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Analysis of a stochastic ratio-dependent one-predator and two-mutualistic-preys model with Markovian switching and Holling type III functional response

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Abstract

In this paper, we propose a stochastic ratio-dependent one-predator and two-mutualistic-preys model perturbed by white and telegraph noise. By the *M*-matrix analysis and Lyapunov functions, sufficient conditions of stochastic permanence and extinction are established. These conditions are all dependent on the parameters of subsystems and the stationary probability distribution of the Markov chain. We also obtain the boundary of limit superior and inferior of the average in time of the solution under stochastic permanence. Finally, we give two examples and numerical simulations to illustrate main results.

Keywords: stochastic permanence; extinction; Markovian switching; predator-prey model; mutualism

1 Introduction

Mutualism plays a key part in ecology, and many researchers have proposed different mathematical models to describe the mutualistic interaction [1–6]. For example, motivated by Holling type II functional response [7], Wright [5] established the Holling type II mutualistic model:

$$\begin{cases} \dot{x}(t) = x(t)\left(a_1 + \frac{b_1 y(t)}{k_1 + y(t)} - c_1 x(t)\right), \\ \dot{y}(t) = y(t)\left(a_2 + \frac{b_2 x(t)}{k_2 + x(t)} - c_2 y(t)\right). \end{cases} \tag{1.1}$$

For the biological meaning of parameters in the above model, please refer to [5, 6, 8].

Besides, many researchers have paid attention to the predator-prey model due to the universality of predator-prey interaction in the natural world. For the predator-prey model, the functional response is critical. Predator-prey models with Holling types I, II and III functional responses were investigated in [9–11]. The functional responses mentioned above are prey-dependent. But some mathematical analysis, laboratory experiments and observations [12–16] showed that in some circumstances, especially when predators had to search for food (and therefore had to share or strive for food), the ratio-dependent form of the functional response was more realistic and suitable. Based on Holling type II functional response $\frac{\alpha x}{\beta + x}$, the predator-prey model with a typical ratio-dependent-type

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functional response $\frac{\alpha x}{x+\beta y}$ has been further studied, such as [17, 18]. However, the Holling type III form $\frac{\alpha x^2}{\beta+x^2}$ is more successful in describing the feeding by vertebral predators compared to the Holling type II form, which is more suitable for the feeding by insects [19]. And researchers have considered the ratio-dependent predator-prey model with Holling type III functional response [10, 20].

Moreover, by considering the coexistence of antagonism, mutualism and competition, Mougi and Kondoh [21] showed that interaction-type diversity generally enhanced stability of complex communities. But limited work is available on predator-prey model with mutualism. Motivated by the above ideas, we consider the following ratio-dependent one-predator and two-mutualistic-preys model with Holling type III functional response:

$$\begin{cases} \dot{x}(t) = x(a_1 + \frac{b_1 y}{k_1+y} - c_1 x - \frac{e_1 xz}{x^2+f_1 z^2}), \\ \dot{y}(t) = y(a_2 + \frac{b_2 x}{k_2+x} - c_2 y - \frac{e_2 yz}{y^2+f_2 z^2}), \\ \dot{z}(t) = z(-a_3 + \frac{e_3 x^2}{x^2+f_1 z^2} + \frac{e_4 y^2}{y^2+f_2 z^2} - c_3 z), \end{cases} \tag{1.2}$$

where species x, y are two mutualistic preys and z is the predator. Please notice that we drop t from $x(t)$, $y(t)$ and $z(t)$ in model (1.2) and do that throughout this paper.

However, it is not enough to only consider certain factors. The biological system is more or less affected by stochastic fluctuations. One of these general fluctuations is white noise. Recently, many authors have studied stochastic models with white noise, such as [11, 22, 23]. This literature showed that taking white noise into account, the system would change significantly. In this paper, we assume that white noise affects the intrinsic birth rate and death rate, that is,

$$a_1 \to a_1 + \sigma_1 \dot{B}(t), \qquad a_2 \to a_2 + \sigma_2 \dot{B}(t), \qquad -a_3 \to -a_3 + \sigma_3 \dot{B}(t),$$

where $B(t)$, denoting white noise, is the standard Brownian motion, and σ_j^2 ($j = 1, 2, 3$) denotes the intensity of white noise.

In addition to white noise, the biological system is inevitably affected by another environment noise, that is, telegraph noise. This noise, which is distinguished by factors such as rain falls and nutrition, can be represented by a switching among two or more regimes of environment [24, 25]. Let $\{r(t), t \geq 0\}$ be a Markov chain controlling the switching among regimes and taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$. Then taking white and telegraph noise into consideration, on the basis of model (1.2), we finally develop the following stochastic ratio-dependent one-predator and two-mutualistic-preys model with Markovian switching and Holling type III functional response:

$$\begin{cases} \mathrm{d}x(t) = x(a_1(r(t)) + \frac{b_1(r(t))y}{k_1(r(t))+y} - c_1(r(t))x - \frac{e_1(r(t))xz}{x^2+f_1(r(t))z^2})\,\mathrm{d}t + \sigma_1(r(t))x\,\mathrm{d}B(t), \\ \mathrm{d}y(t) = y(a_2(r(t)) + \frac{b_2(r(t))x}{k_2(r(t))+x} - c_2(r(t))y - \frac{e_2(r(t))yz}{y^2+f_2(r(t))z^2})\,\mathrm{d}t + \sigma_2(r(t))y\,\mathrm{d}B(t), \\ \mathrm{d}z(t) = z(-a_3(r(t)) + \frac{e_3(r(t))x^2}{x^2+f_1(r(t))z^2} + \frac{e_4(r(t))y^2}{y^2+f_2(r(t))z^2} - c_3(r(t))z)\,\mathrm{d}t + \sigma_3(r(t))z\,\mathrm{d}B(t), \end{cases} \tag{1.3}$$

with the initial data $x(0) > 0$, $y(0) > 0$, $z(0) > 0$, $r(0) = r_0 \in \mathbb{S}$, where all parameters are nonnegative. In regime i ($i \in \mathbb{S}$), system (1.3) obeys

$$\begin{cases} \mathrm{d}x(t) = x(a_1(i) + \frac{b_1(i)y}{k_1(i)+y} - c_1(i)x - \frac{e_1(i)xz}{x^2+f_1(i)z^2})\,\mathrm{d}t + \sigma_1(i)x\,\mathrm{d}B(t), \\ \mathrm{d}y(t) = y(a_2(i) + \frac{b_2(i)x}{k_2(i)+x} - c_2(i)y - \frac{e_2(i)yz}{y^2+f_2(i)z^2})\,\mathrm{d}t + \sigma_2(i)y\,\mathrm{d}B(t), \\ \mathrm{d}z(t) = z(-a_3(i) + \frac{e_3(i)x^2}{x^2+f_1(i)z^2} + \frac{e_4(i)y^2}{y^2+f_2(i)z^2} - c_3(i)z)\,\mathrm{d}t + \sigma_3(i)z\,\mathrm{d}B(t). \end{cases} \tag{1.4}$$

Therefore, equation (1.4) is regarded as a subsystem of system (1.3). In this paper, our main aim is to reveal how two kinds of environment noise, that is, white and telegraph noise, affect permanence and extinction of system (1.3).

The stochastic differential equations controlled by a continuous Markov chain have been applied to the population models with telegraph noise. Li *et al.* [26] investigated the logistic population system without intra-specific competition incorporating white and telegraph noise, and mainly researched stochastic permanence and extinction. A two-dimensional stochastic predator-prey model with Markovian switching was developed by Ouyang and Li [18], and they explored permanence and asymptotical behavior. Nevertheless, for the stochastic predator-prey model with Markovian switching, most of the work focused on two-dimensional systems. To the best of our knowledge, there is no work on 3-dimensional stochastic ratio-dependent predator-prey models with Markovian switching, two mutualistic preys and Holling type III functional responses till now.

We arrange the rest of this paper as follows. In Section 2, we prepare some notations and consider the existence and uniqueness of the solution of system (1.3). By means of the M-matrix analysis and Lyapunov functions, we study stochastically ultimate boundedness and stochastic permanence, and the sufficient condition of stochastic permanence is given in Section 3. Section 4 gives the sample Lyapunov exponent and hence shows the sufficient condition of extinction. We obtain the boundary of limit superior and inferior of the average in time of the solution under stochastic permanence in Section 5. In Section 6, we give two examples and make numerical simulations to illustrate main results. In Section 7, we give conclusions.

2 Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.* it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Denote by $\tilde{\mathbb{R}}^n_+$ the nonnegative cone in $\mathbb{R}^n$, and denote by $\mathbb{R}^n_+$ the positive cone in $\mathbb{R}^n$. Denote by $X(t) = (x(t), y(t), z(t))$ a solution of system (1.3) and its norm is defined by $|X(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)}$.

Assume that $r(t)$ is a right-continuous Markov chain taking values in the finite state space $\mathbb{S}$ with the generator $\Gamma = (\gamma_{mn})_{N\times N}$ defined by

$$\mathbb{P}\big\{r(t+\delta) = n | r(t) = m\big\} = \begin{cases} \gamma_{mn}\delta + o(\delta), & \text{if } n \neq m, \\ 1 + \gamma_{mn}\delta + o(\delta), & \text{if } n = m, \end{cases}$$

where $\delta > 0$. Here γ_{mn} is the transition rate from regime m to regime n and $\gamma_{mn} \geq 0$ if $n \neq m$, while

$$\gamma_{mm} = -\sum_{n=1,n\neq m}^{N} \gamma_{mn}, \quad \forall m \in \mathbb{S}. \tag{2.1}$$

We also assume that the Markov chain $r(t)$ is independent of the Brownian motion $B(t)$ and irreducible (*e.g.*, see [27]). Under this assumption, $r(t)$ has a unique stationary probability distribution $\pi = (\pi_1, \pi_2, \dots, \pi_N) \in \mathbb{R}^{1\times N}$, depending on the equation

$$\pi\Gamma = 0 \tag{2.2}$$

subject to

$$\sum_{i=1}^{N} \pi_i = 1 \quad \text{and} \quad \pi_i > 0, \quad \forall i \in \mathbb{S}.$$

Let $L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ denote the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variables ξ with $\mathbf{E}|\xi|^2 < \infty$ and $L_{\mathcal{F}_t}(\Omega;\mathbb{S})$ denote the family of $\mathbb{S}$-valued $\mathcal{F}_t$-measurable random variables. Now consider an n-dimensional stochastic differential equation with Markovian switching,

$$\mathrm{d}x(t) = f\bigl(x(t), r(t)\bigr)\,\mathrm{d}t + g\bigl(x(t), r(t)\bigr)\,\mathrm{d}B(t),$$

on $t \geq 0$ with initial data $x(0) \in L^2_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)$, $r(0) \in L_{\mathcal{F}_0}(\Omega;\mathbb{S})$ and

$$f : \mathbb{R}^n \times \tilde{\mathbb{R}}_+ \times \mathbb{S} \to \mathbb{R}^n, \qquad g : \mathbb{R}^n \times \tilde{\mathbb{R}}_+ \times \mathbb{S} \to \mathbb{R}^n.$$

