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S-asymptotically periodic solutions for an epidemic model with superlinear perturbation

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Abstract

This paper is concerned with the existence of S-asymptotically periodic solutions for an epidemic model with superlinear perturbation. It seems that this is a first result as regards such a model with superlinear perturbation. We give sufficient conditions to ensure the existence of S-asymptotically periodic solutions for the problem addressed and give an example to show that our sufficient conditions can be satisfied.

MSC: 45G10; 34K14

Keywords: S-asymptotically periodic; delay integral equation; superlinear perturbation

1 Introduction and preliminaries

In [1], Cooke and Kaplan initiated the study of the following nonlinear delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f\bigl(s, x(s)\bigr)\,ds, \quad t \in \mathbb{R}, \tag{1.1}$$

which is a model for the spread of some infectious diseases. Since then, many mathematicians make an extensive study of the existence of periodic solutions and almost periodic solutions for equation (1.1) and its variants. We refer the reader to [2–13] and the references therein for some research work on this topic.

Especially, in 1997, Ait Dads and Ezzinbi considered the existence of positive almost periodic solutions for the following neutral integral equation:

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} f\bigl(s, x(s)\bigr)\,ds, \quad t \in \mathbb{R}, \tag{1.2}$$

where $\gamma \in [0,1)$. Since the work of Ait Dads and Ezzinbi, several authors have made contributions on equation (1.2) and its variants (see, *e.g.*, [6, 8] and the references therein). Especially, stimulated by the work of Ait Dads, Cieutat, and Lhachimi [5], the authors in [8] investigated the existence of positive pseudo almost periodic solution for the following more general neutral integral equation:

$$x(t) = \alpha(t)x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)f\bigl(s, x(s)\bigr)\, ds, \quad t \in \mathbb{R}. \tag{1.3}$$

In fact, both equation (1.2) and equation (1.3) can be seen as a linear perturbation of equation (1.1). Then a natural question arises:

When are there bounded solutions for equation (1.1) *with superlinear perturbations*?

It seems that there is no literature about the existence of bounded solutions for equation (1.1) with superlinear perturbations until now. The aim of this paper is to give some answers to the above problem.

On the other hand, an interesting notion of S-asymptotically periodic functions was recently introduced and studied by several authors (see, *e.g.*, [14] and the references therein). In fact, it turns out that S-asymptotically periodic functions are an important and interesting generalization of asymptotically periodic functions. It has attracted great interest from many authors studying S-asymptotically periodic functions and their applications in differential equations (especially abstract differential equations in Banach spaces). We refer the reader to [15–22] and the references therein for some recent contributions on this topic.

Stimulated by the work on equation (1.1) and the work on S-asymptotically periodic functions, in this paper, we will investigate the existence of S-asymptotically periodic solutions for the following delay integral equation with superlinear perturbations:

$$x(t) = \alpha(t)x^{n}(t-\beta) + \int_{t-\tau(t)}^{t} f\bigl(s, x(s)\bigr)\, ds, \quad t \in \mathbb{R}, \tag{1.4}$$

where $n \geq 1$ and $\beta \geq 0$ are fixed constants, and α, τ, f satisfy some conditions recalled in Section 2. Here, if we define

$$(\mathfrak{D}x)(t) = \alpha(t)x^{n}(t-\beta),$$

then $\mathfrak{D}(\lambda x) = \lambda^{n}\mathfrak{D}x$. In the case of $n = 1$, $\mathfrak{D}$ is a linear operator. Thus, in this paper, we call $\mathfrak{D}$ superlinear in the case of $n > 1$.

Throughout the rest of this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers, and by $\mathbb{R}^{+}$ the set of nonnegative real numbers.

Definition 1.1 A bounded and continuous function $f : \mathbb{R} \to \mathbb{R}$ is called S-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}[f(t+\omega) - f(t)] = 0$. We denote by $\mathrm{SAP}_{\omega}(\mathbb{R})$ the set of all such functions.

Remark 1.2 Note that our definition has a slight difference from [14], where a S-asymptotically periodic function is defined on $\mathbb{R}^{+}$.

