ &= r(t_1;t_0,u_0). \end{aligned}$$

Apply Proposition 1 to the interval $[t_1,s_1] \cap [t_0,T]$ with the initial value $u_0 = r(t_1;t_0,u_0)$ and $V(t_1,\overline{x}(t_1)) \leq r(t_1;t_0,u_0)$ and we obtain $V(t,\overline{x}(t)) \leq r(t)$, $t \in [t_1,s_1] \cap [t_0,T]$. Therefore $V(t,x^*(t)) \leq r(t;t_0,u_0)$, $t \in (t_1,s_1] \cap [t_0,T]$.

Let $T > s_1$ and $t \in (s_1,t_2] \cap [t_0,T]$. From condition 3(ii)

$$V\bigl(t,\phi_1\bigl(t,x^*(s_1-0)\bigr)\bigr) \leq \psi_1\bigl(t,V\bigl(s_1-0,x^*(s_1-0)\bigr)\bigr).$$

From the proof above and monotonicity of ψ_1 we get

$$\psi_1\big(t, V\big(s_1 - 0, x^*(s_1 - 0)\big)\big) \le \psi_1\big(t, r(s_1 - 0; t_0, u_0)\big) = r(t; t_0, u_0),$$

i.e. $V(t, x^*(t)) \le r(t; t_0, u_0)$ for $t \in (s_1, t_2] \cap [t_0, T]$.

Continue this process and an induction argument proves the claim of Lemma 1 is true for $t \in [t_0, T]$. □

Remark 6 Proposition 1 and Lemma 1 are true for $T = \infty$ for the interval $[t_0, \infty)$.

Theorem 1 *Let the following conditions be satisfied*:

1. *Conditions* (H1)-(H3) *are fulfilled*.
2. *There exists a function* $V(t,x) \in \Lambda(\mathbb{R}^+, \mathbb{R}^n)$ *with Lipschitz constant* L *in* S_ρ, $V(t,0) = 0$, *and*:
   (i) *the inequality*

   $$b\big(\|x\|\big) \le V(t,x), \quad x \in \mathbb{R}^n, t \in \mathbb{R}_+,$$

   holds, where $b \in K(\mathbb{R}_+)$;
   (ii) *the inequality* $D^+V(t,x) \le g(t, V(t,x))$, $t \in \bigcup_{k=0}^\infty (t_k, s_k)$, $x \in \mathbb{R}^n$, *holds*;
   (iii) *for any* $k = 1, 2, \ldots$ *the inequality*

   $$V\big(t, \phi_k(t,y)\big) \le \psi_k\big(t, V(s_k - 0, y)\big), \quad t \in (s_k, t_{k+1}], y \in \mathbb{R}^n,$$

   holds.
3. *The zero solution of* (12) *is Lipschitz stable*.

Then the zero solution of (7) *is Lipschitz stable*.

Proof Let $t_0 \ge 0$ be an arbitrary. Without loss of generality we assume $t_0 \in [0, s_0)$. From condition 3 there exist $M \ge 1$, $\delta_1 = \delta_1(t_0, M) > 0$ such that for any $u_0 \in \mathbb{R} : |u_0| < \delta_1$ the inequality

$$\big|u(t; t_0, u_0)\big| \le M|u_0| \quad \text{for } t \ge t_0 \tag{14}$$

holds, where $u(t; t_0, u_0)$ is a solution of (12).

Since $V(t_0, 0) = 0$ there exists a $\delta_2 = \delta_2(t_0, \delta_1) > 0$ such that $V(t_0, x) < \delta_1$ for $\|x\| < \delta_2$. The function $V(t,x)$ is Lipschitz on S_ρ then $\|x\| < \rho$ implies $|V(t,x)| = |V(t,x) - V(t,0)| \le L\|x\|$.

Let $\delta = \min\{\delta_1, \delta_2, \rho\}$ and choose $M_1 \ge 1$ such that $M_1 > ML$ and let $M_2 = q(M_1)$. Note since $M_1 \ge 1$ we have $M_2 \ge 1$ and δ depends on t_0 and M, therefore on M_2.

