

RESEARCH

Open Access



# Fractional calculus of generalized $k$ -Mittag-Leffler function and its applications to statistical distribution

Kottakkaran Sooppy Nisar<sup>1</sup>, Ashraf Fetoh Eata<sup>2,3</sup>, Mujahed Al-Dhaifallah<sup>4</sup> and Junesang Choi<sup>5\*</sup>

\*Correspondence:

junesang@mail.dongguk.ac.kr

<sup>5</sup>Department of Mathematics,  
Dongguk University, Gyeongju,  
38066, Republic of Korea

Full list of author information is  
available at the end of the article

## Abstract

We aim to investigate the MSM-fractional calculus operators, Caputo-type MSM-fractional differential operator, and pathway fractional integral operator of the generalized  $k$ -Mittag-Leffler function. We also investigate certain statistical distribution associated with the generalized  $k$ -Mittag-Leffler function. Certain particular cases of the derived results are considered and indicated to further reduce to some known results.

**MSC:** 33C10; 44A10; 44A20; 33E12

**Keywords:** gamma function; generalized  $k$ -Mittag-Leffler functions; statistical distribution

## 1 Introduction and preliminaries

Throughout this paper, let  $\mathbb{C}$ ,  $\mathbb{R}$ ,  $\mathbb{R}^+$ ,  $\mathbb{Z}_0^-$ , and  $\mathbb{N}$  be the sets of complex numbers, real numbers, positive real numbers, nonpositive integers, and positive integers, respectively, and let  $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ .

Díaz and Pariguan [1] found that the expression

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k) \quad (1.1)$$

has appeared repeatedly in a variety of contexts such as combinatorics of creation, annihilation operators, and perturbative computation of Feynman integrals. Motivated by this observation, they [1] used the Gauss form of the gamma function (see [2], Eq. (6), p.2) to introduce the so-called  $k$ -gamma function

$$\Gamma_k(z) = \lim_{n \rightarrow \infty} \frac{n!k^n (nk)^{\frac{z}{k}-1}}{(z)_{n,k}} \quad (k \in \mathbb{R}^+, z \in \mathbb{C} \setminus k\mathbb{Z}_0^-). \quad (1.2)$$

Starting from this definition, they [1] presented a number of properties for the  $k$ -gamma function. We recall some of them:

$$\Gamma_k(z+k) = z\Gamma_k(z) \quad \text{and} \quad \Gamma_k(k) = 1; \quad (1.3)$$

the Euler integral form:

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt \quad (k \in \mathbb{R}^+; \Re(z) > 0); \tag{1.4}$$

the  $k$ -Pochhammer symbol  $(\lambda)_{n,k}$  defined (for  $\lambda, \nu \in \mathbb{C}; k \in \mathbb{R}$ ) by

$$\begin{aligned} (\lambda)_{\nu,k} &:= \frac{\Gamma_k(\lambda + \nu k)}{\Gamma_k(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \{0\}) \\ &= \begin{cases} 1 & (\nu = 0), \\ \lambda(\lambda + k) \cdots (\lambda + (n-1)k) & (\nu = n \in \mathbb{N}); \end{cases} \end{aligned} \tag{1.5}$$

from (1.4), it is easy to find the following relationship between the gamma function  $\Gamma$  and the  $k$ -gamma function  $\Gamma_k$ :

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right). \tag{1.6}$$

In a number of subsequent works including [1], the  $k$ -gamma function and  $k$ -Pochhammer symbol have been used to extend and investigate such special functions and integral operators as (for example) the  $k$ -beta function,  $k$ -zeta function,  $k$ -hypergeometric function,  $k$ -Mittag-Letter functions,  $k$ -Wright function, and  $k$ -analogue of the Riemann-Liouville fractional integral operator.

In 1903, the Swedish mathematician Gosta Mittag-Leffler [3] (see also [4]) introduced and investigated the so-called Mittag-Leffler function

$$E_\alpha(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z \in \mathbb{C}; \alpha \in \mathbb{R}_0^+). \tag{1.7}$$

Since then, the Mittag-Leffler function  $E_\alpha$  (1.7) has been extended in a number of ways and, together with its extensions, applied in various research areas such as engineering and (in particular) statistics. The Mittag-Leffler functions and related distributions were given in [5]. Further, Mathai and Haubold [6] established connections among generalized Mittag-Leffler functions, pathway model, Tsallis statistics, superstatistics and power law, and the corresponding entropy measures. A statistical perspective of Mittag-Leffler functions and matrix-variate analogues was given by Mathai, who presented the involved results in terms of statistical densities, which are useful in statistical distribution theory and stochastic processes. Also, various pathways were investigated from the exponential and gamma densities to the Mittag-Leffler densities and then from the Mittag-Leffler densities to the Lèvi and Linnik densities [7]. Lin [8] proved that the Mittag-Leffler distributions belong to the class of distributions with completely monotone derivatives. The fundamental properties of the Mittag-Leffler distributions and their extensions, including the tail behavior of distribution and explicit expressions for moments of all orders and for the density functions, are also given.

Here, for an easier reference, we give a brief history of some chosen extensions of the Mittag-Leffler function  $E_\alpha$  (1.7). Wiman [9] presented the following generalization  $E_{\alpha,\beta}$

of  $E_\alpha$ :

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\alpha, \beta \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta)\} > 0). \tag{1.8}$$

Prabhakar [10] introduced the function  $E_{\alpha,\beta}^\gamma$  in the following form:

$$E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n! \Gamma(\alpha n + \beta)} z^n \quad (\alpha, \beta, \gamma \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0). \tag{1.9}$$

Another generalization of the Mittag-Leffler function  $E_\alpha$  was given by Shukla and Prajapati [11]:

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n! \Gamma(\alpha n + \beta)} z^n \quad (\alpha, \beta, \gamma \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0; q \in (0, 1) \cup \mathbb{N}). \tag{1.10}$$

We also recall the following two extensions of the Mittag-Leffler function (see [12], Eqs. (1.6) and (1.9)):

$$E_{\alpha,\beta,p}^{\eta,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{qn}}{\Gamma(\alpha n + \beta)(\delta)_{pn}} z^n \quad (p, q \in \mathbb{R}^+; \min\{\Re(\alpha), \Re(\beta), \Re(\eta), \Re(\delta)\} > 0); \tag{1.11}$$

$$E_{\alpha,\beta,v,\sigma,\delta,p}^{\mu,\rho,\eta,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} (\eta)_{qn}}{\Gamma(\alpha n + \beta)(v)_{\sigma n} (\delta)_{pn}} z^n \quad (p, q \in \mathbb{R}^+; q \leq \Re(\alpha) + p; \min\{\Re(\alpha), \Re(\beta), \Re(\eta), \Re(\delta), \Re(\mu), \Re(v), \Re(\rho), \Re(\sigma)\} > 0). \tag{1.12}$$

For more generalizations of the Mittag-Leffler functions, we refer the reader, for example, to [12–14, 45].

The Fox-Wright hypergeometric function  ${}_p\Psi_q(z)$  is given by the series

$${}_p\Psi_q(z) = {}_p\Psi_q \left[ \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z \right] = \frac{\prod_{j=1}^q \Gamma(\beta_j)}{\prod_{i=1}^p \Gamma(\alpha_i)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \tag{1.13}$$

where  $a_i, b_j \in \mathbb{C}$  and  $\alpha_i, \beta_j \in \mathbb{R}$  ( $i = 1, 2, \dots, p; j = 1, 2, \dots, q$ ). Asymptotic behavior of this function for large values of the argument of  $z \in \mathbb{C}$  was studied in [15], and under the condition

$$\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1, \tag{1.14}$$

was found in [16, 17]. Properties of this generalized Wright function were investigated in [18] (see also [19, 20]).

The generalized hypergeometric function  ${}_pF_q$  is defined as follows [21]:

$${}_pF_q \left[ \begin{matrix} (a_p); \\ (b_q); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n z^n}{\prod_{j=1}^q (b_j)_n n!} \quad (p \leq q, z \in \mathbb{C}; p = q + 1, |z| < 1), \tag{1.15}$$

which obviously is a particular case of the Fox-Wright hypergeometric function  ${}_p\Psi_q(z)$  (1.13) when  $\alpha_i = 1 = \beta_j$  ( $i = 1, 2, \dots, p; j = 1, 2, \dots, q$ ).