Lemma 1.3 *Let $f, g \in \mathrm{SAP}_{\omega}(\mathbb{R})$. Then the following assertions hold*:

(a) $\mathrm{SAP}_{\omega}(\mathbb{R})$ *is a Banach space under the supremum norm.*
(b) $f(\cdot + s) \in \mathrm{SAP}_{\omega}(\mathbb{R})$ *for every $s \in \mathbb{R}$.*
(c) $f \cdot g \in \mathrm{SAP}_{\omega}(\mathbb{R})$.

Proof One can prove (a) by using a very similar proof to that of [14], Proposition 3.5. Moreover, one can show (b) and (c) by directly using the definition of $\mathrm{SAP}_\omega(\mathbb{R})$. We omit the details here. □

2 Main results

We first establish two lemmas about S-asymptotically periodic functions.

Lemma 2.1 *Let $f:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+$ be a continuous function satisfying $f(t,\lambda x)\ge\lambda f(t,x)$ for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^+$. Moreover, for every $x>0$, $f(\cdot,x)\in\mathrm{SAP}_\omega(\mathbb{R})$ and $g\in\mathrm{SAP}_\omega(\mathbb{R})$ with $\inf_{t\in\mathbb{R}}g(t)>0$. Then $f(\cdot,g(\cdot))\in\mathrm{SAP}_\omega(\mathbb{R})$.*

Proof Let $a=\inf_{t\in\mathbb{R}}g(t)$ and $b=\sup_{t\in\mathbb{R}}g(t)$. Then $0<a\le b<+\infty$. Noting that $f(t,\lambda x)\ge\lambda f(t,x)$, by [5], Lemma 3.1, there exists $L>0$ such that

$$\bigl|f(t,x)-f(t,y)\bigr|\le L|x-y|,\quad t\in\mathbb{R},x,y\in[a,b].$$

Then it is easy to see that $f(\cdot,g(\cdot))$ is bounded and continuous.

For every $n\in\mathbb{N}$ and $t\in\mathbb{R}$, there exists $i_t\in\{0,1,2,\dots,n\}$ such that $|g(t)-a-\frac{i_t}{n}(b-a)|\le\frac{b-a}{n}$. Then, for every $n\in\mathbb{N}$ and $t\in\mathbb{R}$, we have

$$\begin{aligned}
&\bigl|f\bigl(t+\omega,g(t+\omega)\bigr)-f\bigl(t,g(t)\bigr)\bigr|\\
&\quad\le\bigl|f\bigl(t+\omega,g(t+\omega)\bigr)-f\bigl(t,g(t+\omega)\bigr)\bigr|+\bigl|f\bigl(t,g(t+\omega)\bigr)-f\bigl(t,g(t)\bigr)\bigr|\\
&\quad\le\bigl|f\bigl(t+\omega,g(t+\omega)\bigr)-f\bigl(t,g(t+\omega)\bigr)\bigr|+L\bigl|g(t+\omega)-g(t)\bigr|\\
&\quad\le\frac{2L(b-a)}{n}+\left|f\left(t+\omega,a+\frac{i_{t+\omega}}{n}(b-a)\right)-f\left(t,a+\frac{i_{t+\omega}}{n}(b-a)\right)\right|\\
&\qquad+L\bigl|g(t+\omega)-g(t)\bigr|\\
&\quad\le\frac{2L(b-a)}{n}+\sum_{i=0}^{n}\left|f\left(t+\omega,a+\frac{i}{n}(b-a)\right)-f\left(t,a+\frac{i}{n}(b-a)\right)\right|\\
&\qquad+L\bigl|g(t+\omega)-g(t)\bigr|,
\end{aligned}$$

combining this with

$$\lim_{t\to\infty}\left|f\left(t+\omega,a+\frac{i}{n}(b-a)\right)-f\left(t,a+\frac{i}{n}(b-a)\right)\right|=0,\quad i=0,1,2,\dots,n,$$

we conclude that $\lim_{t\to\infty}|f(t+\omega,g(t+\omega))-f(t,g(t))|=0$, *i.e.*, $f(\cdot,g(\cdot))\in\mathrm{SAP}_\omega(\mathbb{R})$. □

Lemma 2.2 *Let $f,\tau\in\mathrm{SAP}_\omega(\mathbb{R})$. Then $F\in\mathrm{SAP}_\omega(\mathbb{R})$, where $F(t)=\int_{t-\tau(t)}^{t}f(s)\,ds$ for all $t\in\mathbb{R}$.*