Moreover, let $C^{2,1}(\mathbb{R}^n \times \tilde{\mathbb{R}}_+ \times \mathbb{S};\mathbb{R})$ denote the family of all real-valued functions $V(x,t,i)$ on $\mathbb{R}^n \times \tilde{\mathbb{R}}_+ \times \mathbb{S}$ which are continuously twice differentiable in x and once in t. If $V \in C^{2,1}(\mathbb{R}^n \times \tilde{\mathbb{R}}_+ \times \mathbb{S};\mathbb{R})$, define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \tilde{\mathbb{R}}_+ \times \mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(x,t,i) = {} & V_t(x,t,i) + V_x(x,t,i)f(x,i) \\ & + \frac{1}{2}\operatorname{trace}\bigl[g^T(x,i)V_{xx}(x,t,i)g(x,i)\bigr] + \sum_{j=1}^{N} \gamma_{ij} V(x,t,j), \end{aligned} \tag{2.3}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}, \qquad V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \frac{\partial V(x,t,i)}{\partial x_2}, \ldots, \frac{\partial V(x,t,i)}{\partial x_n}\right)$$

and

$$V_{xx}(x,t,i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_i\,\partial x_j}\right)_{n\times n}.$$

For convenience and simplicity, we give the following notations:

$$\begin{aligned} &\hat{g} = \min_{i\in\mathbb{S}} g(i), \qquad \check{g} = \max_{i\in\mathbb{S}} g(i), \\ &F_1\bigl(x,y,z,r(t)\bigr) = a_1\bigl(r(t)\bigr) + \frac{b_1(r(t))y}{k_1(r(t)) + y} - c_1\bigl(r(t)\bigr)x - \frac{e_1(r(t))xz}{x^2 + f_1(r(t))z^2}, \\ &F_2\bigl(x,y,z,r(t)\bigr) = a_2\bigl(r(t)\bigr) + \frac{b_2(r(t))x}{k_2(r(t)) + x} - c_2\bigl(r(t)\bigr)y - \frac{e_2(r(t))yz}{y^2 + f_2(r(t))z^2}, \\ &F_3\bigl(x,y,z,r(t)\bigr) = -a_3\bigl(r(t)\bigr) + \frac{e_3(r(t))x^2}{x^2 + f_1(r(t))z^2} + \frac{e_4(r(t))y^2}{y^2 + f_2(r(t))z^2} - c_3\bigl(r(t)\bigr)z. \end{aligned}$$

Assumption A1 $c_1(i) > 0$, $c_2(i) > 0$, $c_3(i) > 0$, $\forall i \in \mathbb{S}$.

In order to study the dynamic behavior, we must first guarantee that there exists a unique, positive, and global solution.

Theorem 2.1 *Under Assumption* A1, *there is a unique positive solution of system* (1.3) *on* $t \geq 0$, *which remains in* $\mathbb{R}^3_+$ *with probability* 1.

Proof Define a function $V : \mathbb{R}^3_+ \times \tilde{\mathbb{R}}_+ \times \mathbb{S} \to \mathbb{R}_+$ by

$$V\big(X(t), t, i\big) = (x - 1 - \log x) + (y - 1 - \log y) + (z - 1 - \log z).$$

This proof is standard, please refer to [28, 29] as we omit it. □

3 Stochastic permanence

In this section, we consider stochastic permanence and first study stochastically ultimate boundedness.

Lemma 3.1 *Let the three constants* $\alpha_1, \alpha_2, \alpha_3 > 0$. *Under Assumption* A1, *there is a constant* $H_1 = H_1(\alpha_1, \alpha_2, \alpha_3) > 0$ *such that the solution* $X(t)$ *of system* (1.3) *satisfies*

$$\limsup_{t\to\infty} \mathbf{E}\big(x^{\alpha_1}(t) + y^{\alpha_2}(t) + z^{\alpha_3}(t)\big) \leq H_1.$$

Proof Define a function

$$V(x, y, z, t, i) = x^{\alpha_1} + y^{\alpha_2} + z^{\alpha_3}.$$

By means of the generalized Itô formula (*e.g.*, see [27]), we get

$$\begin{aligned}
&\mathcal{L}V(x, y, z, t, i) \\
&\quad = \frac{1}{2}\alpha_1(\alpha_1 - 1)\sigma_1^2(i)x^{\alpha_1} + \frac{1}{2}\alpha_2(\alpha_2 - 1)\sigma_2^2(i)y^{\alpha_2} + \frac{1}{2}\alpha_3(\alpha_3 - 1)\sigma_3^2(i)z^{\alpha_3} \\
&\qquad + \alpha_1 x^{\alpha_1}F_1(x, y, z, i) + \alpha_2 y^{\alpha_2}F_2(x, y, z, i) + \alpha_3 z^{\alpha_3}F_3(x, y, z, i).
\end{aligned}$$

Note that the coefficients of the higher order terms x^{α_1+1}, y^{α_2+1}, z^{α_3+1} in the above equality are all negative under Assumption A1, there is a positive constant $H_1 := H_1(\alpha_1, \alpha_2, \alpha_3)$ such that

$$V(x, y, z, t, i) + \mathcal{L}V(x, y, z, t, i) \leq H_1.$$

Then applying the generalized Itô formula to $e^t V(x, y, z, i)$, we get

$$\mathcal{L}\big(e^t V(x, y, z, t, i)\big) = e^t\big[V(x, y, z, t, i) + \mathcal{L}V(x, y, z, t, i)\big] \leq H_1 e^t.$$

Integrating both sides of $\mathrm{d}(e^t V(x, y, z, t, i))$ from 0 to t, taking the expectation and taking the limit superior, we finally obtain the desired conclusion. □

Theorem 3.1 *Under Assumption* A1, *system* (1.3) *is stochastically ultimately bounded*.

Proof By the definition of stochastically ultimate boundedness (*e.g.*, see [26]), the conclusion follows from Lemma 3.1 and Chebyshev's inequality. □

Next we investigate stochastic permanence. Based on the above conclusion, we only need to prove another inequality about stochastic permanence (*e.g.*, see [26]). And one of main methods in this section is the M-matrix analysis which was introduced by [27] and used in [18, 26].

Now we give notations, a lemma and some assumptions. Let A be a vector or a matrix. Denote by $A \gg 0$ all elements of A that are positive. Set

$$Z^{N\times N} = \{A = (a_{ij})_{N\times N} : a_{ij} \le 0, i \ne j\}.$$

Lemma 3.2 (*e.g.*, see [27]) *If* $A \in Z^{N\times N}$, *then the following statements are equivalent*:
(i) *A is a nonsingular M-matrix*.
(ii) *A is semi-positive*; *that is*, *there exists* $x \gg 0$ *in* $\mathbb{R}^N$ *such that* $Ax \gg 0$.

Assumption A2 For some $n \in \mathbb{S}$, $\gamma_{in} > 0$, $\forall i \ne n$.

Assumption A3 $f_1(i) > 0, f_2(i) > 0$, $\forall i \in \mathbb{S}$, and $\sum_{i\in\mathbb{S}} \pi_i q(i) > 0$, where

$$\begin{aligned} q(i) = \min&\left\{2a_1(i) - \frac{e_1(i)}{\sqrt{\hat{f}_1}}, 2a_2(i) - \frac{e_2(i)}{\sqrt{\hat{f}_2}}, e_3(i) + e_4(i) - 2a_3(i)\right\} \\ &- \frac{1}{2}\max\{3\sigma_1^2(i) + 3\sigma_2^2(i), 3\sigma_1^2(i) + 3\sigma_3^2(i) + e_3(i) + e_4(i), \\ &3\sigma_2^2(i) + 3\sigma_3^2(i) + e_3(i) + e_4(i)\} \\ &- \max\{\sigma_1^2(i), \sigma_2^2(i), \sigma_3^2(i)\}. \end{aligned}$$

Assumption A4 For some $i \in \mathbb{S}$, $c_j(i) > 0$ $(j = 1, 2, 3)$, $f_k(i) > 0$ $(k = 1, 2)$ and $q(i) > 0$, where $q(i)$ is defined in the above assumption.

The proof of stochastic permanence is rather long and technical. To make it more understandable, we divide the proof into several lemmas.

Lemma 3.3 *Assumptions* A2 *and* A3 *imply that there exists a constant* $\alpha > 0$ *such that the matrix*

$$G(\alpha) = \operatorname{diag}\{\beta_1(\alpha), \beta_2(\alpha), \dots, \beta_N(\alpha)\} - \Gamma \tag{3.1}$$

is a nonsingular M-matrix, *where*

$$\beta_i(\alpha) = q(i)\alpha - 2\max\{\sigma_1^2(i), \sigma_2^2(i), \sigma_3^2(i)\}\alpha^2, \quad i \in \mathbb{S}. \tag{3.2}$$

Proof This proof is common, please refer to [26] as we omit it. □

Lemma 3.4 *Let $\hat{f}_1 > 0$ and $\hat{f}_2 > 0$. If there is a constant $\alpha > 0$ such that $G(\alpha)$ is a nonsingular M-matrix, then the solution $X(t)$ of system* (1.3) *satisfies*

$$\limsup_{t\to\infty} \mathbf{E}\frac{1}{x^{2\alpha}(t)} \le H_1(\alpha), \qquad \limsup_{t\to\infty} \mathbf{E}\frac{1}{y^{2\alpha}(t)} \le H_2(\alpha),$$
$$\limsup_{t\to\infty} \mathbf{E}\frac{1}{z^{2\alpha}(t)} \le H_3(\alpha),$$

where $H_1(\alpha)$, $H_2(\alpha)$, $H_3(\alpha)$ are positive constants.