Let  $\lambda, \lambda', \xi, \xi', \gamma \in \mathbb{C}$  with  $\Re(\gamma) > 0$  and  $x \in \mathbb{R}^+$ . Then the generalized fractional integral operators involving the Appell functions  $F_3$  are defined as follows:

$$(I_{0,+}^{\lambda, \lambda', \xi, \xi', \gamma} f)(x) = \frac{x^{-\lambda}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\lambda'} F_3 \left( \lambda, \lambda', \xi, \xi'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt \tag{1.16}$$

and

$$(I_{-}^{\lambda, \lambda', \xi, \xi', \gamma} f)(x) = \frac{x^{-\lambda'}}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} t^{-\lambda} F_3 \left( \lambda, \lambda', \xi, \xi'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt. \tag{1.17}$$

The generalized fractional integral operators of types (1.16) and (1.17) have been introduced by Marichev [22] and later extended and studied by Saigo and Maeda [23]. These operators are known as the Marichev-Saigo-Maeda operators (MSM-operators). Recently, Mondal and Nisar [24] have investigated the Marichev-Saigo-Maeda fractional integral operators involving generalized Bessel functions (see also [43]).

The corresponding fractional differential operators have their respective forms:

$$(D_{0,+}^{\lambda, \lambda', \xi, \xi', \gamma} f)(x) = \left( \frac{d}{dx} \right)^{[\Re(\gamma)]+1} (I_{0,+}^{-\lambda', -\lambda, -\xi', -\xi + [\Re(\gamma)]+1, -\xi, -\gamma + [\Re(\gamma)]+1} f)(x) \tag{1.18}$$

and

$$(D_{-}^{\lambda, \lambda', \xi, \xi', \gamma} f)(x) = \left( -\frac{d}{dx} \right)^{[\Re(\gamma)]+1} (I_{-}^{-\lambda', -\lambda, -\xi', -\xi + [\Re(\gamma)]+1, -\gamma + [\Re(\gamma)]+1} f)(x). \tag{1.19}$$

The fractional integral operators have many interesting applications in various fields including (for example) a certain class of complex analytic functions (see [25]). For some basic results on fractional calculus, we refer to [26–28, 44].

The following four results will be required (for the first and second, see [23, 29]; for the third and fourth, see [19]).

**Lemma 1.1** *Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho \in \mathbb{C}$  be such that  $\Re(\gamma) > 0$  and*

$$\Re(\rho) > \max\{0, \Re(\lambda + \lambda' + \xi - \gamma), \Re(\lambda' - \xi')\}.$$

*Then*

$$\begin{aligned} & (I_{0,+}^{\lambda, \lambda', \xi, \xi', \gamma} t^{\rho-1})(x) \\ &= \frac{\Gamma(\rho)\Gamma(\rho + \gamma - \lambda - \lambda' - \xi)\Gamma(\rho + \xi' - \lambda')}{\Gamma(\rho + \xi')\Gamma(\rho + \gamma - \lambda - \lambda')\Gamma(\rho + \gamma - \lambda' - \xi)} x^{\rho - \lambda - \lambda' + \gamma - 1}. \end{aligned} \tag{1.20}$$

**Lemma 1.2** *Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho \in \mathbb{C}$  be such that  $\Re(\gamma) > 0$  and*

$$\Re(\rho) > \max\{\Re(\xi), \Re(-\lambda - \lambda' + \gamma), \Re(-\lambda - \xi' + \gamma)\}.$$

*Then*

$$\begin{aligned} & (I_{-}^{\lambda, \lambda', \xi, \xi', \gamma} t^{-\rho})(x) \\ &= \frac{\Gamma(-\xi + \rho)\Gamma(\lambda + \lambda' - \gamma + \rho)\Gamma(\lambda + \xi' - \gamma + \rho)}{\Gamma(\rho)\Gamma(\lambda - \xi + \rho)\Gamma(\lambda + \lambda' + \xi' - \gamma + \rho)} x^{-\lambda - \lambda' + \gamma - \rho}. \end{aligned} \tag{1.21}$$

**Lemma 1.3** *Let  $\lambda, \delta, \gamma, \rho \in \mathbb{C}$  with  $\Re(\lambda) > 0$  and  $\Re(\rho) > \max\{0, \Re(\delta - \gamma)\}$ . Then*

$$(I_{0+}^{\lambda, \delta, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(\rho + \gamma - \delta)}{\Gamma(\rho - \delta)\Gamma(\rho + \lambda + \gamma)} x^{\rho - \delta - 1}. \tag{1.22}$$

*In particular,*

$$(I_{0+}^{\lambda, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho + \gamma)}{\Gamma(\rho + \lambda + \gamma)} x^{\rho-1} \quad (\Re(\lambda) > 0, \Re(\rho) > \max\{0, -\Re(\gamma)\}) \tag{1.23}$$

*and*

$$(I_{0+}^{\lambda} t^{\rho-1})(x) = \frac{\Gamma(\rho)}{\Gamma(\rho + \lambda)} x^{\rho-1} \quad (\min\{\Re(\lambda), \Re(\rho)\} > 0). \tag{1.24}$$

**Lemma 1.4** *Let  $\lambda, \delta, \gamma, \rho \in \mathbb{C}$  with  $\Re(\lambda) > 0$  and  $\Re(\rho) < 1 + \min\{\Re(\delta), \Re(\gamma)\}$ . Then*

$$(I_{-}^{\lambda, \delta, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\delta - \rho + 1)\Gamma(\gamma - \rho + 1)}{\Gamma(1 - \rho)\Gamma(\lambda + \delta + \gamma - \rho + 1)} x^{\rho - \delta - 1}. \tag{1.25}$$

*In particular,*

$$(I_{-}^{\lambda, \gamma} t^{\rho-1})(x) = \frac{\Gamma(\gamma - \rho + 1)}{\Gamma(\lambda + \gamma - \rho + 1)} x^{\rho-1} \quad (\Re(\lambda) > 0, \Re(\rho) < 1 + \Re(\gamma)) \tag{1.26}$$

*and*

$$(I_{-}^{\lambda} t^{\rho-1})(x) = \frac{\Gamma(1 - \rho)}{\Gamma(\lambda - \rho + 1)} x^{\rho-1} \quad (\Re(\lambda) > 0, \Re(\rho) < 1). \tag{1.27}$$

As mentioned before,  $k$ -extensions of the Mittag-Leffler functions have been given and investigated particularly in view of statistics. In this paper, we aim to investigate the MSM-fractional calculus operators, Caputo-type MSM-fractional differential operator, and that pathway fractional integral operator of the generalized  $k$ -Mittag-Leffler function (2.5). We also investigate certain statistical distribution associated with the generalized  $k$ -Mittag-Leffler function (2.5), in which certain particular cases of the derived results are considered and indicated to further reduce to some known results.

### 2 *k*-Mittag-Leffler functions

Here we introduce *k*-Mittag-Leffler functions and their extensions. The simplest *k*-extensions of the Mittag-Leffler functions (1.7) and (1.8) can be given by

$$E_{k,\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n + 1)} \quad (k \in \mathbb{R}^+; \alpha \in \mathbb{R}_0^+) \tag{2.1}$$

and

$$E_{k,\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n + \beta)}$$

$$(k \in \mathbb{R}^+; \min\{\Re(\alpha), \Re(\beta)\} > 0), \tag{2.2}$$

respectively (see, e.g., [30], Eq. (5)). Dorrego and Cerutti [31] introduced the *k*-Mittag-Leffler function

$$E_{k,\alpha,\beta}^{\eta}(z) := \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}$$

$$(k \in \mathbb{R}^+; \alpha, \beta, \eta \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta)\} > 0) \tag{2.3}$$

and investigated some properties associated with the definition itself and the Riemann-Liouville fractional calculus operators. Saxena et al. [32] extended the *k*-Mittag-Leffler function (2.3) slightly as follows:

$$E_{k,\alpha,\beta}^{\eta,\tau}(z) := \sum_{n=0}^{\infty} \frac{(\eta)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}$$

$$(k \in \mathbb{R}^+; \alpha, \beta, \eta, \tau \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta)\} > 0). \tag{2.4}$$

They derived its Euler transform, Laplace transform, Whittaker transform, and fractional Fourier transform of order  $\alpha$  ( $0 < \alpha \leq 1$ ). Daiya and Ram [33] investigated the statistical density of the *k*-Mittag-Leffler function (2.4). Gupta and Parihar [30] defined a further extension of the *k*-Mittag-Leffler functions,

$$E_{k,\alpha,\beta,p}^{\eta,\delta,q}(z) := \sum_{n=0}^{\infty} \frac{(\eta)_{qn,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}} z^n$$

$$(k, p, q \in \mathbb{R}^+; \alpha, \beta, \eta, \delta \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\eta), \Re(\delta)\} > 0; q \leq \Re(\alpha) + p), \tag{2.5}$$

presented its properties, including differentiation, the fractional Fourier transform, Laplace transform, and *k*-Beta transform, and determined the *k*-Riemann-Liouville fractional integral and differentiation.