Proof Note that

$$\begin{aligned}
\bigl|F(t+\omega)-F(t)\bigr|&=\left|\int_{t+\omega-\tau(t+\omega)}^{t+\omega}f(s)\,ds-\int_{t-\tau(t)}^{t}f(s)\,ds\right|\\
&=\left|\int_{t-\tau(t+\omega)}^{t}f(s+\omega)\,ds-\int_{t-\tau(t)}^{t}f(s)\,ds\right|
\end{aligned}$$

$$\leq \left|\int_{t-\tau(t+\omega)}^{t-\tau(t)} f(s+\omega)\,ds\right| + \left|\int_{t-\tau(t)}^{t} \bigl[f(s+\omega)-f(s)\bigr]\,ds\right|$$
$$\leq \|f\|\cdot\bigl|\tau(t+\omega)-\tau(t)\bigr| + \|\tau\|\cdot \sup_{t-\|\tau\|\leq s\leq t+\|\tau\|}\bigl|f(s+\omega)-f(s)\bigr|,$$

it follows from $f,\tau\in \mathrm{SAP}_\omega(\mathbb{R})$ that $\lim_{t\to\infty}[F(t+\omega)-F(t)]=0$. Also, it is not difficult to see that F is bounded and continuous. This completes the proof. □

For convenience, we list some assumptions.

(H0) $\alpha,\tau\in \mathrm{SAP}_\omega(\mathbb{R})$ are nonnegative.

(H1) $f:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function satisfying that $f(t,\cdot)$ is nondecreasing in $\mathbb{R}^+$ for every $t\in\mathbb{R}$, $f(\cdot,x)\in \mathrm{SAP}_\omega(\mathbb{R})$ for every $x\in\mathbb{R}^+$, and there exists $\phi:(0,1)\to(0,1]$ such that $f(t,\lambda x)\geq\phi(\lambda)f(t,x)$ for all $t\in\mathbb{R}$, $\lambda\in(0,1)$ and $x\in\mathbb{R}^+$.

(H2) There exist two constants $M>\varepsilon>0$ such that for all $t\in\mathbb{R}$,

$$\alpha(t)\varepsilon^n+\int_{t-\tau(t)}^{t} f(s,\varepsilon)\,ds\geq\varepsilon \quad\text{and}\quad \alpha(t)M^n+\int_{t-\tau(t)}^{t} f(s,M)\,ds\leq M.$$

(H3) For every $\lambda\in(0,1)$, $\phi(\lambda)>\lambda+r_0(\lambda-\lambda^n)$, where

$$r_0=\frac{\overline{\alpha}\cdot M^n}{\inf_{t\in\mathbb{R}}\int_{t-\tau(t)}^{t} f(s,\frac{\varepsilon^2}{M})\,ds}<+\infty \quad\text{and}\quad \overline{\alpha}=\sup_{t\in\mathbb{R}}\alpha(t).$$

Theorem 2.3 *Assume that* (H0)-(H3) *hold. Then equation* (1.4) *has a S-asymptotically periodic solution* x^* *with* $\inf_{t\in\mathbb{R}}x^*(t)>0$.

Proof Let

$$E=\Bigl\{x\in \mathrm{SAP}_\omega(\mathbb{R}):\inf_{t\in\mathbb{R}}x(t)>0\Bigr\},$$
$$(Bx)(t)=\int_{t-\tau(t)}^{t} f\bigl(s,x(s)\bigr)\,ds,\qquad (Cx)(t)=\alpha(t)x^n(t-\beta),\quad t\in\mathbb{R},x\in E,$$

and

$$(Ax)(t)=(Bx)(t)+(Cx)(t),\quad t\in\mathbb{R},x\in E.$$

Noting that $r_0=\frac{\overline{\alpha}\cdot M^n}{\inf_{t\in\mathbb{R}}\int_{t-\tau(t)}^{t} f(s,\frac{\varepsilon^2}{M})\,ds}$, for every $x\in E$ with $\frac{\varepsilon^2}{M}\leq x(t)\leq M$ for all $t\in\mathbb{R}$, we have

$$(Cx)(t)\leq\overline{\alpha}\cdot M^n=r_0\inf_{t\in\mathbb{R}}\int_{t-\tau(t)}^{t} f\left(s,\frac{\varepsilon^2}{M}\right)ds\leq r_0(Bx)(t),\quad t\in\mathbb{R}.$$