Proof Define

$$u = \frac{1}{x}, \qquad v = \frac{1}{y}, \qquad w = \frac{1}{z}.$$

By the generalized Itô formula, we have

$$\begin{cases} \mathrm{d}u(t) = u(-a_1(r(t)) + \sigma_1^2(r(t)) - \frac{b_1(r(t))}{k_1(r(t))v+1} + \frac{c_1(r(t))}{u} + \frac{e_1(r(t))uw}{w^2+f_1(r(t))u^2})\,\mathrm{d}t \\ \qquad - \sigma_1(r(t))u\,\mathrm{d}B(t), \\ \mathrm{d}v(t) = v(-a_2(r(t)) + \sigma_2^2(r(t)) - \frac{b_2(r(t))}{k_2(r(t))u+1} + \frac{c_2(r(t))}{v} + \frac{e_2(r(t))vw}{w^2+f_2(r(t))v^2})\,\mathrm{d}t \\ \qquad - \sigma_2(r(t))v\,\mathrm{d}B(t), \\ \mathrm{d}w(t) = w(a_3(r(t)) + \sigma_3^2(r(t)) - \frac{e_3(r(t))w^2}{w^2+f_1(r(t))u^2} - \frac{e_4(r(t))w^2}{w^2+f_2(r(t))v^2} + \frac{c_3(r(t))}{w})\,\mathrm{d}t \\ \qquad - \sigma_3(r(t))w\,\mathrm{d}B(t). \end{cases} \tag{3.3}$$

For given $\alpha > 0$, by Lemma 3.2, there exists a vector $\vec{\eta} = (\eta_1, \dots, \eta_N)^T \gg 0$ such that $G(\alpha)\vec{\eta} \gg 0$, that is,

$$\beta_i(\alpha)\eta_i - \sum_{j=1}^{N} \gamma_{ij}\eta_j > 0, \quad i \in \mathbb{S}. \tag{3.4}$$

Define again

$$V(u,v,w,t,i) = \eta_i\bigl(1 + \check{f}_1 u^2 + \check{f}_2 v^2 + w^2\bigr)^\alpha.$$

By the generalized Itô formula, we have

$$\begin{aligned} &\mathcal{L}V(u,v,w,t,i) \\ &\quad = \alpha\eta_i\check{f}_1\bigl(1 + \check{f}_1 u^2 + \check{f}_2 v^2 + w^2\bigr)^{\alpha-1} \\ &\qquad \cdot \left(2\bigl(-a_1(i) + \sigma_1^2(i)\bigr)u^2 - \frac{2b_1(i)u^2}{k_1(i)v+1} + 2c_1(i)u + \frac{2e_1(i)u^3w}{w^2+f_1(i)u^2}\right) \\ &\qquad + \alpha\eta_i\check{f}_2\bigl(1 + \check{f}_1 u^2 + \check{f}_2 v^2 + w^2\bigr)^{\alpha-1} \\ &\qquad \cdot \left(2\bigl(-a_2(i) + \sigma_2^2(i)\bigr)v^2 - \frac{2b_2(i)v^2}{k_2(i)u+1} + 2c_2(i)v + \frac{2e_2(i)v^3w}{w^2+f_2(i)v^2}\right) \\ &\qquad + \alpha\eta_i\bigl(1 + \check{f}_1 u^2 + \check{f}_2 v^2 + w^2\bigr)^{\alpha-1} \\ &\qquad \cdot \left(2\bigl(a_3(i) + \sigma_3^2(i)\bigr)w^2 - \frac{2e_3(i)w^4}{w^2+f_1(i)u^2} - \frac{2e_4(i)w^4}{w^2+f_2(i)v^2} + 2c_3(i)w\right) \end{aligned}$$

$$
\begin{aligned}
&+ \alpha\eta_i\bigl(1+\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^{\alpha-2}\bigl[\bigl(\check{f}_1+(2\alpha-1)\check{f}_1^2u^2+\check{f}_1\check{f}_2v^2+\check{f}_1w^2\bigr)\sigma_1^2(i)u^2 \\
&+ \bigl(\check{f}_2+\check{f}_1\check{f}_2u^2+(2\alpha-1)\check{f}_2^2v^2+\check{f}_2w^2\bigr)\sigma_2^2(i)v^2+\bigl(1+\check{f}_1u^2+\check{f}_2v^2+(2\alpha-1)w^2\bigr)\sigma_3^2(i)w^2 \\
&+ (4\alpha-4)\check{f}_1\check{f}_2\sigma_1(i)\sigma_2(i)u^2v^2+(4\alpha-4)\check{f}_1\sigma_1(i)\sigma_3(i)u^2w^2 \\
&+ (4\alpha-4)\check{f}_2\sigma_2(i)\sigma_3(i)v^2w^2\bigr]+\sum_{j=1}^{N}\gamma_{ij}\eta_j\bigl(1+\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^{\alpha} \\
\le{}& \bigl(1+\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^{\alpha-2}\Biggl\{\alpha\eta_i\check{f}_1\bigl(\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)u^2\Biggl(\frac{e_1(i)}{\sqrt{\hat{f}_1}}-2a_1(i)+2\sigma_1^2(i)\Biggr) \\
&+ \alpha\eta_i\check{f}_2\bigl(\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)v^2\Biggl(\frac{e_2(i)}{\sqrt{\hat{f}_2}}-2a_2(i)+2\sigma_2^2(i)\Biggr) \\
&+ \alpha\eta_i\bigl[w^4\bigl(2a_3(i)+2\sigma_3^2(i)-e_3(i)-e_4(i)\bigr)+\check{f}_1u^2w^2\bigl(2a_3(i)+2\sigma_3^2(i)\bigr) \\
&+ \check{f}_2v^2w^2\bigl(2a_3(i)+2\sigma_3^2(i)\bigr)\bigr] \\
&+ \alpha\eta_i\bigl[(2\alpha-1)f_1^2(i)\sigma_1^2(i)u^4+(2\alpha-1)f_2^2(i)\sigma_2^2(i)v^4+(2\alpha-1)\sigma_3^2(i)w^4 \\
&+ \check{f}_1\check{f}_2u^2v^2\bigl(\sigma_1^2(i)+\sigma_2^2(i)\bigr)+\check{f}_1u^2w^2\bigl(\sigma_1^2(i)+\sigma_3^2(i)\bigr)+\check{f}_2v^2w^2\bigl(\sigma_2^2(i)+\sigma_3^2(i)\bigr) \\
&+ (4\alpha-2)\sigma_1(i)\sigma_2(i)\check{f}_1\check{f}_2u^2v^2+(4\alpha-2)\sigma_1(i)\sigma_3(i)\check{f}_1u^2w^2 \\
&+ (4\alpha-2)\sigma_2(i)\sigma_3(i)\check{f}_2v^2w^2\bigr]+\sum_{j=1}^{N}\gamma_{ij}\eta_j\bigl(\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^2+H_1(u,v,w,i)\Biggr\} \\
\le{}& \bigl(1+\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^{\alpha-2}\Bigl\{2\alpha^2\eta_i\bigl(\sigma_1(i)\check{f}_1u^2+\sigma_2(i)\check{f}_2v^2+\sigma_3(i)w^2\bigr)^2 \\
&+ \alpha\eta_i\bigl(\sigma_1(i)\check{f}_1u^2-\sigma_2(i)\check{f}_2v^2-\sigma_3(i)w^2\bigr)^2 \\
&+ \alpha\eta_i\bigl(\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)\Biggl[\check{f}_1u^2\Biggl(\frac{e_1(i)}{\sqrt{\hat{f}_1}}-2a_1(i)\Biggr) \\
&+ \check{f}_2v^2\Biggl(\frac{e_2(i)}{\sqrt{\hat{f}_2}}-2a_2(i)\Biggr)+w^2\bigl(2a_3(i)-e_3(i)-e_4(i)\bigr)\Biggr] \\
&+ \frac{\alpha\eta_i}{2}\max\bigl\{3\sigma_1^2(i)+3\sigma_2^2(i),3\sigma_1^2(i)+3\sigma_3^2(i)+e_3(i)+e_4(i), \\
&3\sigma_2^2(i)+3\sigma_3^2(i)+e_3(i)+e_4(i)\bigr\}\bigl(\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^2 \\
&+ \sum_{j=1}^{N}\gamma_{ij}\eta_j\bigl(\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^2+H_1(u,v,w,i)\Bigr\} \\
\le{}& \bigl(1+\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^{\alpha-2}\Bigl\{\bigl(\check{f}_1u^2+\check{f}_2v^2+w^2\bigr)^2 \\
&\cdot\Biggl[-\alpha\eta_i\Biggl(\min\Biggl\{2a_1(i)-\frac{e_1(i)}{\sqrt{\hat{f}_1}},2a_2(i)-\frac{e_2(i)}{\sqrt{\hat{f}_2}},e_3(i)+e_4(i)-2a_3(i)\Biggr\} \\
&- \frac{1}{2}\max\bigl\{3\sigma_1^2(i)+3\sigma_2^2(i),3\sigma_1^2(i)+3\sigma_3^2(i)+e_3(i)+e_4(i),
\end{aligned}
$$

$$3\sigma_2^2(i) + 3\sigma_3^2(i) + e_3(i) + e_4(i)\big\} - \max\big\{\sigma_1^2(i), \sigma_2^2(i), \sigma_3^2(i)\big\}\bigg)$$
$$+ 2\alpha^2\eta_i \max\big\{\sigma_1^2(i), \sigma_2^2(i), \sigma_3^2(i)\big\} + \sum_{j=1}^{N}\gamma_{ij}\eta_j\Bigg] + H_1(u,v,w,i)\Bigg\},$$

where $H_1(u,v,w,i)$ is a cubic polynomial as regards u, v, w. Under (3.4), there is a sufficiently small constant $l > 0$ such that $G(\alpha)\vec{\eta} - l\vec{\eta} \gg 0$, that is,

$$\beta_i(\alpha)\eta_i - \sum_{j=1}^{N}\gamma_{ij}\eta_j - l\eta_i > 0, \quad i \in \mathbb{S}. \tag{3.5}$$

Applying the generalized Itô formula to $e^{lt}V(u,v,w,t,i)$ and noticing (3.2), we obtain

$$\begin{aligned}
&\mathcal{L}\big[e^{lt}V(u,v,w,t,i)\big] \\
&\quad = le^{lt}V(u,v,w,t,i) + e^{lt}\mathcal{L}V(u,v,w,t,i) \\
&\quad \le e^{lt}\Bigg\{l\eta_i\big(1 + \check{f}_1u^2 + \check{f}_2v^2 + w^2\big)^\alpha + \big(1 + \check{f}_1u^2 + \check{f}_2v^2 + w^2\big)^{\alpha-2}\Bigg[\big(\check{f}_1u^2 + \check{f}_2v^2 + w^2\big)^2 \\
&\qquad \cdot \Bigg(-\beta_i(\alpha)\eta_i + \sum_{j=1}^{N}\gamma_{ij}\eta_j\Bigg) + H_1(u,v,w,i)\Bigg]\Bigg\} \\
&\quad \le e^{lt}\big(1 + \check{f}_1u^2 + \check{f}_2v^2 + w^2\big)^{\alpha-2}\Bigg\{-\Bigg[\beta_i(\alpha)\eta_i - \sum_{j=1}^{N}\gamma_{ij}\eta_j - l\eta_i\Bigg] \\
&\qquad \cdot \big(\check{f}_1u^2 + \check{f}_2v^2 + w^2\big)^2 + H_2(u,v,w,i)\Bigg\},
\end{aligned}$$

where $H_1(u,v,w,i)$ is also a cubic polynomial as regards u, v, w. By (3.5), it is obvious that $\mathcal{L}[e^{lt}V(u,v,w,t,i)] \le H_{22}e^{lt}$, where

$$\begin{aligned}
H_{22} = \max_{i\in\mathbb{S}}\Bigg\{&\sup_{t\ge-\tau}\big(1 + \check{f}_1u^2 + \check{f}_2v^2 + w^2\big)^{\alpha-2}\Bigg\{-\Bigg[\beta_i(\alpha)\eta_i - \sum_{j=1}^{N}\gamma_{ij}\eta_j - l\eta_i\Bigg] \\
&\cdot \big(\check{f}_1u^2 + \check{f}_2v^2 + w^2\big)^2 + H_2(u,v,w,i)\Bigg\}, 1\Bigg\}.
\end{aligned}$$