### 3 MSM fractional integral representations of (2.5)

Here we present MSM fractional integral representations of the generalized *k*-Mittag-Leffler function (2.5) and consider some particular cases.

**Theorem 1** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho, \alpha, \beta, \eta, \delta \in \mathbb{C}$  with  $\Re(\gamma) > 0$  and

$$\Re(\rho) > \max\{0, \Re(\lambda + \lambda' + \xi - \gamma), \Re(\lambda' - \xi')\}.$$

Also, let  $k, p, q, x \in \mathbb{R}^+$ . Then

$$\begin{aligned} (I_{0,+}^{\lambda,\lambda',\xi,\xi',\gamma} t^{\rho-1} E_{k,\alpha,\beta,p}^{\eta,\delta,q}(t))(x) &= \frac{\Gamma(\delta/k)}{\Gamma(\eta/k)} \frac{x^{\rho-\lambda-\lambda'+\gamma-1}}{k^{\frac{\beta}{k}-1}} \\ &\times {}_5\Psi_5 \left[ \begin{matrix} (\frac{\eta}{k}, q), (\rho, 1), (\rho + \gamma - \lambda - \lambda' - \xi, 1), \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (\frac{\delta}{k}, p), (\rho + \xi', 1), (\rho + \gamma - \lambda - \lambda', 1), \\ (\rho + \xi' - \lambda', 1), (1, 1); \\ (\rho + \gamma - \lambda' - \xi, 1); \end{matrix} \quad k^{q-p-\frac{\alpha}{k}} x \right]. \end{aligned} \tag{3.1}$$

*Proof* Let  $\mathcal{L}_1$  be the left-hand side of (3.1). Then, using (2.5), we have

$$\mathcal{L}_1 = \left( I_{0,+}^{\lambda,\lambda',\xi,\xi',\gamma} t^{\rho-1} \sum_{n=0}^{\infty} \frac{(\eta)_{nq,k}}{\Gamma_k(\alpha n + \beta)} \frac{t^n}{(\delta)_{pn,k}} \right)(x).$$

Interchanging the summation and integration, which is verified under the conditions in this theorem, we get

$$\mathcal{L}_1 = \sum_{n=0}^{\infty} \frac{(\eta)_{nq,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}} (I_{0,+}^{\lambda,\lambda',\xi,\xi',\gamma} t^{\rho+n-1})(x).$$

Applying Lemma 1.1, we obtain

$$\begin{aligned} \mathcal{L}_1 &= \sum_{n=0}^{\infty} \frac{(\eta)_{nq,k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}} \\ &\times \frac{\Gamma(\rho + n)\Gamma(\rho + \gamma - \lambda - \lambda' - \xi + n)\Gamma(\rho + \xi' - \lambda' + n)}{\Gamma(\rho + \xi' + n)\Gamma(\rho + \gamma - \lambda - \lambda' + n)\Gamma(\rho + \gamma - \lambda' - \xi + n)} \\ &\times x^{\rho+n-\lambda-\lambda'+\gamma-1}. \end{aligned}$$

Now, using relations (1.5) and (1.6), we get

$$\begin{aligned} \mathcal{L}_1 &= x^{\rho-\lambda-\lambda'+\gamma-1} \\ &\times \sum_{n=0}^{\infty} \frac{k^{nq} \Gamma(\frac{\eta}{k} + nq) \Gamma(\frac{\delta}{k})}{\Gamma(\frac{\eta}{k}) k^{\frac{\alpha n + \beta}{k} - 1} \Gamma(\frac{\alpha n + \beta}{k}) \Gamma(\frac{\delta}{k} + np) k^{np}} \\ &\times \frac{\Gamma(\rho + n)\Gamma(\rho + \gamma - \lambda - \lambda' - \xi + n)\Gamma(\rho + \xi' - \lambda' + n)\Gamma(n + 1) x^n}{\Gamma(\rho + \xi' + n)\Gamma(\rho + \gamma - \lambda - \lambda' + n)\Gamma(\rho + \gamma - \lambda' - \xi + n) n!}, \end{aligned}$$

which, in view of (1.13), leads to the right-hand side of (3.1). This completes the proof.  $\square$

**Corollary 3.1** Let  $\lambda, \xi, \gamma, \rho, \alpha, \beta, \eta, \delta \in \mathbb{C}$  with  $\Re(\gamma) > 0$  and

$$\Re(\rho) > \max\{0, \Re(\xi - \gamma)\}.$$

Also, let  $k, p, q, x \in \mathbb{R}^+$ . Then

$$\begin{aligned}
 & (I_{0,+}^{\lambda,\xi,\gamma} t^{\rho-1} E_{k,\alpha,\beta,p}^{\eta,\delta,q}(t))(x) \\
 &= \frac{\Gamma(\delta/k) x^{\rho-\xi-1}}{\Gamma(\eta/k) k^{\frac{\beta}{k}-1}} \\
 & \quad \times {}_4\Psi_4 \left[ \begin{matrix} (\frac{\eta}{k}, q), (\rho + \gamma - \xi, 1), (\rho, 1), (1, 1); \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (\frac{\delta}{k}, p), (\rho - \xi, 1), (\rho + \gamma + \lambda, 1); \end{matrix} \quad k^{q-p-\frac{\alpha}{k}} x \right]. \tag{3.2}
 \end{aligned}$$

**Theorem 2** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho, \alpha, \beta, \eta, \delta \in \mathbb{C}$  be such that  $\Re(\lambda) > 0$  and

$$\Re(\rho) > \max\{\Re(\xi), \Re(-\lambda - \lambda' + \gamma), \Re(-\lambda - \xi' + \gamma)\}.$$

Also, let  $k, p, q \in \mathbb{R}^+$ . Then

$$\begin{aligned}
 & (I_{-}^{\lambda,\lambda',\xi,\xi',\gamma} t^{\rho-1} E_{k,\alpha,\beta,p}^{\eta,\delta,q}(t))(x) = \frac{\Gamma(\delta/k) x^{-\lambda-\lambda'+\gamma-\rho}}{\Gamma(\eta/k) k^{\frac{\beta}{k}-1}} \\
 & \quad \times {}_5\Psi_5 \left[ \begin{matrix} (\frac{\eta}{k}, q), (-\xi + \rho, 1), (\lambda + \lambda' - \gamma + \rho, 1), \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (\frac{\delta}{k}, p), (\rho, 1), (\lambda - \xi + \rho, 1), \\ (\lambda + \xi' - \gamma + \rho, 1), (1, 1); \\ (\lambda + \lambda' + \xi' - \gamma + \rho, 1); \end{matrix} \quad k^{q-p-\frac{\alpha}{k}} x \right]. \tag{3.3}
 \end{aligned}$$

*Proof* We can establish (3.3) by a similar argument as in the proof of (3.1), using Lemma 1.2 instead of Lemma 1.1. We omit the details.  $\square$

**Corollary 3.2** Let  $\lambda, \xi, \gamma, \rho, \alpha, \beta, \eta, \delta \in \mathbb{C}$  be such that  $\Re(\lambda) > 0$  and

$$\Re(\rho) > \max\{\Re(-\xi), \Re(-\gamma)\}.$$

Also, let  $k, p, q \in \mathbb{R}^+$ . Then

$$\begin{aligned}
 & (I_{-}^{\lambda,\xi,\gamma} t^{\rho-1} E_{k,\alpha,\beta,p}^{\eta,\delta,q}(t))(x) \\
 &= \frac{\Gamma(\delta/k) x^{\rho-\xi-1}}{\Gamma(\eta/k) k^{\frac{\beta}{k}-1}} \\
 & \quad \times {}_4\Psi_4 \left[ \begin{matrix} (\frac{\eta}{k}, q), (\xi - \rho + 1, 1), (\gamma - \rho + 1, 1), (1, 1); \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (\frac{\delta}{k}, p), (1 - \rho, 1), (\lambda + \xi + \gamma - \rho + 1, 1); \end{matrix} \quad k^{q-p-\frac{\alpha}{k}} x \right]. \tag{3.4}
 \end{aligned}$$

**4 MSM-fractional differential operator of (2.5)**

Here we derive the Marichev-Saigo-Maeda fractional differentiation of the generalized  $k$ -Mittag-Leffler function (2.5). The following lemmas will be required (see [34]).