Now let the initial value be such that $\|x_0\| < \delta$. Consider a solution $x(t) = x(t; t_0, x_0)$ of system (7). Let $u_0^* = V(t_0, x_0)$. Then from the choice of x_0 it follows that $u_0^* = V(t_0, x_0) < \delta_1$ for $\|x_0\| < \delta$. Therefore, the function $u^*(t)$ satisfies (14) for $t \ge t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t; t_0, u_0^*)$ is a solution of (12).

Using condition 2(ii) and applying Lemma 1 for $\Delta = \mathbb{R}^n$, $T = \infty$ we get

$$V\big(t, x^*(t)\big) \le u^*(t) \quad \text{for } t \ge t_0. \tag{15}$$

From inequalities (14), (15), Lipschitz property of $V(t,x)$, condition 2(i), and Lemma 1 we obtain, for any $t \geq t_0$,

$$b(\|x^*(t)\|) \leq V(t,x^*(t)) \leq |u^*(t)| \leq M|u_0^*| = MV(t_0,x_0) \\ \leq ML\|x_0\| < M_1\|x_0\|. \tag{16}$$

From the properties of $b \in K$ and $M_1 > 1$ it follows that $b^{-1}(M_1 u) < M_1 q(u)$, and

$$\|x^*(t)\| \leq b^{-1}(M_1\|x_0\|) \leq \|x_0\| q(M_1) = M_2\|x_0\|.$$

From $M_1 \geq 1$ its follows that $q(M_1) \geq 1$ and therefore

$$\|x^*(t;t_0,x_0)\| \leq M_2|x_0|, \quad t \geq t_0. \qquad \square$$

Corollary 1 *Let the conditions of Theorem* 1 *be satisfied with* $b(u) = K_1 u$, $K_1 > 0$.
Then the zero solution of (7) *is Lipschitz stable*.

Proof The proof is similar to the one of Theorem 1 with $M_1 \geq 1 : M_1 > M\frac{L}{K_1}$ and $M_2 = M_1$. $\square$

Theorem 2 *Let the following conditions be satisfied*:

1. *Conditions* (H1)-(H3) *are fulfilled*.
2. *There exists a function* $V(t,x) \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *and*:
   (i) *the inequalities*

   $$b(\|x\|) \leq V(t,x) \leq a(\|x\|), \quad x \in S_\rho, t \in \mathbb{R}_+$$

   holds, where $b \in K([0,\rho])$, $a \in M([0,\rho])$, $\rho > 0$;
   (ii) *the inequality* $D^+V(t,x) \leq g(t, V(t,x))$, $t \in \bigcup_{k=0}^{\infty}(t_k, s_k)$, $x \in S_\rho$ *holds*;
   (iii) *for any* $k = 0,1,2,\ldots$ *the inequality*

   $$V(t,\phi_k(t,y)) \leq \psi_k(t, V(s_k - 0, y)), \quad t \in (s_k, t_{k+1}], y \in S_\rho,$$

   holds.
3. *The zero solution of* (12) *is uniformly Lipschitz stable* (*uniformly globally Lipschitz stable*).

Then the zero solution of (7) *is uniformly Lipschitz stable* (*uniformly globally Lipschitz stable*).

Proof Let the zero solution of (12) be uniformly Lipschitz stable. Let $t_0 \geq 0$ be an arbitrary. Without loss of generality we assume $t_0 \in [0, s_0)$. From condition 3 there exist $M \geq 1$, $\delta_1 > 0$ such that for any $t_0 \in \bigcup_{k=0}^{\infty}[s_k, t_{k+1})$ and any $u_0 \in \mathbb{R} : |u_0| < \delta_1$ the inequality

$$|u(t;t_0,u_0)| \leq M|u_0| \quad \text{for } t \geq t_0 \tag{17}$$

holds, where $u(t;t_0,u_0)$ is a solution of (12).