Combining this with (H1), for all $t\in\mathbb{R}$, $\lambda\in(0,1)$, and $x\in E$ with $\frac{\varepsilon^2}{M}\leq x(t)\leq M$ for all $t\in\mathbb{R}$, we have

$$A(\lambda x)(t)=B(\lambda x)(t)+C(\lambda x)(t)\geq\phi(\lambda)(Bx)(t)+\lambda^n(Cx)(t)$$
$$=\lambda(Ax)(t)+\bigl[\phi(\lambda)-\lambda\bigr](Bx)(t)+\bigl[\lambda^n-\lambda\bigr](Cx)(t)$$

$$\begin{aligned}
&\geq \lambda(Ax)(t) + \bigl[\phi(\lambda) - \lambda - \bigl(\lambda - \lambda^n\bigr)r_0\bigr](Bx)(t) \\
&\geq \left[\lambda + \frac{\phi(\lambda) - \lambda - (\lambda - \lambda^n)r_0}{1 + r_0}\right](Ax)(t) \\
&= \psi(\lambda)(Ax)(t),
\end{aligned} \tag{2.1}$$

where $\psi(\lambda) = \lambda + \frac{\phi(\lambda)-\lambda-(\lambda-\lambda^n)r_0}{1+r_0} > \lambda$ for every $\lambda \in (0,1)$ by (H3). Especially, it follows from (2.1) that, for all $t \in \mathbb{R}$, $\lambda \in (0,1]$ and $x \in E$ with $\frac{\varepsilon^2}{M} \leq x(t) \leq M$ for all $t \in \mathbb{R}$, we have

$$A(\lambda x)(t) \geq \lambda A(x)(t). \tag{2.2}$$

Let $u_0(t) \equiv \varepsilon$, $v_0(t) \equiv M$, and

$$u_k(t) = (Au_{k-1})(t), \qquad v_k(t) = (Av_{k-1})(t), \quad t \in \mathbb{R}, k = 1, 2, \ldots.$$

By (H2), we know that $u_1(t) \geq \varepsilon$ and $v_1(t) \leq M$ for every $t \in \mathbb{R}$. Then, by using the fact that $f(t,\cdot)$ is nonincreasing in $\mathbb{R}^+$ for every $t \in \mathbb{R}$, we conclude that

$$\varepsilon \leq u_1(t) \leq u_2(t) \leq \cdots \leq u_k(t) \leq \cdots \leq v_k(t) \leq \cdots \leq v_2(t) \leq v_1(t) \leq M, \quad t \in \mathbb{R}.$$

In addition, combining (H1) with Lemma 1.3, Lemma 2.1, and Lemma 2.2, we can conclude that for every $k \in \mathbb{N}$, $u_k \in \mathrm{SAP}_\omega(\mathbb{R})$ and $v_k \in \mathrm{SAP}_\omega(\mathbb{R})$.

Let $\mu_k = \sup\{\mu > 0 : u_k(t) \geq \mu v_k(t), t \in \mathbb{R}\}$. Then

$$\frac{\varepsilon}{M} \leq \mu_1 \leq \cdots \leq \mu_k \leq 1.$$

Set $\lim_{k\to\infty} \mu_k = \mu_0$. It is easy to see that $\mu_0 \in [\frac{\varepsilon}{M}, 1]$. We claim that $\mu_0 = 1$. In fact, if $\mu_0 < 1$, noting that $\frac{\varepsilon^2}{M} \leq \mu_0 v_k(t) \leq M$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$, by (2.2) and (2.1), we have

$$\begin{aligned}
u_{k+1}(t) &= (Au_k)(t) \\
&\geq A(\mu_k v_k)(t) \\
&= A\left(\frac{\mu_k}{\mu_0}\mu_0 v_k\right)(t) \\
&\geq \frac{\mu_k}{\mu_0} A(\mu_0 v_k)(t) \\
&\geq \frac{\mu_k}{\mu_0}\psi(\mu_0) v_{k+1}(t),
\end{aligned}$$

which means that $\mu_{k+1} \geq \frac{\mu_k}{\mu_0}\psi(\mu_0)$, *i.e.*,

$$\frac{\mu_{k+1}}{\mu_k} \geq \frac{\psi(\mu_0)}{\mu_0} > 1.$$

Then we have

$$\mu_{k+1} \geq \mu_1 \left[\frac{\psi(\mu_0)}{\mu_0}\right]^k \to \infty,$$

and thus $\lim_{k\to\infty} \mu_k = \infty$. This is a contradiction.