**Lemma 4.1** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho \in \mathbb{C}$  be such that

$$\Re(\rho) > \max\{0, \Re(-\lambda + \xi), \Re(-\lambda - \lambda' - \xi' + \gamma)\}.$$



Then

$$(D_{0+}^{\lambda, \lambda', \xi, \xi', \gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(-\xi + \lambda + \rho)\Gamma(\lambda + \lambda' + \xi' - \gamma + \rho)}{\Gamma(-\xi + \rho)\Gamma(\lambda + \lambda' - \gamma + \rho)\Gamma(\lambda + \xi' - \gamma + \rho)} x^{\lambda + \lambda' - \gamma + \rho - 1}. \tag{4.1}$$

**Lemma 4.2** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho \in \mathbb{C}$  be such that

$$\Re(\rho) > \max\{\Re(-\xi'), \Re(\lambda' + \xi - \gamma), \Re(\lambda + \lambda' - \gamma) + [\Re(\gamma)] + 1\}.$$

Then

$$(D_{-}^{\lambda, \lambda', \xi, \xi', \gamma} t^{-\rho})(x) = \frac{\Gamma(\xi' + \rho)\Gamma(-\lambda - \lambda' + \gamma + \rho)\Gamma(-\lambda' - \xi + \gamma + \rho)}{\Gamma(\rho)\Gamma(-\lambda' + \xi' + \rho)\Gamma(-\lambda - \lambda' - \xi + \gamma + \rho)} x^{\lambda + \lambda' - \gamma - \rho}. \tag{4.2}$$

**Theorem 3** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho, \alpha, \beta, \eta, \delta \in \mathbb{C}$  be such that

$$\Re(\rho) > \max\{0, \Re(-\lambda + \xi), \Re(-\lambda - \lambda' - \xi' + \gamma)\}.$$

Also, let  $k, p, q \in \mathbb{R}^+$ . Then

$$\begin{aligned} (D_{0+}^{\lambda, \lambda', \xi, \xi', \gamma} t^{\rho-1} E_{k, \alpha, \beta, p}^{\eta, \delta, q}(t))(x) &= \frac{\Gamma(\delta/k)}{\Gamma(\eta/k)} \frac{x^{\lambda + \lambda' - \gamma + \rho - 1}}{k^{\frac{\beta}{k} - 1}} \\ &\times {}_5\Psi_5 \left[ \begin{matrix} (\frac{\eta}{k}, q), (\rho, 1), (-\xi + \lambda + \rho, 1), \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (\frac{\delta}{k}, p), (-\xi + \rho, 1), (\lambda + \lambda' - \gamma + \rho, 1), \\ (\lambda + \lambda' + \xi' - \gamma + \rho, 1), (1, 1); \\ (\lambda + \xi' - \gamma + \rho, 1); \end{matrix} \right. \\ &\left. k^{q-p-\frac{\alpha}{k}} x \right]. \end{aligned} \tag{4.3}$$

*Proof* Let  $\mathcal{L}_2$  be the left-hand side of (4.3). Taking the MSM differential operator on (2.5) and interchanging the differentiation and summation, which is verified under the conditions in this theorem, we have

$$\mathcal{L}_2 = \sum_{n=0}^{\infty} \frac{(\eta)_{nq, k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn, k}} (D_{0+}^{\lambda, \lambda', \xi, \xi', \gamma} t^{\rho+n-1})(x).$$

Using (1.5), (1.6), and Lemma 4.1, we get

$$\begin{aligned} \mathcal{L}_2 &= \sum_{n=0}^{\infty} \frac{k^{nq} \Gamma(\frac{\eta}{k} + nq) \Gamma(\frac{\delta}{k})}{\Gamma(\frac{\eta}{k}) k^{\frac{\alpha n + \beta}{k} - 1} \Gamma(\frac{\alpha n + \beta}{k}) \Gamma(\frac{\delta}{k} + np) k^{np}} \\ &\times \frac{\Gamma(\rho + n)\Gamma(-\xi + \lambda - \rho + n)\Gamma(\lambda + \lambda' + \xi' - \gamma + \rho + n)}{\Gamma(-\xi - \rho + n)\Gamma(\lambda + \lambda' - \gamma + \rho + n)\Gamma(\lambda + \xi' - \gamma + \rho + n)} \\ &\times x^{\lambda + \lambda' - \gamma + \rho + n - 1}, \end{aligned}$$

which, in view of (1.13), is equal to the right-hand side of (4.3). □

**Theorem 4** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho, \alpha, \beta, \eta, \delta \in \mathbb{C}$  be such that

$$\Re(\rho) > \max\{\Re(-\xi'), \Re(\lambda' + \xi - \gamma), \Re(\lambda + \lambda' - \gamma) + [\Re(\gamma)] + 1\}.$$

Also, let  $k, p, q \in \mathbb{R}^+$ . Then

$$\begin{aligned}
 (D_-^{\lambda, \lambda', \xi, \xi', \gamma} t^{-\rho} E_{k, \alpha, \beta, p}^{\eta, \delta, q}(t))(x) &= \frac{x^{\lambda + \lambda' - \gamma - \rho} \Gamma(\delta/k)}{k^{\frac{\beta}{k} - 1} \Gamma(\eta/k)} \\
 &\times {}_5\Psi_5 \left[ \begin{matrix} (\frac{\eta}{k}, q), (\xi' + \rho, 1), (-\lambda - \lambda' + \gamma + \rho, 1), \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (\frac{\delta}{k}, p), (\rho, 1), (-\lambda' + \xi' + \rho, 1), \\ (-\lambda - \xi + \gamma + \rho, 1), (1, 1); \\ (-\lambda - \lambda' - \xi + \gamma + \rho, 1); \end{matrix} \quad k^{q-p-\frac{\alpha}{k}} x \right]. \tag{4.4}
 \end{aligned}$$

*Proof* The proof runs parallel to that of Theorem 3, using Lemma 4.2 instead of Lemma 4.1. We omit the details. □

**Remark 4.1** The results in Theorems 1 to 4 can be easily reduced to yield some corresponding formulas involving simpler fractional calculus operators such as the Erdélyi-Kober fractional calculus operators.

**5 Caputo-type MSM fractional differentiation of (2.5)**

Rao et al. [35] introduced the Caputo-type fractional derivatives that have the Gauss hypergeometric function in the kernel. The left- and right-hand sided Caputo fractional differential operators associated with the Gauss hypergeometric function are defined, respectively, by

$$(D_{0+}^{\lambda, \xi, \gamma} f)(x) = (I_{0+}^{-\lambda + [\Re(\lambda)] + 1, -\xi - [\Re(\lambda)] - 1, \lambda + \gamma - [\Re(\lambda)] - 1} f^{([\Re(\lambda)] + 1)})(x)$$

and

$$({}^c D_-^{\lambda, \xi, \gamma} f)(x) = (-1)^{[\Re(\lambda)] + 1} (I_-^{-\lambda + [\Re(\lambda)] + 1, -\xi - [\Re(\lambda)] - 1, \lambda + \gamma - [\Re(\lambda)] - 1} f^{([\Re(\lambda)] + 1)})(x),$$

where  $\lambda, \xi, \gamma \in \mathbb{C}$  with  $\Re(\lambda) > 0$  and  $x \in \mathbb{R}^+$ .

The left- and right-hand sided Caputo-type MSM fractional differential operators associated with the Appell function  $F_3$  are defined, respectively, by

$$({}^c D_{0+}^{\lambda, \lambda', \xi, \xi', \gamma} f)(x) = (I_{0+}^{-\lambda', -\lambda, -\xi' + [\Re(\gamma)] + 1, -\xi, -\gamma + [\Re(\gamma)] + 1} f^{([\Re(\gamma)] + 1)})(x)$$

and

$$({}^c D_-^{\lambda, \lambda', \xi, \xi', \gamma} f)(x) = (-1)^{[\Re(\gamma)] + 1} (I_-^{-\lambda', -\lambda, -\xi', -\xi + [\Re(\gamma)] + 1, -\gamma + [\Re(\gamma)] + 1} f^{([\Re(\gamma)] + 1)})(x),$$

where  $\lambda, \lambda', \xi, \xi', \gamma, \rho \in \mathbb{C}$  with  $\Re(\gamma) > 0$  and  $x \in \mathbb{R}^+$ .