Thus,

$$\limsup_{t\to\infty}\mathbf{E}\big(1 + \check{f}_1u^2(t) + \check{f}_2v^2(t) + w^2(t)\big)^\alpha \le \frac{H_{22}}{l\hat{\eta}}.$$

Hence,

$$\limsup_{t\to\infty}\mathbf{E}\frac{1}{x^{2\alpha}(t)} \le \frac{H_{22}}{l\hat{\eta}\check{f}_1^\alpha} := H_1(\alpha), \qquad \limsup_{t\to\infty}\mathbf{E}\frac{1}{y^{2\alpha}(t)} \le \frac{H_{22}}{l\hat{\eta}\check{f}_2^\alpha} := H_2(\alpha),$$
$$\limsup_{t\to\infty}\mathbf{E}\frac{1}{z^{2\alpha}(t)} \le \frac{H_{22}}{l\hat{\eta}} := H_3(\alpha).$$

□

Theorem 3.2 *Under Assumptions* A1, A2 *and* A3, *system* (1.3) *is stochastically permanent*.

Proof The desired conclusion can be directly obtained by Lemma 3.3, Lemma 3.4, Chebyshev's inequality and Theorem 3.1. □

By a similar method to Theorem 3.2, we directly get the following conclusion for subsystem (1.4).

Corollary 3.1 *Under Assumption* A4, *subsystem* (1.4) *is stochastically permanent*.

4 Extinction

In this section, we discuss the sample Lyapunov exponent of system (1.3) and hence get the sufficient condition for three species to be extinct.

Theorem 4.1 *The solution* $X(t)$ *of system* (1.3) *has the property*:

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\ln x(t)}{t}\leq\sum_{i\in\mathbb{S}}\pi_i\left(a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i)\right)\quad a.s.,\\
&\limsup_{t\to\infty}\frac{\ln y(t)}{t}\leq\sum_{i\in\mathbb{S}}\pi_i\left(a_2(i)+b_2(i)-\frac{1}{2}\sigma_2^2(i)\right)\quad a.s.,\\
&\limsup_{t\to\infty}\frac{\ln z(t)}{t}\leq\sum_{i\in\mathbb{S}}\pi_i\left(e_3(i)+e_4(i)-a_3(i)-\frac{1}{2}\sigma_3^2(i)\right)\quad a.s.
\end{aligned}$$

Particularly, if

$$\max\left\{\sum_{i\in\mathbb{S}}\pi_i\left(a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i)\right),\sum_{i\in\mathbb{S}}\pi_i\left(a_2(i)+b_2(i)-\frac{1}{2}\sigma_2^2(i)\right)\right\}<0,$$

then

$$\lim_{t\to\infty}x(t)=0,\qquad\lim_{t\to\infty}y(t)=0,\qquad\lim_{t\to\infty}z(t)=0\quad a.s.$$

Proof By the generalized Itô formula, we have

$$\begin{aligned}
\mathrm{d}\ln x(t)&=\left(a_1\bigl(r(t)\bigr)+\frac{b_1(r(t))y}{k_1(r(t))+y}-c_1\bigl(r(t)\bigr)x-\frac{e_1(r(t))xz}{x^2+f_1(r(t))z^2}-\frac{1}{2}\sigma_1^2\bigl(r(t)\bigr)\right)\mathrm{d}t\\
&\quad+\sigma_1\bigl(r(t)\bigr)\,\mathrm{d}B(t)\\
&\leq\left(a_1\bigl(r(t)\bigr)+b_1\bigl(r(t)\bigr)-\frac{1}{2}\sigma_1^2\bigl(r(t)\bigr)\right)\mathrm{d}t+\check{\sigma}_1\,\mathrm{d}B(t).
\end{aligned}$$

Integrating from 0 to t on both sides of the above inequality, taking the limit superior, by the strong law of large numbers and the ergodic property of Markov chain (*e.g.*, see [27]), note that $\lim t\to\infty B(t)/t=0$, we obtain

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t}\leq\sum_{i\in\mathbb{S}}\pi_i\left(a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i)\right)\quad\text{a.s.}$$

By the above same methods and procedures, we have

$$\limsup_{t\to\infty}\frac{\ln y(t)}{t}\le\sum_{i\in\mathbb{S}}\pi_i\left(a_2(i)+b_2(i)-\frac{1}{2}\sigma_2^2(i)\right)\quad\text{a.s.},$$
$$\limsup_{t\to\infty}\frac{\ln z(t)}{t}\le\sum_{i\in\mathbb{S}}\pi_i\left(e_3(i)+e_4(i)-a_3(i)-\frac{1}{2}\sigma_3^2(i)\right)\quad\text{a.s.}$$

Particularly, if

$$\max\left\{\sum_{i\in\mathbb{S}}\pi_i\left(a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i)\right),\sum_{i\in\mathbb{S}}\pi_i\left(a_2(i)+b_2(i)-\frac{1}{2}\sigma_2^2(i)\right)\right\}<0,$$

then

$$\lim_{t\to\infty}x(t)=0,\qquad\lim_{t\to\infty}y(t)=0\quad\text{a.s.}\tag{4.1}$$

Noticing that the third equation of system (1.3), it is clear that if (4.1) holds, then $\lim_{t\to\infty}z(t)=0$ a.s. □

On the basis of the above theorem, we directly give the following corollary as regards the subsystem's extinction.

Corollary 4.1 *For subsystem* (1.4), *if the solution* $(x(t),y(t),z(t))$ *satisfies* $\max\{a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i)<0,a_2(i)+b_2(i)-\frac{1}{2}\sigma_2^2(i)\}<0$, *then*

$$\lim_{t\to\infty}x(t)=0,\qquad\lim_{t\to\infty}y(t)=0,\qquad\lim_{t\to\infty}z(t)=0\quad\textit{a.s.}$$

5 Asymptotic properties

In this section, we consider asymptotic properties of system (1.3) and then obtain the boundary of limit superior and inferior of the average in time of the solution under stochastic permanence.

Lemma 5.1 *Under Assumption* A1, *the solution* $X(t)$ *of system* (1.3) *satisfies*

$$\limsup_{t\to\infty}\frac{\ln[x(t)+y(t)+z(t)]}{\ln t}\le 1\quad\textit{a.s.}$$

Proof By the generalized Itô formula, we have

$$\begin{aligned}
&\mathrm{d}\big[x(t)+y(t)+z(t)\big]\\
&\quad=\big(F_1\big(x,y,z,r(t)\big)+F_2\big(x,y,z,r(t)\big)+F_3\big(x,y,z,r(t)\big)\big)\,\mathrm{d}t\\
&\qquad+\big(\sigma_1\big(r(t)\big)x+\sigma_2\big(r(t)\big)y+\sigma_3\big(r(t)\big)z\big)\,\mathrm{d}B(t)\\
&\quad\le\big[(\check{a}_1+\check{b}_1)x+(\check{a}_2+\check{b}_2)y+(\check{e}_3+\check{e}_4)z\big]\,\mathrm{d}t\\
&\qquad+\big(\sigma_1\big(r(t)\big)x+\sigma_2\big(r(t)\big)y+\sigma_3\big(r(t)\big)z\big)\,\mathrm{d}B(t).
\end{aligned}$$

Then let $g \triangleq \max\{\check{a}_1 + \check{b}_1, \check{a}_2 + \check{b}_2, \check{e}_3 + \check{e}_4\}$, we get

$$\begin{aligned}
&\mathbf{E}\Big(\sup_{t\le u\le t+1}\big[x(u)+y(u)+z(u)\big]\Big) \\
&\quad\le \mathbf{E}\big(x(t)+y(t)+z(t)\big) + g\int_t^{t+1}\mathbf{E}\big(x(s)+y(s)+z(s)\big)\,\mathrm{d}s \\
&\qquad + \mathbf{E}\Big(\sup_{t\le u\le t+1}\int_t^u \big(\sigma_1\big(r(s)\big)x(s)+\sigma_2\big(r(s)\big)y(s)+\sigma_3\big(r(s)\big)z(s)\big)\,\mathrm{d}B(s)\Big).
\end{aligned} \tag{5.1}$$

By Lemma 3.1, there is a positive constant H such that

$$\lim_{t\to\infty}\mathbf{E}\big[x(t)+y(t)+z(t)\big] \le H. \tag{5.2}$$

By the special case of Burkholder-Davis-Gundy inequality (*e.g.*, see [27], p.76), and let $h \triangleq \max\{\check{\sigma}_1^2, \check{\sigma}_2^2, \check{\sigma}_3^2\}$, we obtain

$$\begin{aligned}
&\mathbf{E}\Big(\sup_{t\le u\le t+1}\int_t^u \big(\sigma_1\big(r(s)\big)x(s)+\sigma_2\big(r(s)\big)y(s)+\sigma_3\big(r(s)\big)z(s)\big)\,\mathrm{d}B(s)\Big) \\
&\quad\le 3\mathbf{E}\Big(\int_t^{t+1} h\big(x(s)+y(s)+z(s)\big)^2\,\mathrm{d}s\Big)^{1/2} = \mathbf{E}\Big(9h\int_t^{t+1}\big(x(s)+y(s)+z(s)\big)^2\,\mathrm{d}s\Big)^{1/2} \\
&\quad\le \mathbf{E}\Big(\sup_{t\le u\le t+1}\big[x(u)+y(u)+z(u)\big]9h\int_t^{t+1}\big(x(s)+y(s)+z(s)\big)\,\mathrm{d}s\Big)^{1/2} \\
&\quad\le \mathbf{E}\Big(\frac{1}{2}\sup_{t\le u\le t+1}\big[x(u)+y(u)+z(u)\big] + \frac{9}{2}h\int_t^{t+1}\big(x(s)+y(s)+z(s)\big)\,\mathrm{d}s\Big) \\
&\quad\le \frac{1}{2}\mathbf{E}\Big(\sup_{t\le u\le t+1}\big[x(u)+y(u)+z(u)\big]\Big) + \frac{9}{2}h\int_t^{t+1}\mathbf{E}\big(x(s)+y(s)+z(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Substituting the above inequality into (5.1), we have

$$\begin{aligned}
&\mathbf{E}\Big(\sup_{t\le u\le t+1}\big[x(u)+y(u)+z(u)\big]\Big) \\
&\quad\le 2\mathbf{E}\big(x(t)+y(t)+z(t)\big) + (2g+9h)\int_t^{t+1}\mathbf{E}\big(x(s)+y(s)+z(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Let $t\to\infty$ and by (5.2), we get

$$\lim_{t\to\infty}\mathbf{E}\Big(\sup_{t\le u\le t+1}\big[x(u)+y(u)+z(u)\big]\Big) \le (2+2g+9h)H.$$