For every $k, m \in \mathbb{N}$ with $k > m$, we have

$$0 \le u_k(t) - u_m(t) \le v_k(t) - u_m(t) \le v_m(t) - u_m(t) \le (1 - \mu_m)v_m(t) \le (1 - \mu_m)M, \quad t \in \mathbb{R},$$

which yields

$$\sup_{t\in\mathbb{R}} \left| u_k(t) - u_m(t) \right| \le (1 - \mu_m)M \to 0, \quad m \to \infty.$$

Thus, there exists $x^* \in \mathrm{SAP}_\omega(\mathbb{R})$ such that $u_k \to x^*$ in $\mathrm{SAP}_\omega(\mathbb{R})$ as $k \to \infty$. It is easy to see that for all $t \in \mathbb{R}$,

$$\varepsilon \le u_1(t) \le u_2(t) \le \cdots \le u_k(t) \le \cdots \le x^*(t) \le \cdots \le v_k(t) \le \cdots \le v_2(t) \le v_1(t) \le M.$$

So we have

$$0 \le v_k(t) - x^*(t) \le v_k(t) - u_k(t) \le (1 - \mu_k)M, \quad t \in \mathbb{R},$$

which means that $v_k \to x^*$ in $\mathrm{SAP}_\omega(\mathbb{R})$ as $k \to \infty$. For all $k \in \mathbb{N}$ and $t \in \mathbb{R}$, since $u_k(t) \le x^*(t) \le v_k(t)$, we have

$$u_{k+1}(t) = (Au_k)(t) \le \left(Ax^*\right)(t) \le (Av_k)(t) = v_{k+1}(t).$$

Letting $k \to \infty$, we get $(Ax^*)(t) = x^*(t)$ for all $t \in \mathbb{R}$, *i.e.*,

$$x^*(t) = \alpha(t)\left[x^*(t - \beta)\right]^n + \int_{t-\tau(t)}^{t} f\left(s, x^*(s)\right) ds, \quad t \in \mathbb{R}.$$

Thus x^* is a S-asymptotically periodic solution of equation (1.4). □

Next, we show that our assumptions can be satisfied by a simple example, which does not aim at generality.

Example 2.4 Let $n = \frac{3}{2}$, $\alpha(t) \equiv \frac{1}{20}$, $\beta = 0$, $\tau(t) \equiv 1$, and

$$f(t, x) = a(t)\sqrt{x},$$

where $a \in \mathrm{SAP}_\omega(\mathbb{R})$ with $1 \le \inf_{t\in\mathbb{R}} a(t) \le \sup_{t\in\mathbb{R}} a(t) \le \frac{13}{10}$.

Obviously, (H0) holds. It is easy to see that (H1) holds with $\phi(\lambda) = \sqrt{\lambda}$. By a direct calculation, one can show that (H2) holds with $\varepsilon = 1$ and $M = 2$. In addition, we have

$$r_0 = \frac{\overline{\alpha} \cdot M^n}{\inf_{t\in\mathbb{R}} \int_{t-\tau(t)}^{t} f(s, \frac{\varepsilon^2}{M})\, ds} \le \frac{1}{5},$$

and for every $\lambda \in (0, 1)$,

$$\frac{\phi(\lambda) - \lambda}{\lambda - \lambda^n} = \frac{\sqrt{\lambda} - \lambda}{\lambda - \lambda^{\frac{3}{2}}} = \frac{1}{\sqrt{\lambda}} > 1,$$

which means that (H3) holds.

Remark 2.5 By using the approach in Theorem 2.3, one can also prove some similar results as regards the existence of almost periodic type solutions for equation (1.4).

Competing interests
None of the authors have any competing interests in the manuscript.

Authors' contributions
All authors contribute equally to this work.

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