In this section, we investigate the Caputo-type MSM fractional differential operator of the generalized  $k$ -Mittag-Leffler function (2.5). The following lemmas will be required (see [34]).

**Lemma 5.1** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho \in \mathbb{C}$  and  $m = [\Re(\gamma)] + 1$  with

$$\Re(\rho) - m > \max\{0, \Re(-\lambda + \xi), \Re(-\lambda - \lambda' - \xi' + \gamma)\}.$$

Then

$$\begin{aligned}
 ({}^c D_{0+}^{\lambda, \lambda', \xi, \xi', \gamma} t^{\rho-1})(x) &= \frac{\Gamma(\rho)\Gamma(\lambda - \xi + \rho - m)\Gamma(\lambda + \lambda' + \xi' - \gamma + \rho - m)}{\Gamma(-\xi + \rho - m)\Gamma(\lambda + \lambda' - \gamma + \rho)\Gamma(\lambda + \xi' - \gamma + \rho - m)} \\
 &\times x^{\lambda + \lambda' - \gamma + \rho + 1}.
 \end{aligned}
 \tag{5.1}$$

**Lemma 5.2** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho \in \mathbb{C}$  and  $m = [\Re(\gamma)] + 1$  with

$$\Re(\rho) + m > \max\{\Re(-\xi'), \Re(\lambda' + \xi - \gamma), \Re(\lambda + \lambda' - \gamma) + [\Re(\gamma)] + 1\}.$$

Then

$$\begin{aligned}
 ({}^c D_{-}^{\lambda, \lambda', \xi, \xi', \gamma} t^{-\rho})(x) &= \frac{\Gamma(\xi' + \rho + m)\Gamma(-\lambda - \lambda' + \gamma + \rho)\Gamma(-\lambda' - \xi + \gamma + \rho + m)}{\Gamma(\rho)\Gamma(-\lambda' + \xi' + \rho + m)\Gamma(-\lambda - \lambda' - \xi + \gamma + \rho + m)} \\
 &\times x^{\lambda + \lambda' - \gamma - \rho}.
 \end{aligned}
 \tag{5.2}$$

**Theorem 5** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho, \alpha, \beta, \eta, \delta \in \mathbb{C}$  and  $m = [\Re(\gamma)] + 1$  with

$$\Re(\rho) - m > \max\{0, \Re(-\lambda + \xi), \Re(-\lambda - \lambda' - \xi' + \gamma)\}.$$

Also, let  $k, p, q \in \mathbb{R}^+$ . Then

$$\begin{aligned}
 ({}^c D_{0+}^{\lambda, \lambda', \xi, \xi', \gamma} t^{\rho-1} E_{k, \alpha, \beta, p}^{\eta, \delta, q}(t))(x) &= \frac{x^{\lambda + \lambda' - \gamma + \rho + 1}}{k^{\frac{\beta}{k} - 1}} \frac{\Gamma(\delta/k)}{\Gamma(\eta/k)} \\
 &\times {}_5\Psi_5 \left[ \begin{matrix} (\frac{\eta}{k}, q), (\rho, 1), (\lambda - \xi + \rho - m, 1), \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (\frac{\delta}{k}, p), (-\xi - m + \rho, 1), (\lambda + \lambda' - \gamma + \rho, 1), \\ (\lambda + \lambda' + \xi' - \gamma + \rho - m, 1), (1, 1); \\ (\lambda + \xi' - \gamma + \rho - m, 1); \end{matrix} \right. \\
 &\left. k^{q-p-\frac{\alpha}{k}} x \right].
 \end{aligned}
 \tag{5.3}$$

*Proof* Let  $\mathcal{L}_3$  be the left-hand side of (5.3). Then using (2.5) and interchanging the order of summation and differentiation, which is verified under the conditions in this theorem, we have

$$\mathcal{L}_3 = \sum_{n=0}^{\infty} \frac{(\eta)_{nq, k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn, k}} ({}^c D_{0+}^{\lambda, \lambda', \delta, \delta', \gamma} t^{\rho+n-1}).$$

Applying Lemma 5.1 together with (1.5) and (1.6), we get

$$\begin{aligned}
 \mathcal{L}_3 &= \sum_{n=0}^{\infty} \frac{k^{nq} \Gamma(\frac{\eta}{k} + nq) \Gamma(\frac{\delta}{k})}{\Gamma(\frac{\eta}{k}) k^{\frac{\alpha n + \beta}{k} - 1} \Gamma(\frac{\alpha n + \beta}{k}) \Gamma(\frac{\delta}{k} + np) k^{np}} \\
 &\times \frac{\Gamma(\rho + n)\Gamma(\lambda - \xi + \rho + n - m)\Gamma(\lambda + \lambda' + \xi' - \gamma + \rho + n - m)}{\Gamma(-\xi + \rho + n - m)\Gamma(\lambda + \lambda' - \gamma + \rho + n)\Gamma(\lambda + \xi' - \gamma + \rho + n - m)} \\
 &\times x^{\lambda + \lambda' - \gamma + \rho + 1 + n},
 \end{aligned}$$

which, in view of (1.13), leads to the right-hand side of (5.3). □

**Theorem 6** Let  $\lambda, \lambda', \xi, \xi', \gamma, \rho, \alpha, \beta, \gamma, \eta, \delta \in \mathbb{C}$  and  $m = [\Re(\gamma)] + 1$  with

$$\Re(\rho) + m > \max\{\Re(-\xi'), \Re(\lambda + \lambda' - \gamma) + m\}.$$

Also, let  $k, p, q \in \mathbb{R}^+$ . Then

$$\begin{aligned} ({}^c D_{-}^{\lambda, \lambda', \xi, \xi', \gamma} t^{-\rho} E_{k, \alpha, \beta, p}^{\eta, \delta, q}(t))(x) &= \frac{x^{\lambda + \lambda' - \gamma + \rho} \Gamma(\delta/k)}{k^{\frac{\beta}{k} - 1} \Gamma(\eta/k)} \\ &\times {}_5\Psi_5 \left[ \begin{matrix} \left(\frac{\eta}{k}, q\right), (\xi' + \rho + m, 1), (-\lambda - \lambda' + \gamma + \rho, 1), \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), \left(\frac{\delta}{k}, p\right), (\rho, 1), (-\lambda' + \xi' + \rho + m, 1), \\ (-\lambda' - \xi + \gamma + \rho + m, 1), (1, 1); \end{matrix} \right. \\ &\left. \begin{matrix} k^{q-p-\frac{\alpha}{k}} \\ x \end{matrix} \right]. \end{aligned} \tag{5.4}$$

*Proof* We establish the result by a similar argument as in the proof of Theorem 5, using Lemma 5.2 instead of Lemma 5.1. We omit the details.  $\square$

**Remark 5.1** The results in Theorems 5 and 6 can be easily specialized to yield the corresponding formulas involving simpler Caputo-type fractional derivatives with (2.5) such as the left-hand-sided generalized Caputo fractional differentiation  ${}^c D_{0+}^{\lambda, \xi, \gamma} f$ , the left-hand-sided Caputo-type Erdelyi-Kober fractional differentiation  ${}^c D_{\gamma, \lambda}^+ f$ , the right-hand-sided generalized Caputo fractional differentiation  ${}^c D_{-}^{\lambda, \xi, \gamma} f$ , and the right-hand-sided Caputo-type Erdelyi-Kober fractional differentiation  ${}^c D_{\gamma, \lambda}^- f$  (see, e.g., [36]).

### 6 Pathway integral representation of (2.5)

Recently, Nair [37] introduced the pathway fractional integral operator by using the pathway idea of Mathai [6], developed further by Mathai and Haubold [38] and defined as follows (cf. [39]).