Then, for $k = 1, 2, \ldots$, there exists a positive constant $\bar{H}$ such that

$$\mathbf{E}\Big(\sup_{k\le t\le k+1}\big[x(t)+y(t)+z(t)\big]\Big) \le \bar{H}.$$

Let $\epsilon > 0$ be arbitrary. Then, for $k = 1, 2, \ldots$, by Chebyshev inequality, we get

$$\mathbb{P}\Big\{\sup_{k\le t\le k+1}\big[x(t)+y(t)+z(t)\big] > k^{1+\epsilon}\Big\} \le \frac{\mathbf{E}(\sup_{k\le t\le k+1}[x(t)+y(t)+z(t)])}{k^{1+\epsilon}} \le \frac{\bar{H}}{k^{1+\epsilon}}.$$

By Borel-Cantelli lemma (*e.g.*, see [27]), there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $w \in \Omega_0$, there is an integer $k_0 = k_0(w)$ such that

$$x(t) + y(t) + z(t) \leq k^{1+\epsilon}$$

for all $k \leq t \leq k+1$ and $k \geq k_0(w)$. Therefore, for any $k \leq t \leq k+1$ and $k \geq k_0(w)$,

$$\frac{\ln[x(t) + y(t) + z(t)]}{\ln t} \leq 1 + \epsilon.$$

Thus,

$$\limsup_{t\to\infty} \frac{\ln[x(t) + y(t) + z(t)]}{\ln t} \leq 1 + \epsilon \quad \text{a.s.}$$

Let $\epsilon \to 0$, we get the desired conclusion. □

Lemma 5.2 *Let $\hat{f}_1 > 0$ and $\hat{f}_2 > 0$. If there is a constant $\alpha > 0$ such that $G(\alpha)$ is a nonsingular M-matrix, then the solution $X(t)$ of system* (1.3) *satisfies*

$$\liminf_{t\to\infty} \frac{\ln x(t)}{\ln t} \geq -\frac{1}{\alpha}, \qquad \liminf_{t\to\infty} \frac{\ln y(t)}{\ln t} \geq -\frac{1}{\alpha}, \qquad \liminf_{t\to\infty} \frac{\ln z(t)}{\ln t} \geq -\frac{1}{\alpha} \quad \textit{a.s.}$$

Proof For the given constant $\alpha > 0$, applying the generalized Itô formula to $(1 + \check{f}_1 u(t) + \check{f}_2 v(t) + w(t))^\alpha$, it follows from (3.3) that

$$\begin{aligned}
&\mathrm{d}\bigl(1 + \check{f}_1 u(t) + \check{f}_2 v(t) + w(t)\bigr)^\alpha \\
&\quad = \alpha(1 + \check{f}_1 u + \check{f}_2 v + w)^{\alpha-1} \\
&\qquad \cdot \biggl[\check{f}_1 u\biggl(-a_1\bigl(r(t)\bigr) + \sigma_1^2\bigl(r(t)\bigr) - \frac{b_1(r(t))}{k_1(r(t))v + 1} + \frac{c_1(r(t))}{u} + \frac{e_1(r(t))uw}{w^2 + f_1(r(t))u^2}\biggr) \\
&\qquad + \check{f}_2 v\biggl(-a_2\bigl(r(t)\bigr) + \sigma_2^2\bigl(r(t)\bigr) - \frac{b_2(r(t))}{k_2(r(t))u + 1} + \frac{c_2(r(t))}{v} + \frac{e_2(r(t))vw}{w^2 + f_2(r(t))v^2}\biggr) \\
&\qquad + w\biggl(a_3\bigl(r(t)\bigr) + \sigma_3^2\bigl(r(t)\bigr) - \frac{e_3(r(t))w^2}{w^2 + f_1(r(t))u^2} - \frac{e_4(r(t))w^2}{w^2 + f_2(r(t))v^2} + \frac{c_3(r(t))}{w}\biggr)\biggr]\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\alpha(\alpha-1)(1 + \check{f}_1 u + \check{f}_2 v + w)^{\alpha-2}\bigl(\sigma_1\bigl(r(t)\bigr)\check{f}_1 u + \sigma_2\bigl(r(t)\bigr)\check{f}_2 v + \sigma_3\bigl(r(t)\bigr)w\bigr)^2\,\mathrm{d}t \\
&\qquad - \alpha(1 + \check{f}_1 u + \check{f}_2 v + w)^{\alpha-1}\bigl(\sigma_1\bigl(r(t)\bigr)\check{f}_1 u + \sigma_2\bigl(r(t)\bigr)\check{f}_2 v + \sigma_3\bigl(r(t)\bigr)w\bigr)\,\mathrm{d}B(t) \\
&\quad \leq \alpha(1 + \check{f}_1 u + \check{f}_2 v + w)^{\alpha-1} \\
&\qquad \cdot \biggl[\check{f}_1 u\biggl(\check{\sigma}_1^2 + \frac{\check{e}_1}{2\sqrt{\hat{f}_1}}\biggr) + \check{f}_2 v\biggl(\check{\sigma}_2^2 + \frac{\check{e}_2}{2\sqrt{\hat{f}_2}}\biggr) + w\bigl(\check{\sigma}_3^2 + \check{a}_3\bigr) + \check{f}_1\check{c}_1 + \check{f}_2\check{c}_2 + \check{c}_3\biggr]\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\alpha^2\check{\sigma}^2(1 + \check{f}_1 u + \check{f}_2 v + w)^{\alpha-2}(\check{f}_1 u + \check{f}_2 v + w)^2\,\mathrm{d}t \\
&\qquad - \alpha\bigl(1 + \check{f}_1 u(t) + \check{f}_2 v(t) + w(t)\bigr)^{\alpha-1} \\
&\qquad \cdot \bigl(\sigma_1\bigl(r(t)\bigr)\check{f}_1 u + \sigma_2\bigl(r(t)\bigr)\check{f}_2 v + \sigma_3\bigl(r(t)\bigr)w\bigr)\,\mathrm{d}B(t) \\
&\quad \leq \alpha\lambda(1 + \check{f}_1 u + \check{f}_2 v + w)^\alpha\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&-\alpha(1+\check{f}_1 u+\check{f}_2 v+w)^{\alpha-1}\\
&\cdot\big(\sigma_1\big(r(t)\big)\check{f}_1 u(t)+\sigma_2\big(r(t)\big)\check{f}_2 v(t)+\sigma_3\big(r(t)\big)w(t)\big)\,\mathrm{d}B(t),
\end{aligned}
\tag{5.3}
$$

where $\check{\sigma}=\max_{i\in\mathbb{S},j=\{1,2,3\}}\sigma_j(i)$ and $\lambda=\check{\sigma}_1^2+\frac{\check{e}_1}{2\sqrt{\check{f}_1}}+\check{\sigma}_2^2+\frac{\check{e}_2}{2\sqrt{\check{f}_2}}+\check{\sigma}_3^2+\check{a}_3+\check{f}_1\check{c}_1+\check{f}_2\check{c}_2+\check{c}_3+\frac{1}{2}\alpha\check{\sigma}^2$. By the conclusion of Lemma 3.4 and Hölder inequality, there is a positive constant $M:=M(\alpha)$ such that

$$
\mathbf{E}\big[\big(1+\check{f}_1 u(t)+\check{f}_2 v(t)+w(t)\big)^{\alpha}\big]\le\frac{M}{2},\quad t\ge 0.
\tag{5.4}
$$

Choose a sufficiently small $\delta>0$ such that

$$
\alpha\big(\lambda\delta+3\check{\sigma}\delta^{\frac{1}{2}}\big)<1.
\tag{5.5}
$$

It follows from (5.3) that

$$
\begin{aligned}
&\mathbf{E}\Big(\sup_{k\delta\le t\le(k+1)\delta}\big(1+\check{f}_1 u(t)+\check{f}_2 v(t)+w(t)\big)^{\alpha}\Big)\\
&\quad\le\mathbf{E}\big(\big(1+\check{f}_1 u(k\delta)+\check{f}_2 v(k\delta)+w(k\delta)\big)^{\alpha}\big)\\
&\qquad+\mathbf{E}\Big(\sup_{k\delta\le t\le(k+1)\delta}\Big|\int_{k\delta}^{t}\alpha\lambda\big(1+\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^{\alpha}\,\mathrm{d}s\Big|\Big)\\
&\qquad+\mathbf{E}\Big(\sup_{k\delta\le t\le(k+1)\delta}\Big|\int_{k\delta}^{t}\alpha\big(1+\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^{\alpha-1}\\
&\qquad\cdot\big(\sigma_1\big(r(s)\big)\check{f}_1 u(s)+\sigma_2\big(r(s)\big)\check{f}_2 v(s)+\sigma_3\big(r(s)\big)w(s)\big)\,\mathrm{d}B(s)\Big|\Big).
\end{aligned}
\tag{5.6}
$$

It can be computed that

$$
\begin{aligned}
&\mathbf{E}\Big(\sup_{k\delta\le t\le(k+1)\delta}\Big|\int_{k\delta}^{t}\alpha\lambda\big(1+\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^{\alpha}\,\mathrm{d}s\Big|\Big)\\
&\quad\le\alpha\lambda\mathbf{E}\Big(\int_{k\delta}^{(k+1)\delta}\big|\big(1+\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^{\alpha}\big|\,\mathrm{d}s\Big)\\
&\quad\le\alpha\lambda\mathbf{E}\Big(\int_{k\delta}^{(k+1)\delta}\sup_{k\delta\le s\le(k+1)\delta}\big(1+\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^{\alpha}\,\mathrm{d}s\Big)\\
&\quad\le\alpha\lambda\delta\mathbf{E}\Big(\sup_{k\delta\le t\le(k+1)\delta}\big(1+\check{f}_1 u(t)+\check{f}_2 v(t)+w(t)\big)^{\alpha}\Big).
\end{aligned}
\tag{5.7}
$$