From the inclusions $b \in K([0,\rho])$ and $a \in M([0,\rho])$ there exist a function $q_b(u)$ and a positive constant K_a. Choose $M_1 \geq 1$ such that $M_1 > q_b(M)K_a$ and $\delta_2 \leq \frac{\rho}{M_1}$. Therefore, $\delta_2 \leq \rho$.

Let $\delta = \min\{\delta_1, \delta_2, \frac{\delta_1}{K_a}\}$. Choose the initial value x_0: $\|x_0\| < \delta$. Therefore, $\|x_0\| < \delta \leq \delta_2 \leq \rho$, *i.e.* $x_0 \in S_\rho$. Consider the solution $x(t) = x(t;t_0,x_0)$ of system (7) for the chosen initial data. Let $u_0^* = V(t_0,x_0)$. From the choice of x_0 and the properties of the function $a(u)$ applying condition 2(i) we get $u_0^* = V(t_0,x_0) \leq a(\|x_0\|) \leq K_a\|x_0\| < K_a\delta \leq \delta_1$. Therefore, the function $u^*(t)$ satisfies (17) for $t \geq t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t;t_0,u_0^*)$ is a solution of (12).

We will prove

$$\bigl\|x(t)\bigr\| \leq M_1\|x_0\|, \quad t \geq 0. \tag{18}$$

Assume (18) is not true. Therefore, there exists a point $T > t_0$ such that $\|x(t)\| \leq M_1\|x_0\|$ for $t \in [t_0,T]$, $\|x(T)\| = M_1\|x_0\|$ and $\|x(t)\| > M_1\|x_0\|$ for $t \in (T, T+\epsilon]$, where $\epsilon > 0$ is a small enough number. Then for $t \in [t_0,T]$ the inequalities $\|x(t)\| \leq M_1\|x_0\| < M_1\delta \leq M_1\delta_2 \leq \rho$ hold, *i.e.* $x(t) \in S_\rho$ for $t \in [t_0,T]$.

Using condition 2(ii) and applying Lemma 1 on $[t_0,T]$ for $\Delta = S_\rho$ we get

$$V\bigl(t,x^*(t)\bigr) \leq u^*(t) \quad \text{for } t \in [t_0,T]. \tag{19}$$

From inequality (19) and condition 2(i) we obtain

$$\begin{aligned} M_1\|x_0\| &= \bigl\|x(T)\bigr\| \leq b^{-1}\bigl(V\bigl(T,x(T)\bigr)\bigr) \leq b^{-1}\bigl(\bigl|u^*(T)\bigr|\bigr) \\ &\leq b^{-1}\bigl(M\bigl|u_0^*\bigr|\bigr) = b^{-1}\bigl(MV(t_0,x_0)\bigr) \\ &\leq q_b(M)V(t_0,x_0) \leq q_b(M)a\bigl(\|x_0\|\bigr) \\ &\leq q_b(M)K_a\|x_0\| < M_1\|x_0\|. \end{aligned} \tag{20}$$

The contradiction obtained proves the validity of (18).

The proof of globally uniformly Lipschitz stability is analogous and we omit it. □

Corollary 2 *Let* (H1)-(H3) *and condition* 2 *of Theorem* 2 *be satisfied with* $g(t,x) \equiv 0$ *and* $\psi_k(t,x) \equiv x$.

Then the zero solution of (7) *is uniformly Lipschitz stable*.

Corollary 3 *Let* (H1), (H2) *be satisfied and the inequality*

$$xf(t,x) \leq 0, \quad t \in \bigcup_{k=0}^{\infty}(t_k,s_k), x \in \mathbb{R}^n,$$

holds and for any $k = 0,1,2,\ldots$ *the inequality*

$$\bigl(\phi_k(t,y)\bigr)^2 \leq y^2, \quad y \in \mathbb{R}^n, t \in (s_k,t_{k+1}],$$

holds.

Then the zero solution of (7) *is uniformly Lipschitz stable*.