Let  $f \in L(a, b)$ ,  $\eta \in \mathbb{C}$  with  $\Re(\eta)$ ,  $a \in \mathbb{R}^+$ , and  $\sigma < 1$  be the pathway parameter. Then

$$(P_{0+}^{(\eta, \sigma)} f)(x) := x^\eta \int_0^{\lceil \frac{x}{a(1-\sigma)} \rceil} \left[ 1 - \frac{a(1-\sigma)t}{x} \right]^{\frac{\eta}{1-\sigma}} f(t) dt. \tag{6.1}$$

For a real scalar  $\sigma$ , the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{\nu-1} [1 - a(1-\sigma)|x|^\xi]^{-\frac{\lambda}{1-\sigma}},$$

provided that  $x \in \mathbb{R}$ ,  $\nu, \xi \in \mathbb{R}^+$ ,  $\lambda \in \mathbb{R}_0^+$ ,  $1 - a(1-\sigma)|x|^\xi > 0$ . Here  $c$  is the normalizing constant, and  $\sigma$  is called the pathway parameter.

For  $\sigma > 1$ , (6.1) can be written as follows:

$$(P_{0+}^{(\eta, \sigma)} f)(x) = x^\eta \int_0^{\lceil \frac{x}{-a(1-\sigma)} \rceil} \left[ 1 + \frac{a(\sigma-1)t}{x} \right]^{-\frac{\eta}{(\sigma-1)}} f(t) dt, \tag{6.2}$$

and

$$f(x) = c|x|^{\nu-1} [1 + a(\sigma-1)|x|^\xi]^{-\frac{\lambda}{(\sigma-1)}}, \tag{6.3}$$

provided that  $x \in \mathbb{R}, \nu, \xi \in \mathbb{R}^+, \lambda \in \mathbb{R}_0^+$ .

Moreover, as  $\sigma \rightarrow 1-$ , the operator (6.1) reduces to the Laplace integral transform, and when  $\sigma = 0$  and  $a = 1$ , replacing  $\eta$  by  $\eta - 1$ , the operator (6.1) reduces to the Riemann-Liouville fractional integral operator. For more details on the pathway model and its particular cases, the interested reader may refer to the recent works [6, 37, 38].

It is observed that the pathway fractional integral operator (6.1) can lead to other interesting examples of fractional calculus operators regarding some probability density functions and applications in statistics. Recently, Nisar et al. [40] studied the pathway fractional integral operator associated with the Struve function of the first kind [41] and the  $k$ -Mittag-Leffler function.

Here we investigate the pathway integral operator of the generalized  $k$ -Mittag-Leffler function (2.5).

**Theorem 7** *Let  $\rho, \gamma, \beta, \eta \in \mathbb{C}$  with  $\min\{\Re(\rho), \Re(\beta), \Re(\eta)\} > 0$  and  $\Re(\frac{\gamma}{1-\sigma}) > -1$ . Also, let  $k, w, \sigma \in \mathbb{R}$  with  $\sigma < 1, p, q \in \mathbb{R}^+$ , and  $\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-$ . Then*

$$P_{0+}^{(\gamma, \sigma)} \left[ t^{\frac{\beta}{k}-1} E_{k, \rho, \beta, p}^{\eta, \delta, q} \left( w t^{\frac{\rho}{k}} \right) \right] (x) = x^{\gamma + \frac{\beta}{k}} k^{1 + \frac{\gamma}{1-\sigma}} \frac{\Gamma(1 + \frac{\gamma}{1-\sigma})}{[a(1-\sigma)]^{\frac{\beta}{k}}} \times E_{k, \rho, \beta + k(1 + \frac{\gamma}{1-\sigma}), p}^{\eta, \delta, q} \left[ w \left( \frac{x}{a(1-\sigma)} \right)^{\frac{\rho}{k}} \right]. \tag{6.4}$$

*Proof* Let  $\mathcal{L}_4$  be the left-hand side of (6.4). Using (6.1) and interchanging the order of the integration and summation, which is verified under the conditions in this theorem, we get

$$\mathcal{L}_4 = x^\gamma \sum_{n=0}^\infty \frac{(\eta)_{nq, k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn, k}} \int_0^{\frac{x}{a(1-\sigma)}} \left[ 1 - \frac{a(1-\sigma)t}{x} \right]^{\frac{\gamma}{1-\sigma}} t^{\frac{\beta}{k} + \frac{\rho}{k} - 1} dt.$$

Evaluating the inner integral using the beta function (see, e.g., [2], p.8), we get

$$\begin{aligned} \mathcal{L}_4 &= x^\gamma \sum_{n=0}^\infty \frac{(\eta)_{nq, k}}{\Gamma_k(\alpha n + \beta)(\delta)_{pn, k}} \left( \frac{x}{a(1-\sigma)} \right)^{\frac{\rho}{k} n + \frac{\beta}{k}} \frac{\Gamma(1 + \frac{\gamma}{1-\sigma}) \Gamma(\frac{\rho}{k} n + \frac{\beta}{k})}{\Gamma(\frac{\rho}{k} n + \frac{\beta}{k} + 1 + \frac{\gamma}{1-\sigma})} \\ &= \frac{x^{\gamma + \frac{\beta}{k}} \Gamma(1 + \frac{\gamma}{1-\sigma})}{[a(1-\sigma)]^{\frac{\beta}{k}}} \sum_{n=0}^\infty \frac{(\eta)_{nq, k} [w(\frac{x}{a(1-\sigma)})^{\frac{\rho}{k}}]^n}{\Gamma(\frac{\rho}{k} n + \frac{\beta}{k} + 1 + \frac{\gamma}{1-\sigma})(\delta)_{pn, k}} \frac{k^{-\frac{\rho}{k} n} \Gamma(\frac{\rho}{k} n + \frac{\beta}{k})}{\Gamma(\frac{\rho}{k} n + \frac{\beta}{k})}. \end{aligned}$$

Using (1.6), we obtain

$$\mathcal{L}_4 = \frac{x^{\gamma + \frac{\beta}{k}} \Gamma(1 + \frac{\gamma}{1-\sigma}) k^{1 + \frac{\gamma}{1-\sigma}}}{[a(1-\sigma)]^{\frac{\beta}{k}}} \sum_{n=0}^\infty \frac{(\eta)_{nq, k} [w(\frac{x}{a(1-\sigma)})^{\frac{\rho}{k}}]^n}{\Gamma_k(\rho n + \beta + (1 + \frac{\gamma}{1-\sigma})k)(\delta)_{pn, k}},$$

which, with the aid of (2.5), is seen to reach the right-hand side of (6.4). □

Setting  $\delta = q = 1$  and  $k = 1$  in Theorem 7, we obtain the following known result (see [37]).

**Corollary 6.1** *Let  $\rho, \gamma, \beta, \eta \in \mathbb{C}$  with  $\min\{\Re(\rho), \Re(\beta), \Re(\eta)\} > 0$  and  $\Re(\frac{\gamma}{1-\sigma}) > -1$ . Also, let  $w, \sigma \in \mathbb{R}$  with  $\sigma < 1, p \in \mathbb{R}^+$ . Then*

$$\begin{aligned}
 & P_{0+}^{(\eta, \sigma)} [t^{\beta-1} E_{k, \rho, 1, 1}^{\gamma, 1}(wt^\rho)](x) \\
 &= x^{\eta+\beta} \frac{\Gamma(1 + \frac{\eta}{1-\sigma})}{[a(1-\sigma)]^\beta} E_{\rho, \beta+1+\frac{\eta}{1-\sigma}}^\gamma \left[ \frac{wx^\rho}{(a(1-\sigma))^\rho} \right].
 \end{aligned} \tag{6.5}$$

We give Theorem 8 by considering the case  $\sigma > 1$  and using equation (6.2), without its proof, since the proof is similar to that in Theorem 7.

**Theorem 8** *Let  $\rho, \gamma, \beta, \eta \in \mathbb{C}$  with  $\min\{\Re(\rho), \Re(\beta), \Re(\eta)\} > 0$  and  $\Re(1 - \frac{\eta}{\sigma-1}) > 0$ . Also, let  $k, w, \sigma \in \mathbb{R}$  with  $\sigma > 1, p, q \in \mathbb{R}^+$ , and  $\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-$ . Then*

$$\begin{aligned}
 & P_{0+}^{(\eta, \sigma)} [t^{\frac{\beta}{k}-1} E_{k, \rho, \beta, p}^{\eta, \delta, q}(wt^{\frac{\rho}{k}})](x) = x^{\gamma+\frac{\beta}{k}} k^{(1-\frac{\gamma}{1-\sigma})} \frac{\Gamma(1 - \frac{\gamma}{\sigma-1})}{[-a(1-\sigma)]^{\frac{\beta}{k}}} \\
 & \quad \times E_{k, \rho, \beta+k(1-\frac{\gamma}{\sigma-1}), p}^{\eta, \delta, q} \left[ w \left( \frac{x}{-a(1-\sigma)} \right)^{\frac{\rho}{k}} \right].
 \end{aligned} \tag{6.6}$$

The particular case of Theorem 8 when  $\delta = k = q = 1$  reduces to the following known result (see [37]).