By the special case of Burkholder-Davis-Gundy inequality, we obtain

$$
\begin{aligned}
&\mathbf{E}\Big(\sup_{k\delta\le t\le(k+1)\delta}\Big|\int_{k\delta}^{t}\alpha\big(1+\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^{\alpha-1}\\
&\qquad\cdot\big(\sigma_1\big(r(s)\big)\check{f}_1 u(s)+\sigma_2\big(r(s)\big)\check{f}_2 v(s)+\sigma_3\big(r(s)\big)w(s)\big)\,\mathrm{d}B(s)\Big|\Big)\\
&\quad\le 3\mathbf{E}\Big(\int_{k\delta}^{(k+1)\delta}\alpha^2\check{\sigma}^2\big(1+\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^{2\alpha-2}\big(\check{f}_1 u(s)+\check{f}_2 v(s)+w(s)\big)^2\,\mathrm{d}s\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le 3\alpha\check{\sigma}\mathbf{E}\biggl(\int_{k\delta}^{(k+1)\delta}\bigl(1+\check{f}_1u(s)+\check{f}_2v(s)+w(s)\bigr)^{2\alpha}\,\mathrm{d}s\biggr)^{\frac{1}{2}} \\
&\le 3\alpha\check{\sigma}\delta^{\frac{1}{2}}\mathbf{E}\Bigl(\sup_{k\delta\le t\le(k+1)\delta}\bigl(1+\check{f}_1u(t)+\check{f}_2v(s)+w(t)\bigr)^{2\alpha}\Bigr)^{\frac{1}{2}} \\
&\le 3\alpha\check{\sigma}\delta^{\frac{1}{2}}\mathbf{E}\Bigl(\sup_{k\delta\le t\le(k+1)\delta}\bigl(1+\check{f}_1u(t)+\check{f}_2v(s)+w(t)\bigr)^{\alpha}\Bigr).
\end{aligned}
\tag{5.8}
$$

Substituting (5.7) and (5.8) into (5.6) and noting that (5.4) and (5.5), we have

$$
\begin{aligned}
&\mathbf{E}\Bigl(\sup_{k\delta\le t\le(k+1)\delta}\bigl(1+\check{f}_1u(t)+\check{f}_2v(t)+w(t)\bigr)^{\alpha}\Bigr) \\
&\quad\le \mathbf{E}\bigl(\bigl(1+\check{f}_1u(k\delta)+\check{f}_2v(k\delta)+w(k\delta)\bigr)^{\alpha}\bigr) \\
&\qquad+\alpha\bigl(\lambda\delta+3\check{\sigma}\delta^{\frac{1}{2}}\bigr)\mathbf{E}\Bigl(\sup_{k\delta\le t\le(k+1)\delta}\bigl(1+\check{f}_1u(t)+\check{f}_2v(s)+w(t)\bigr)^{\alpha}\Bigr) \\
&\quad\le M.
\end{aligned}
$$

Let $\epsilon > 0$ be arbitrary. Then, for $k = 1, 2, \ldots$, by Chebyshev inequality, we get

$$
\mathbb{P}\Bigl\{\sup_{k\delta\le t\le(k+1)\delta}\bigl(1+\check{f}_1u(t)+\check{f}_2v(t)+w(t)\bigr)^{\alpha} > (k\delta)^{1+\epsilon}\Bigr\} \le \frac{M}{(k\delta)^{1+\epsilon}}.
$$

By Borel-Cantelli lemma, there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $w \in \Omega_0$, there is an integer $k_0 = k_0(w)$ such that

$$
\bigl(1+\check{f}_1u(t)+\check{f}_2v(t)+w(t)\bigr)^{\alpha} \le (k\delta)^{1+\epsilon}
$$

for all $k\delta \le t \le (k+1)\delta$ and $k \ge k_0(w)$. Therefore, for any $k\delta \le t \le (k+1)\delta$ and $k \ge k_0(w)$,

$$
\frac{\ln(1+\check{f}_1u(t)+\check{f}_2v(t)+w(t))^{\alpha}}{\ln t} \le 1+\epsilon.
$$

Thus,

$$
\limsup_{t\to\infty}\frac{\ln(1+\check{f}_1u(t)+\check{f}_2v(t)+w(t))^{\alpha}}{\ln t} \le 1+\epsilon \quad \text{a.s.}
$$

Let $\epsilon \to 0$; we get the desired conclusion

$$
\limsup_{t\to\infty}\frac{\ln(\check{f}_1u(t))^{\alpha}}{\ln t} \le 1 \quad \text{a.s.},
$$

namely,

$$
\liminf_{t\to\infty}\frac{\ln x(t)}{\ln t} \ge -\frac{1}{\alpha} \quad \text{a.s.}
$$

Similarly, we have

$$
\liminf_{t\to\infty}\frac{\ln y(t)}{\ln t} \ge -\frac{1}{\alpha}, \qquad \liminf_{t\to\infty}\frac{\ln z(t)}{\ln t} \ge -\frac{1}{\alpha} \quad \text{a.s.} \qquad \square
$$

Theorem 5.1 *Under Assumptions* A1, A2 *and* A3, *the solution* $X(t)$ *of system* (1.3) *has the following property*:

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,\mathrm{d}s\le\frac{1}{\hat{c}_1}\sum_{i\in\mathbb{S}}\pi_i\left(a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i)\right),$$
$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,\mathrm{d}s\ge\frac{1}{\check{c}_1}\sum_{i\in\mathbb{S}}\pi_i\left(a_1(i)-\frac{e_1(i)}{2\sqrt{f_1(i)}}-\frac{1}{2}\sigma_1^2(i)\right),$$
$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,\mathrm{d}s\le\frac{1}{\hat{c}_2}\sum_{i\in\mathbb{S}}\pi_i\left(a_2(i)+b_2(i)-\frac{1}{2}\sigma_2^2(i)\right),$$
$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,\mathrm{d}s\ge\frac{1}{\check{c}_2}\sum_{i\in\mathbb{S}}\pi_i\left(a_2(i)-\frac{e_2(i)}{2\sqrt{f_2(i)}}-\frac{1}{2}\sigma_2^2(i)\right),$$
$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t z(s)\,\mathrm{d}s\le\frac{1}{\hat{c}_3}\sum_{i\in\mathbb{S}}\pi_i\left(e_3(i)+e_4(i)-a_3(i)-\frac{1}{2}\sigma_3^2(i)\right).$$

Proof By Lemma 5.1, Lemma 3.3 and Lemma 5.2, we have

$$\lim_{t\to\infty}\frac{\ln x(t)}{t}=0,\qquad\lim_{t\to\infty}\frac{\ln y(t)}{t}=0,\qquad\lim_{t\to\infty}\frac{\ln z(t)}{t}=0\quad\text{a.s.}\tag{5.9}$$

By the generalized Itô formula, we get

$$\begin{aligned}\frac{\ln[x(t)/x(0)]}{t}&=\frac{1}{t}\int_0^t\left(a_1\big(r(s)\big)+\frac{b_1(r(s))y}{k_1(r(s))+y}-c_1\big(r(s)\big)x-\frac{e_1(r(s))xz}{x^2+f_1(r(s))z^2}\right)\mathrm{d}s\\&\quad+\frac{1}{t}\int_0^s\sigma_1\big(r(s)\big)\,\mathrm{d}B(s),\\\frac{\ln[y(t)/y(0)]}{t}&=\frac{1}{t}\int_0^t\left(a_2\big(r(s)\big)+\frac{b_2(r(s))x}{k_2(r(s))+x}-c_2\big(r(s)\big)y-\frac{e_2(r(s))yz}{y^2+f_2(r(s))z^2}\right)\mathrm{d}s\\&\quad+\frac{1}{t}\int_0^s\sigma_2\big(r(s)\big)\,\mathrm{d}B(s),\\\frac{\ln[z(t)/z(0)]}{t}&=\frac{1}{t}\int_0^t\left(-a_3\big(r(s)\big)+\frac{e_3(r(s))x^2}{x^2+f_1(r(s))z^2}+\frac{e_4(r(s))y^2}{y^2+f_2(r(s))z^2}-c_3\big(r(s)\big)z\right)\mathrm{d}s\\&\quad+\frac{1}{t}\int_0^s\sigma_3\big(r(s)\big)\,\mathrm{d}B(s).\end{aligned}$$

Let $t\to\infty$, by the strong law of large numbers of local martingales and the ergodicity of the Markov chain, noticing that (5.9), we finally obtain the desired conclusion. □

6 Examples and numerical simulations

In this section, we give two examples and make some numerical simulations to support main results. By the method mentioned in [30], the discrete form of system (1.3) can be given by

$$\begin{aligned}x_{n+1}&=x_n+x_n\left(a_1(r_n)+\frac{b_1(r_n)y_n}{k_1(r_n)+y_n}-c_1(r_n)x_n-\frac{e_1(r_n)x_nz_n}{x_n^2+f_1(r_n)z_n^2}\right)\triangle t\\&\quad+\sigma_1(r_n)x_n\sqrt{\triangle t}\zeta_n+\frac{\sigma_1^2(r_n)}{2}x_n\big(\zeta_n^2-1\big)\triangle t,\end{aligned}$$

$$y_{n+1} = y_n + y_n\left(a_2(r_n) + \frac{b_2(r_n)x_n}{k_2(r_n)+x_n} - c_2(r_n)y_n - \frac{e_2(r_n)y_nz_n}{y_n^2+f_2(r_n)z_n^2}\right)\triangle t$$
$$+ \sigma_2(r_n)y_n\sqrt{\triangle t}\zeta_n + \frac{\sigma_2^2(r_n)}{2}y_n(\zeta_n^2-1)\triangle t,$$
$$z_{n+1} = z_n + z_n\left(-a_3(r_n) + \frac{e_3(r_n)x_n^2}{x_n^2+f_1(r_n)z_n^2} + \frac{e_4(r_n)y_n^2}{y_n^2+f_2(r_n)z_n^2} - c_3(r_n)z_n\right)\triangle t$$
$$+ \sigma_3(r_n)z_n\sqrt{\triangle t}\zeta_n + \frac{\sigma_3^2(r_n)}{2}z_n(\zeta_n^2-1)\triangle t,$$

where ζ_n is a Gaussian random variable that follows $N(0,1)$. For the procedure of generating the discrete Markov chain $\{r_n, n = 0,1,2,\ldots\}$, please refer to [27].

Throughout this section, we assume that $\triangle t = 0.01$ and let the initial data be $x(0) = 1.5$, $y(0) = 1.3$, $z(0) = 1$, $r(0) = 1$.