Proof Consider the quadratic Lyapunov function $V(t,x) = x^2$ for which $D^+V(t,x) = 2xf(t,x)$ and condition 2 of Theorem 2 is satisfied with $K_1 \leq 1$, $K_2 \geq 1$, $g(t,x) = 0$, and $\psi_k(t,x) \equiv x$. □

Theorem 3 *Let the conditions of Theorem* 2 *be satisfied where* 2(i) *is replaced by*

2. (i) *the inequalities* $\lambda_1(t)\|x\|^2 \leq V(t,x) \leq \lambda_2(t)\|x\|^2$, $x \in S_\rho$, $t \in \mathbb{R}^+$ *holds, where* $\lambda_1, \lambda_2 \in C(\mathbb{R}_+, (0,\infty))$ *and there exist positive constants* A_1, A_2: $A_1 < A_2$ *such that* $\lambda_1(t) \geq A_1$, $\lambda_2(t) \leq A_2$ *for* $t \geq 0$, *and* $\rho > 0$.

If the zero solution of (12) *is uniformly Lipschitz stable* (*uniformly globally Lipschitz stable*) *then the zero solution of* (7) *is uniformly Lipschitz stable*(*uniformly globally Lipschitz stable*).

Proof The proof is similar to the one of Theorem 2 where $M_1 = \sqrt{M\frac{A_2}{A_1}}$. □

4 Applications

Let two increasing sequences of points $\{t_i\}_{i=1}^\infty$ and $\{s_i\}_{i=0}^\infty$ be given such that $t_0 = 0$, $0 < s_0 < t_i \leq s_i < t_{i+1}$, $i = 1,2,\ldots$, and $\lim_{k\to\infty} t_k = \infty$. Consider the following single species model exhibiting the so-called Allee effect in which the per-capita growth rate is a quadratic function of the density:

$$\begin{aligned} &N'(t) = N(t)\big(-a - bN(t) + cN^2(t)\big) \quad \text{for } t \in (t_k, s_k], k = 0,1,2,\ldots, \\ &N(t) = \psi_k\big(t, N(t_k - 0)\big) \quad \text{for } t \in (s_k, t_{k+1}], k = 0,1,2,\ldots, \end{aligned} \tag{21}$$

where $a, c > 0$, $b \in \mathbb{R}$. The impulsive functions $\psi_k(t,x) \leq C_k x$, $k = 0,1,2,\ldots$, where $C_k \in (0,1]$.

Define the function $V(t,x) = x^2$.

Then condition 2(i) of Theorem 3 is satisfied for $\lambda_1(t) = 0.5$, $\lambda_2(t) = 1.5$.

For any $x : |x| \leq \rho$, $\rho = |\frac{b-\sqrt{b^2+4ac}}{2c}| > 0$ we have $D^+V(t,x) = 2x^2(cx^2 - bx - a) \leq 0$, $t \in \bigcup_{k=0}^\infty (t_k, s_k)$, $x \in S_\rho$. Therefore, condition 2(ii) is satisfied with $g(t,x) \equiv 0$.

The condition 2(ii) is satisfied for $\psi_k(t,x) \equiv C_k x$.

Therefore, the comparison equation is

$$\begin{aligned} &u'(t) = 0 \quad \text{for } t \in (t_k, s_k], k = 0,1,2,\ldots, \\ &u(t) = C_k u(s_k - 0) \quad \text{for } t \in (s_k, t_{k+1}], k = 0,1,2,\ldots, \end{aligned} \tag{22}$$

which solution is $u(t; t_0, u_0) = u_0 \prod_{i=1}^k C_i$, for $t \in (s_k, s_{k+1}]$, $k = 0,1,2,\ldots$. Therefore, the zero solution of (21) according to Corollary 2 is uniformly Lipschitz stable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors SH and RT contributed to each part of the work equally and read and approved the final version of the manuscript.

Acknowledgements
The research is partially supported by the Fund NPD, Plovdiv University.

Received: 17 October 2016 Accepted: 28 November 2016 Published online: 09 December 2016

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