**Corollary 6.2** *Let  $\rho, \gamma, \beta, \eta \in \mathbb{C}$  with  $\min\{\Re(\rho), \Re(\beta), \Re(\eta)\} > 0$  and  $\Re(1 - \frac{\eta}{\sigma-1}) > 0$ . Also, let  $w, \sigma \in \mathbb{R}$  with  $\sigma > 1, p \in \mathbb{R}^+$ . Then*

$$\begin{aligned}
 & P_{0+}^{(\eta, \sigma)} [t^{\beta-1} E_{1, \rho, \beta}^{\gamma, 1}(wt^\rho)](x) \\
 &= \frac{x^{\eta+\beta} \Gamma(1 - \frac{\eta}{\sigma-1})}{[-a(\sigma-1)]^\beta} E_{\rho, \beta+(1-\frac{\eta}{\sigma-1})}^\gamma \left[ w \left( \frac{x}{-a(\sigma-1)} \right)^\rho \right].
 \end{aligned} \tag{6.7}$$

**7 Generalized  $k$ -Mittag-Leffler function and statistical distribution**

In this section, we investigate the density function for (2.5) stated in Theorem 9. We also consider some particular cases of Theorem 9, which are connected with some possible known results (if any).

**Theorem 9** *Let  $k, p, q, \mu, x \in \mathbb{R}^+$  with  $0 < \mu \leq 1$  and  $q \leq \mu + p$ . Also, let  $\gamma, \delta \in \mathbb{C}$  with  $\min\{\Re(\gamma), \Re(\delta)\} > 0$ . Let*

$$\mathcal{F}_x(x) = 1 - E_{k, \mu, k, p}^{\gamma, \delta, q}(-x^\mu).$$

Then the density function  $f(x)$  of  $\mathcal{F}_x(x)$  is given as follows:

$$f(x) = \mu x^{\mu-1} (\gamma)_{q, k} \sum_{n=0}^{\infty} \frac{(n+1)(\gamma + qk)_{nq, k}}{\Gamma_k(\mu n + \mu + k)} \frac{(-x^\mu)^n}{(\delta)_{(n+1)p, k}} \tag{7.1}$$

$$= \frac{(\gamma)_{q, k}}{(\delta)_{p, k}} x^{\mu-1} E_{k, \mu, \mu, p}^{\gamma+qk, \delta+pk, q}(-x^\mu). \tag{7.2}$$

*Proof* Using (2.5), we have

$$\mathcal{F}_x(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\gamma)_{nq,k}}{\Gamma_k(\mu n + k)} \frac{x^{\mu n}}{(\delta)_{pn,k}}. \tag{7.3}$$

Differentiating each side of (7.3) with respect to  $x$  gives the density function

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\gamma)_{nq,k}}{\Gamma_k(\mu n + k)} \frac{\mu n x^{\mu n-1}}{(\delta)_{pn,k}},$$

which, upon replacing  $n$  by  $n + 1$ , yields

$$f(x) = x^{\mu-1} \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_{nq+q,k}}{\Gamma_k(\mu n + \mu + k)} \frac{(\mu n + \mu)x^{\mu n}}{(\delta)_{p(n+1),k}}. \tag{7.4}$$

Applying the relation

$$(\gamma)_{n+j,k} = (\gamma)_{j,k}(\gamma + jk)_{n,k} \tag{7.5}$$

to the factor in the numerator of (7.4) and using (1.3) on the denominator, we get

$$\begin{aligned} f(x) &= x^{\mu-1} \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_{q,k}(\gamma + qk)_{nq,k}}{(\mu n + \mu)\Gamma_k(\mu n + \mu)} \frac{(\mu n + \mu)x^{\mu n}}{(\delta)_{p(n+1),k}} \\ &= (\gamma)_{q,k} x^{\mu-1} \sum_{n=0}^{\infty} \frac{(\gamma + qk)_{nq,k}}{\Gamma_k(\mu n + \mu)} \frac{(-x^\mu)^n}{(\delta)_{p(n+1),k}} \\ &= x^{\mu-1}(\gamma)_{q,k} \sum_{n=0}^{\infty} \frac{(\gamma + qk)_{nq,k}}{\frac{\Gamma_k(\mu n + \mu + k)}{(\mu n + \mu)}} \frac{(-x^\mu)^n}{(\delta)_{p(n+1),k}}, \end{aligned} \tag{7.6}$$

which is just the desired result (7.1).

Next, employing the same process as in the proof of (7.1), we find from (7.4) that

$$f(x) = x^{\mu-1} \sum_{n=0}^{\infty} \frac{(-1)^n(\gamma)_{nq+q,k}}{\Gamma_k(\mu n + \mu + k)} \frac{(\mu n + \mu)x^{\mu n}}{(\delta)_{p(n+1),k}}.$$

Applying (7.5) to both numerator and denominator, we get

$$f(x) = \frac{(\gamma)_{q,k}}{(\delta)_{p,k}} x^{\mu-1} \sum_{n=0}^{\infty} \frac{(\gamma + qk)_{nq,k}}{\Gamma_k(\mu n + \mu)} \frac{(-x^\mu)^n}{(\delta + pk)_{np,k}}, \tag{7.7}$$

which, in view of (2.5), can be expressed as the desired result (7.2). □

Here we consider some particular known cases of Theorem 9.

Taking  $\delta = p = k = 1$  in Theorem 9, we get the following result (cf. [33], Eq. (19)).

**Corollary 7.1** *Let  $q, \mu, x \in \mathbb{R}^+$  with  $0 < \mu \leq 1$  and  $q \leq \mu + 1$ . Also, let  $\Re(\gamma) > 0$ . Let*

$$\mathcal{F}_x(x) = 1 - E_{\mu,1}^{\gamma,q}(-x^\mu).$$

Then the density function  $f(x)$  of  $\mathcal{F}_x(x)$  is given as follows:

$$f(x) = (\gamma)_q x^{\mu-1} E_{\mu,\mu,1}^{\gamma+q,2,q}(-x^\mu).$$

Setting  $q = k = \gamma = 1$  in Theorem 9, we obtain the following result (cf. [42], Eq. (20)).

**Corollary 7.2** Let  $p, \mu, x \in \mathbb{R}^+$  with  $0 < \mu \leq 1$  and  $1 \leq \mu + p$ . Also, let  $\gamma, \delta \in \mathbb{C}$  with  $\min\{\Re(\gamma), \Re(\delta)\} > 0$ . Let

$$\mathcal{F}_x(x) = 1 - E_{\mu,1,p}^{\gamma,\delta}(-x^\mu).$$

Then the density function  $f(x)$  of  $\mathcal{F}_x(x)$  is given as follows:

$$f(x) = \frac{\gamma}{(\delta)_p} x^{\mu-1} E_{\mu,\mu,p}^{\gamma+1,\delta+p}(-x^\mu). \tag{7.8}$$

Here

$$E_{\alpha,\beta,p}^{\eta,\delta}(z) := E_{\alpha,\beta,p}^{\eta,\delta,1}(z). \tag{7.9}$$

Setting  $q = k = \gamma = 1$  in Theorem 9, we get the following result (see also [7]).

**Corollary 7.3** Let  $p, \mu, x \in \mathbb{R}^+$  with  $0 < \mu \leq 1$  and  $1 \leq \mu + p$ . Also, let  $\Re(\delta) > 0$ . Let

$$\mathcal{F}_x(x) = 1 - E_{\mu,1,p}^{1,\delta}(-x^\mu).$$

Then the density function  $f(x)$  of  $\mathcal{F}_x(x)$  is given as follows:

$$f(x) = \frac{1}{(\delta)_p} x^{\mu-1} E_{\mu,\mu,p}^{2,\delta+p}(-x^\mu). \tag{7.10}$$

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

The authors have contributed equally to this manuscript. They read and approved the final manuscript.