Example 6.1 Let $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S} = \{1,2\}$. System (1.3) may be regarded as the result of the following two subsystems:

$$\begin{cases} \mathrm{d}x(t) = x(\frac{7}{2} + \frac{y}{30+10y} - 3x - \frac{xz}{x^2+4z^2})\,\mathrm{d}t + \frac{1}{2}x\,\mathrm{d}B(t), \\ \mathrm{d}y(t) = y(4 + \frac{x}{10+5x} - 5y - \frac{2yz}{y^2+6z^2})\,\mathrm{d}t + \frac{1}{2}y\,\mathrm{d}B(t), \\ \mathrm{d}z(t) = z(-\frac{1}{4} + \frac{3x^2}{x^2+4z^2} + \frac{4y^2}{y^2+6z^2} - z)\,\mathrm{d}t + \frac{1}{2}z\,\mathrm{d}B(t), \end{cases} \tag{6.1}$$

$$\begin{cases} \mathrm{d}x(t) = x(2 + \frac{y}{2+y} - x - \frac{2xz}{x^2+9z^2})\,\mathrm{d}t + 3x\,\mathrm{d}B(t), \\ \mathrm{d}y(t) = y(\frac{3}{2} + \frac{2x}{3+x} - \frac{3}{2}y - \frac{yz}{y^2+16z^2})\,\mathrm{d}t + 3y\,\mathrm{d}B(t), \\ \mathrm{d}z(t) = z(-\frac{1}{8} + \frac{x^2}{x^2+9z^2} + \frac{2y^2}{y^2+16z^2} - z)\,\mathrm{d}t + \frac{1}{5}z\,\mathrm{d}B(t), \end{cases} \tag{6.2}$$

switching from one to another according to the movement of the Markov chain $r(t)$.

Then we compute that $q(1) = 2 > 0$, $a_1(1)+b_1(1)-\frac{1}{2}\sigma_1^2(1) = \frac{139}{40}$, $a_2(1)+b_2(1)-\frac{1}{2}\sigma_2^2(1) = \frac{163}{40}$. Therefore, by Corollary 3.1, subsystem (6.1) is stochastically permanent.

Compute also $q(2) = -\frac{7{,}283}{218}$, $a_1(2) + b_1(2) - \frac{1}{2}\sigma_1^2(2) = -\frac{3}{2} < 0$, $a_2(2) + b_2(2) - \frac{1}{2}\sigma_2^2(2) = -1 < 0$. Therefore, by Corollary 4.1, subsystem (6.2) is extinct.

Case 6.1.1 Assume that the generator of Markov chain $r(t)$ is

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 17 & -17 \end{bmatrix}.$$

By solving equation (2.2), we get the unique stationary distribution

$$\pi = \left(\frac{17}{18}, \frac{1}{18}\right).$$

Then compute that $\sum_{i=1}^N \pi_i q(i) = \frac{43}{1{,}308} > 0$. Therefore, by Theorem 3.2, the overall system (1.3) is stochastically permanent. By Theorem 5.1, we have

$$\frac{1{,}207}{1{,}296} \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,\mathrm{d}s \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,\mathrm{d}s \le \frac{2{,}303}{720},$$
$$\frac{1{,}141}{1{,}840} \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)\,\mathrm{d}s \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)\,\mathrm{d}s \le \frac{2{,}731}{1{,}080},$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t z(s)\,\mathrm{d}s\le\frac{2{,}887}{450}.$$

Case 6.1.2 Assume that the generator of the Markov chain $r(t)$ is

$$\Gamma=\begin{bmatrix}-5 & 5\\ 1 & -1\end{bmatrix}.$$

By solving equation (2.2), we get the unique stationary distribution

$$\pi=\left(\frac{1}{6},\frac{5}{6}\right).$$

Then we compute that $\sum_{i\in\mathbb{S}}\pi_i(a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i))=\frac{139}{40}\pi_1-\frac{3}{2}\pi_2=-\frac{161}{240}<0$, $\sum_{i\in\mathbb{S}}\pi_i(a_2(i)+b_2(i)-\frac{1}{2}\sigma_3^2(i))=\frac{163}{40}\pi_1-\pi_2=-\frac{37}{240}<0$. Therefore, by Theorem 4.1, the overall system (1.3) is extinct.

Example 6.2 Let $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S}=\{1,2,3\}$. system (1.3) may be regarded as the result of the following three subsystems:

$$\begin{cases}\mathrm{d}x(t)=x(\frac{1}{4}+\frac{y}{12+4y}-\frac{1}{2}x-\frac{xz}{10x^2+4{,}000z^2})\,\mathrm{d}t+\frac{11}{10}x\,\mathrm{d}B(t),\\ \mathrm{d}y(t)=y(\frac{3}{10}+\frac{x}{40+20x}-\frac{3}{5}y-\frac{yz}{5y^2+125z^2})\,\mathrm{d}t+\frac{9}{10}y\,\mathrm{d}B(t),\\ \mathrm{d}z(t)=z(-\frac{6}{50}+\frac{x^2}{5x^2+2{,}000z^2}+\frac{3y^2}{5y^2+125z^2}-\frac{7}{10}z)\,\mathrm{d}t+\frac{1}{5}z\,\mathrm{d}B(t),\end{cases}\tag{6.3}$$

$$\begin{cases}\mathrm{d}x(t)=x(\frac{3}{10}+\frac{y}{60+20y}-\frac{1}{2}x-\frac{xz}{10x^2+80z^2})\,\mathrm{d}t+\frac{19}{20}x\,\mathrm{d}B(t),\\ \mathrm{d}y(t)=y(\frac{1}{5}+\frac{x}{20+10x}-\frac{1}{5}y-\frac{yz}{10y^2+150z^2})\,\mathrm{d}t+\frac{9}{10}y\,\mathrm{d}B(t),\\ \mathrm{d}z(t)=z(-\frac{1}{10}+\frac{4x^2}{50x^2+400z^2}+\frac{7y^2}{100y^2+1{,}500z^2}-\frac{3}{10}z)\,\mathrm{d}t+\frac{1}{2}z\,\mathrm{d}B(t),\end{cases}\tag{6.4}$$

$$\begin{cases}\mathrm{d}x(t)=x(\frac{4}{5}+\frac{y}{300+100y}-\frac{9}{10}x-\frac{3xz}{5x^2+80z^2})\,\mathrm{d}t+\frac{1}{5}x\,\mathrm{d}B(t),\\ \mathrm{d}y(t)=y(\frac{7}{10}+\frac{x}{300+100x}-\frac{1}{2}y-\frac{4yz}{5y^2+320z^2})\,\mathrm{d}t+\frac{1}{5}y\,\mathrm{d}B(t),\\ \mathrm{d}z(t)=z(-\frac{1}{10}+\frac{4x^2}{5x^2+80z^2}+\frac{7y^2}{10y^2+640z^2}-\frac{1}{2}z)\,\mathrm{d}t+\frac{1}{5}z\,\mathrm{d}B(t),\end{cases}\tag{6.5}$$

switching from one to another according to the movement of Markov chain $r(t)$.

We compute that $q(1)=-\frac{3{,}983}{1{,}055}$, $a_1(1)+b_1(1)-\frac{1}{2}\sigma_1^2(1)=-\frac{21}{200}<0$, $a_2(1)+b_2(1)-\frac{1}{2}\sigma_2^2(1)=-\frac{11}{200}<0$. Therefore, by Corollary 4.1, subsystem (6.3) is extinct. See Figure 1.

Compute $q(2)=-\frac{2{,}817}{800}$, $a_1(2)+b_1(2)-\frac{1}{2}\sigma_1^2(2)=-\frac{81}{800}<0$, $a_2(2)+b_2(2)-\frac{1}{2}\sigma_2^2(2)=-\frac{21}{200}<0$. Therefore, by Corollary 4.1, subsystem (6.4) is extinct. See Figure 2.

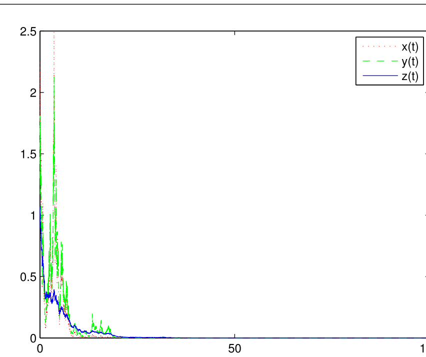


Figure 1 Extinction. Trajectories of the solution $(x(t), y(t), z(t))$ for subsystem (6.3). (Color online.)

Figure 2 Extinction. Trajectories of the solution ($x(t)$, $y(t)$, $z(t)$) for subsystem (6.4). (Color online.)

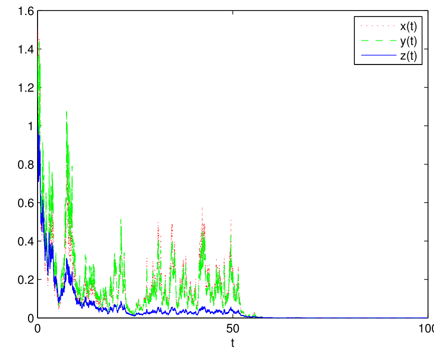
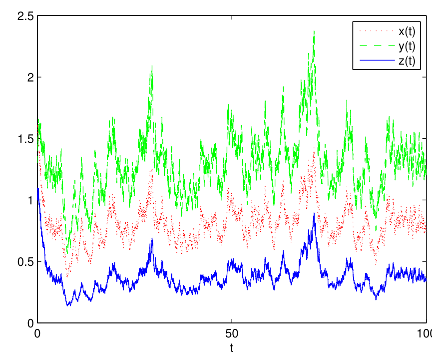


Figure 3 Stochastic permanence. Trajectories of the solution ($x(t)$, $y(t)$, $z(t)$) for subsystem (6.5). (Color online.)



Compute again $q(3) = \frac{659}{2{,}325} > 0$, $a_1(3) + b_1(3) - \frac{1}{2}\sigma_1^2(3) = \frac{79}{100}$, $a_2(3) + b_2(3) - \frac{1}{2}\sigma_2^2(3) = \frac{69}{100}$. Therefore, by Corollary 3.1, subsystem (6.5) is stochastically permanent. See Figure 3.

Case 6.2.1 Assume that the generator of the Markov chain $r(t)$ is

$$\Gamma = \begin{bmatrix} -28 & 5 & 23 \\ 1 & -5 & 4 \\ 1 & 0 & -1 \end{bmatrix}.$$

By solving equation (2.2), we get the unique stationary distribution

$$\pi = \left(\frac{1}{29}, \frac{1}{29}, \frac{27}{29}\right).$$

Then compute that $\sum_{i=1}^{N} \pi_i q(i) = \frac{135}{10{,}988} > 0$. Therefore, by Theorem 3.2, the overall system (1.3) is stochastically permanent. See Figures 4-6. By Theorem 5.1, we have

$$\begin{aligned} &\frac{256}{361} \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,\mathrm{d}s \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,\mathrm{d}s \le \frac{877}{602}, \\ &\frac{614}{641} \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)\,\mathrm{d}s \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)\,\mathrm{d}s \le \frac{1{,}847}{580}, \\ &\limsup_{t\to\infty} \frac{1}{t}\int_0^t z(s)\,\mathrm{d}s \le \frac{87}{20}. \end{aligned}$$

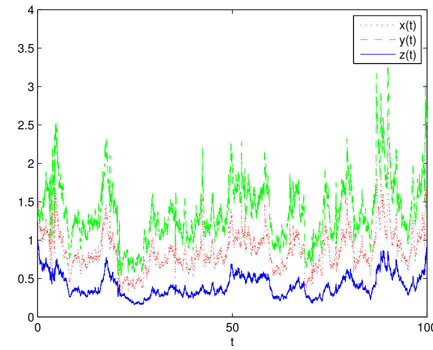


Figure 4 Stochastic permanence. Trajectories of the solution $(x(t), y(t), z(t))$ for the overall system (1.3) in Case 6.2.1. (Color online.)

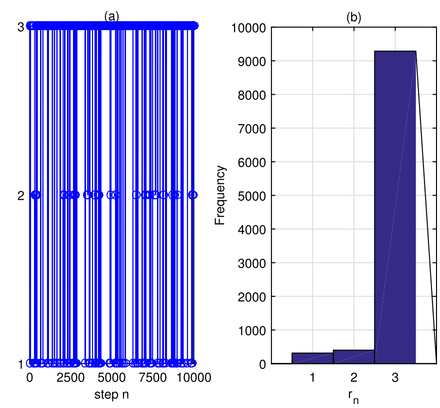


Figure 5 Trajectory and frequency of the discrete Markov chain. Subgraphs **(a)** and **(b)** denote the trajectory and frequency of the discrete Markov chain r_n taking value in $\{1, 2, 3\}$, respectively. $r_n = 1$, $r_n = 2$ and $r_n = 3$ mean that the overall system (1.3) switches to subsystem (6.3), (6.4) and (6.5) in step n, respectively. The change of r_n in (a) means the process of regime switching of system (1.3) among subsystems (6.3), (6.4) and (6.5). The frequency of r_n in (b) shows the result of regime switching-the total number of steps of the overall system (1.3) switching to (6.3), (6.4) and (6.5). This graph shows that under the control of the Markov chain, the overall system (1.3) mostly switches to subsystem (6.5) in Case 6.2.1.

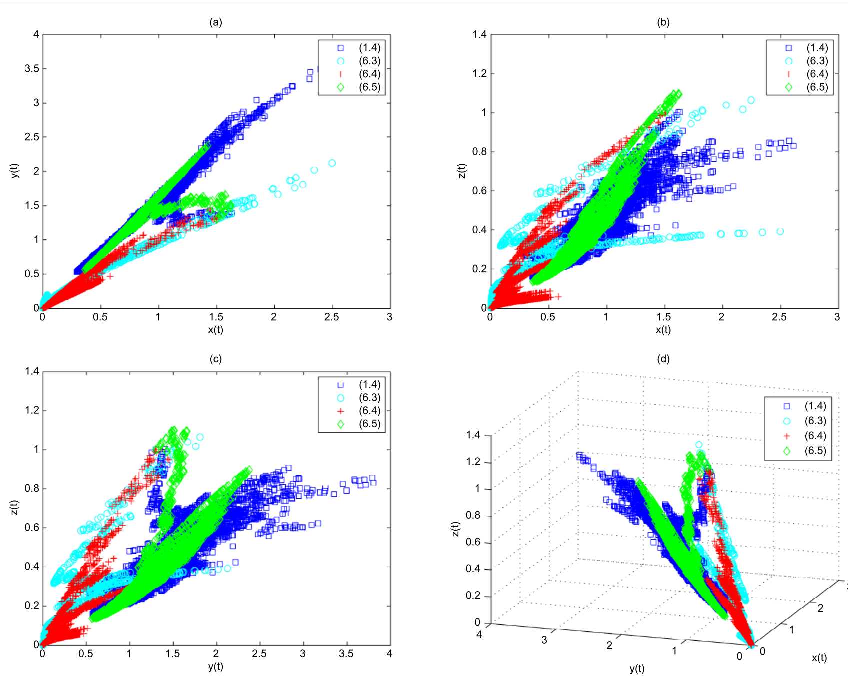


Figure 6 The discrete point distribution. Subgraphs **(a)**, **(b)**, **(c)** and **(d)** denote the discrete point distribution of three subsystems and the overall system in *xy*, *xz*, *yz* and *xyz*, respectively. The blue, cyan, red and green areas represent the overall system (1.3), subsystem (6.3), (6.4) and (6.5), respectively. Most points of the cyan and red areas lie in the origin and this means extinction. The green area is far away from the origin and this means stochastic permanence. Under the control of Markov chain, the blue area also keeps away from the origin and this means stochastic permanence in Case 6.2.1. (Color online.)

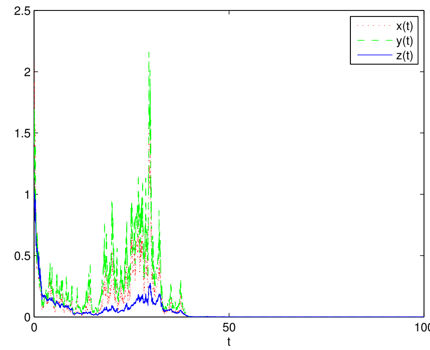


Figure 7 Extinction. Trajectories of the solution ($x(t)$, $y(t)$, $z(t)$) for the overall system (1.3) in Case 6.2.2. (Color online.)

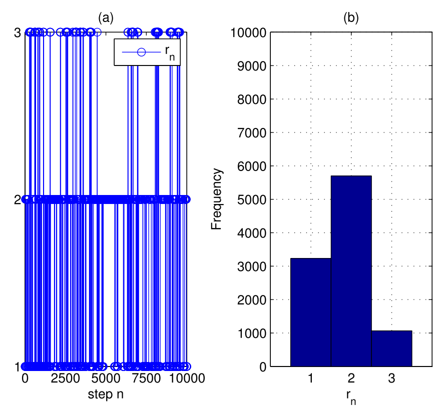


Figure 8 Trajectory and frequency of the discrete Markov chain. The subgraphs have the same notations as in Figure 5. This graph means that, under the control of the Markov chain, the overall system (1.3) mostly switches to subsystem (6.4) in Case 6.2.2.

Case 6.2.2 Assume that the generator of the Markov chain $r(t)$ is

$$\Gamma = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix}.$$

By solving equation (2.2), we get the unique stationary distribution

$$\pi = \left(\frac{3}{10}, \frac{3}{5}, \frac{1}{10}\right).$$

Then we compute that $\sum_{i\in\mathbb{S}} \pi_i(a_1(i) + b_1(i) - \frac{1}{2}\sigma_1^2(i)) = -\frac{53}{4{,}000} < 0$, $\sum_{i\in\mathbb{S}} \pi_i(a_2(i) + b_2(i) - \frac{1}{2}\sigma_2^2(i)) = -\frac{21}{2{,}000} < 0$. Therefore, by Theorem 4.1, the overall system (1.3) is extinct. See Figures 7-9.

7 Conclusions

In this paper, we investigate dynamical behaviors of a stochastic ratio-dependent one-predator and two-mutualistic-preys model perturbed by white and telegraph noise.

Theorem 3.2 and Theorem 4.1 give sufficient conditions of stochastic permanence and extinction for system (1.3), respectively. These conditions are all dependent on both parameters of each subsystem (1.4) and the stationary distribution probability. This means that if some subsystems are stochastically permanent and others are extinct, under the control of Markov chain, the overall system (1.3) is stochastically per-

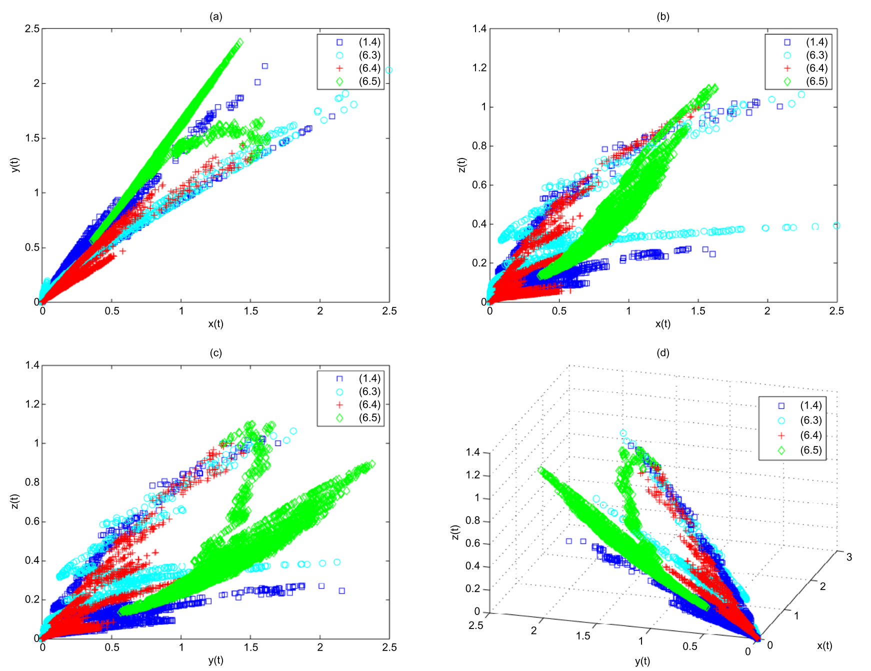


Figure 9 The discrete point distribution. The above subgraphs have the same notations as in Figure 6. Under the control of Markov chain, most points of the blue area lie in the origin and this means extinction in Case 6.2.2. (Color online.)

manent and extinct, determined by the sign of $\sum_{i\in\mathbb{S}}\pi_i q(i)$ and $\max\{\sum_{i\in\mathbb{S}}\pi_i(a_1(i)+b_1(i)-\frac{1}{2}\sigma_1^2(i)),\sum_{i\in\mathbb{S}}\pi_i(a_2(i)+b_2(i)-\frac{1}{2}\sigma_2^2(i))\}$, respectively. This explanation can be verified by Cases 6.1.1-6.1.2 or Cases 6.2.1-6.2.2.

When system (1.3) is stochastically permanent, we obtain boundaries of limit superior and inferior of the average in time of the solution in Theorem 5.1. These boundaries also all depend on both parameters of each subsystem (1.4) and the stationary distribution probability.

In addition to the one-predator and two-mutualistic-preys model, there are other three-species models such as tri-trophic food-chain model [31], herbivore-plant-pollinator model [8]. At the same time, besides white and telegraph noise, Lévy noise is inevitable in nature [32]. Therefore, the above three-species models with Lévy noise deserve further investigation and we may consider them in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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