**Author details**

<sup>1</sup>Department of Mathematics, College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Al dawaser, Riyadh region 11991, Saudi Arabia. <sup>2</sup>Department of Applied Statistics & Insurance, Mansoura University, Mansoura, Egypt. <sup>3</sup>College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Al dawaser, Riyadh region 11991, Saudi Arabia. <sup>4</sup>Electrical Engineering Department, College of Engineering-Wadi Aldawaser, Prince Sattam bin Abdulaziz University, Wadi Al dawaser, Riyadh region 11991, Saudi Arabia. <sup>5</sup>Department of Mathematics, Dongguk University, Gyeongju, 38066, Republic of Korea.

**Acknowledgements**

This project was supported by the Deanship of Scientific Research at Prince Sattam Bin Abdulaziz University under the research project No. 2016/01/6637.

Received: 20 July 2016 Accepted: 17 November 2016 Published online: 29 November 2016

**References**

1. Diaz, R, Pariguan, E: On hypergeometric functions and  $k$ -Pochhammer symbol. *Divulg. Mat.* **15**(2), 179-192 (2007)
2. Srivastava, HM, Choi, J: Zeta and  $q$ -Zeta Functions and Associated Series and Integrals. Elsevier Science, Amsterdam (2012)
3. Mittag-Leffler, GM: Sur la nouvelle fonction  $E_\alpha(x)$ . *C. R. Acad. Sci. Paris* **137**, 554-558 (1903)



4. Mittag-Leffler, GM: Sur la représentation analytique d'une branche uniforme d'une fonction monogène. *Acta Math.* **29**, 101-181 (1905)
5. Pillai, RN: On Mittag-Leffler functions and related distributions. *Ann. Inst. Stat. Math.* **42**(1), 157-161 (1990)
6. Mathai, M, Haubold, HJ: Mittag-Leffler functions to pathway model to Tsallis statistics. *Integral Transforms Spec. Funct.* **21**(11), 867-875 (2010)
7. Mathai, AM: Some properties of Mittag-Leffler functions and matrix variate analogues: a statistical perspective. *Fract. Calc. Appl. Anal.* **13**(2), 113-132 (2010)
8. Lin, GW: On Mittag-Leffler distributions. *J. Stat. Plan. Inference* **74**, 1-9 (1998)
9. Wiman, A: Über den Fundamentalsatz in der Theorie der Funktionen  $E_\alpha(x)$ . *Acta Math.* **29**, 191-201 (1905)
10. Prabhakar, TR: A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* **19**, 7-15 (1971)
11. Shukla, AK, Prajapati, JC: On a generalization of Mittag-Leffler function and its properties. *J. Math. Anal. Appl.* **336**, 797-811 (2007)
12. Khan, MA, Ahmed, S: On some properties of the generalized Mittag-Leffler function. *SpringerPlus* **2**, 337 (2013)
13. Salim, TO: Some properties relating to the generalized Mittag-Leffler function. *Adv. Appl. Math. Anal.* **4**, 21-30 (2009)
14. Srivastava, HM, Tomovski, Z: Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Appl. Math. Comput.* **211**, 198-210 (2009)
15. Fox, C: The asymptotic expansion of generalized hypergeometric functions. *Proc. Lond. Math. Soc.* **27**(4), 389-400 (1928)
16. Wright, EM: The asymptotic expansion of integral functions defined by Taylor series. *Philos. Trans. Roy. Soc. London, Ser. A.* **238**, 423-451 (1940)
17. Wright, EM: The asymptotic expansion of the generalized hypergeometric function. *Proc. Lond. Math. Soc.* **46**, 389-408 (1940)
18. Kilbas, AA, Saigo, M, Trujillo, JJ: On the generalized Wright function. *Fract. Calc. Appl. Anal.* **5**(4), 437-460 (2002)
19. Kilbas, AA, Sebastian, N: Generalized fractional integration of Bessel function of the first kind. *Integral Transforms Spec. Funct.* **19**(11-12), 869-883 (2008)
20. Kilbas, AA, Sebastian, N: Fractional integration of the product of Bessel function of the first kind. *Fract. Calc. Appl. Anal.* **13**(2), 159-175 (2010)
21. Rainville, ED: *Special Functions*. Macmillan Company, New York (1960)
22. Marichev, OI: Volterra equation of Mellin convolution type with a horn function in the kernel. *Izvestiya Akademii Nauk BSSR. Seriya Fiziko-Matematicheskikh Nauk* **1**, 128-129 (1974) (Russian)
23. Saigo, M, Maeda, N: More generalization of fractional calculus. In: *Transform Methods & Special Functions*, Varna '96, pp. 386-400. Bulgarian Academy of Sciences, Bulgaria (1998)
24. Mondal, SR, Nisar, KS: Marichev-Saigo-Maeda fractional integration operators involving generalized Bessel functions. *Math. Probl. Eng.* **2014**, Article ID 274093 (2014)
25. Kim, YC, Lee, KS, Srivastava, HM: Some applications of fractional integral operators and Ruscheweyh derivatives. *J. Math. Anal. Appl.* **197**(2), 505-517 (1996)
26. Kiryakova, V: All the special functions are fractional differintegrals of elementary functions. *J. Phys. A, Math. Gen.* **30**(14), 5085-5103 (1997)
27. Miller, KS, Ross, B: *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York (1993)
28. Srivastava, HM, Lin, S-D, Wang, P-Y: Some fractional calculus results for the  $H$ -function associated with a class of Feynman integrals. *Russ. J. Math. Phys.* **13**(1), 94-100 (2006)
29. Saxena, RK, Saigo, M: Generalized fractional calculus of the  $H$ -function associated with the Appell function  $F_3$ . *J. Fract. Calc.* **19**, 89-104 (2001)
30. Gupta, A, Parihar, CL:  $k$ -New generalized Mittag-Leffler function. *J. Fract. Calc. Appl.* **5**, 165-176 (2014)
31. Dorrego, GA, Cerutti, RA: The  $k$ -Mittag-Leffler function. *Int. J. Contemp. Math. Sci.* **7**(15), 705-716 (2012)
32. Saxena, RK, Daiya, J, Singh, A: Integral transforms of the  $k$ -generalized Mittag-Leffler function  $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$ . *Matematicheski LIX(Fasc. II)*, 7-16 (2014)
33. Daiya, J, Ram, J:  $k$ -Generalized Mittag-Leffler statistical distribution. *Journal of Research in Mathematical Archives* **1**(12), 61-68 (2013)
34. Kataria, KK, Vellaisamy, P: The generalized  $k$ -Wright function and Marichev-Saigo-Maeda fractional operators (17 Aug 2014) arXiv:1408.4762v1 [math.CA]
35. Rao, A, Garg, M, Kalla, SL: Caputo-type fractional derivative of a hypergeometric integral operator. In: *Kuwait J. Sci. Eng.* vol. 37.1A, pp. 15-29 (2010)
36. Kilbas, AA, Srivastava, HM, Trujillo, JJ: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematical Studies, vol. 204. Elsevier, Amsterdam (2006)
37. Nair, SS: Pathway fractional integration operator. *Fract. Calc. Appl. Anal.* **12**(3), 237-252 (2009)
38. Mathai, AM, Haubold, HJ: On generalized distributions and pathways. *Phys. Lett. A* **372**, 2109-2113 (2008)
39. Baleanu, D, Agarwal, P: A composition formula of the pathway integral transform operator. *Note Mat.* **34**(2), 145-155 (2014)
40. Nisar, KS, Purohit, SD, Abouzaid, MS, Qurashi, M, Baleanu, D: Generalized  $k$ -Mittag Leffler function and its composition with pathway integral operators. *J. Nonlinear Sci. Appl.* **9**, 3519-3526 (2016)
41. Nisar, KS, Mondal, SR, Agarwal, P: Pathway fractional integral operator associated with Struve function of first kind. *Adv. Stud. Contemp. Math., Kyungshang* **26**(1), 63-70 (2016)
42. Ram, C, Choudhary, P, Gehlot, KS: Statistical distribution of  $k$ -Mittag-Leffler function. *Palestine J. Math.* **3**(2), 269-272 (2014)
43. Nisar, KS, Mondal, SR: Marichev-Saigo-Maeda fractional integration operators and generalized Mittag-Leffler function. *Thai J. Math.* (2016) (in press)
44. Podlubny, I: *Fractional Differential Equations*. Academic Press, San Diego (1999)
45. Sha, PV, Jana, RK, Patel, AD, Salehbbhai, IA: Some result based on generalized Mittag-Leffler function. *Int. J. Math. Anal.* **6**, 503-508 (2012)