Sun et al. *Advances in Difference Equations* (2016) 2016:222
DOI 10.1186/s13662-016-0947-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH
Open Access

Certain fractional q-symmetric integrals and q-symmetric derivatives and their application

Mingzhe Sun, Yuanfeng Jin and Chengmin Hou*

*Correspondence:
cmhou@foxmail.com
Department of Mathematics,
Yanbian University, Yanji, 133002,
P.R. China

Abstract

The q-symmetric analogs of Cauchy's formulas for multiple integrals are obtained. We introduce the concepts of the fractional q-symmetric integrals and fractional q-symmetric derivatives and discuss some of their properties. By using some properties of q-symmetric fractional integrals and fractional difference operators, we study a boundary value problem with nonlocal boundary conditions.

MSC: 26A33; 34B15

Keywords: fractional q-symmetric integrals; fractional q-symmetric derivatives; boundary value problem

1 Introduction

The q-quantum calculus is an old subject that was first developed by Jackson [1, 2]. It plays an important role in several fields of physics, such as cosmic strings and black holes [3], conformal quantum mechanics [4], nuclear and high energy physics [5], and so on. As a survey of this calculus we refer to [6]. Starting from the q-analog of cauchy formula [7], Al-Salam started the fitting of the concept of q-fractional calculus. After that he [8, 9] and Agarwal [10] continued by studying certain q-fractional integrals and derivatives. Recently, perhaps due to the explosion in research within the fractional calculus setting, new developments in this theory of fractional q-difference calculus were made, specifically, q-analogs of the integral and differential fractional operators properties such as the q-Laplace transform, and q-Taylor's formula [11, 12]. More recently, the authors in [13, 14] studied the problems of q-fractional initial value and approximation solutions by means of the generalized type of the q-Mittag-Leffler function introduced. Baleanu and Agarwal [15] established some inequalities involving the Saigo fractional q-integral operator in the theory of quantum calculus. There are also many papers dealing with the existence of solutions for q-fractional boundary value problems (see, *e.g.*, [16–23]).

The q-symmetric quantum calculus has proven to be useful in several fields, in particular in quantum mechanics [24]. As noticed in [25], consistently with the q-deformed theory, the standard q-symmetric integral must be generalized to the basic integral defined. However, to the best of the authors' knowledge there are no results available in the literature introducing basic definitions for fractional q-symmetric integrals and fractional

q-symmetric derivatives. The basic theory of q-symmetric quantum calculus needs to be explored. The object of this paper is to define a fractional q-symmetric operator corresponding to the q-symmetric analog of $\int_0^x f(\tau)\widetilde{d}_q\tau$. Besides this we shall investigate the fundamental properties of this operator. A study of these fractional q-symmetric operators is expected to be of great importance in the development of the q-function theory, which plays an important role in combinatory analysis.

2 The q-symmetric analogs of Cauchy's formulas

For a real parameter $q \in \mathbb{R}^+\backslash\{1\}$, we introduce a q-real number $\overline{[a]}_q$ by

$$\overline{[a]}_q = \frac{1-q^{2a}}{1-q^2} \quad (a \in \mathbb{R}).$$

For a nonnegative integer n, let

$$\overline{[0]}_q! = 1, \qquad \overline{[n]}_q! = \overline{[n]}_q\overline{[n-1]}_q \cdots \overline{[1]}_q.$$

Also, the q-symmetric analog of the power $(a-b)^k$ is

$$\overline{(a-b)}^{(0)} = 1, \qquad \overline{(a-b)}^{(k)} = \prod_{i=0}^{k-1}\bigl(a - bq^{2i+1}\bigr) \quad (k \in \mathbb{N}, a, b \in \mathbb{R}).$$

Their natural expansions to reals are

$$\overline{(a-b)}^{(\alpha)} = a^\alpha \frac{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^{2i+1})}{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^{2(i+\alpha)+1})} \quad (\alpha \in \mathbb{R}, a \neq 0). \tag{2.1}$$

The q-symmetric gamma function is defined by

$$\begin{aligned}\widetilde{\Gamma}_q(x) &= \frac{\prod_{i=0}^{\infty}(1-q^{2i+2})}{\prod_{i=0}^{\infty}(1-q^{2(i+x-1)+2})}\bigl(1-q^2\bigr)^{1-x} \\ &= \overline{(1-q)}^{(x-1)}\bigl(1-q^2\bigr)^{1-x} \quad \bigl(x \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}\bigr).\end{aligned} \tag{2.2}$$

Obviously,

$$\widetilde{\Gamma}_q(1) = \overline{(1-q)}^{(0)}\bigl(1-q^2\bigr)^0 = 1, \qquad \widetilde{\Gamma}_q(x+1) = \overline{[x]}_q\widetilde{\Gamma}_q(x).$$

The basic q-symmetric integrals are defined through the relations

$$(\widetilde{I}_{q,0}f)(t) = \int_0^x f(t)\widetilde{d}_qt = x\bigl(1-q^2\bigr)\sum_{k=0}^{\infty} q^{2k}f\bigl(xq^{2k+1}\bigr), \tag{2.3}$$

$$(\widetilde{I}_{q,a}f)(t) = \int_a^x f(t)\widetilde{d}_qt = \int_0^x f(t)\widetilde{d}_qt - \int_0^a f(t)\widetilde{d}_qt.$$

The q-symmetric derivative of a function $f(x)$ is defined as

$$(\widetilde{D}_qf)(x) = \frac{f(qx)-f(q^{-1}x)}{x(q-q^{-1})}, \qquad (\widetilde{D}_qf)(0) := f'(0), \tag{2.4}$$

and the q-symmetric derivatives of higher order as

$$\big(\widetilde{D}_q^0 f\big)(x) = f(x), \qquad \big(\widetilde{D}_q^n f\big)(x) = \big(\widetilde{D}_q \widetilde{D}_q^{n-1} f\big)(x), \quad n \in \mathbb{N}^+.$$

As for q-symmetric derivatives, we can define an operator $\widetilde{I}_q^n$ by

$$\big(\widetilde{I}_{q,a}^0 f\big)(x) = f(x), \qquad \big(I_{q,a}^n f\big)(x) = \big(\widetilde{I}_{q,a} \widetilde{I}_{q,a}^{n-1} f\big)(x), \quad n \in \mathbb{N}^+. \tag{2.5}$$

For operators defined in this manner, the following is valid:

$$(\widetilde{D}_q \widetilde{I}_{q,0}) f(x) = f(x), \qquad (\widetilde{I}_{q,0} \widetilde{D}_q f)(x) = f(x) - f(0). \tag{2.6}$$

The formula for q-symmetric integration by parts is

$$\int_a^b u(qx)(\widetilde{D}_q v)(x)\,\widetilde{d}_q x = \big[u(x)v(x)\big]\big|_a^b - \int_a^b v\big(q^{-1}x\big)(\widetilde{D}_q u)(x)\,\widetilde{d}_q x.$$

Using (2.1) and (2.4), we may obtain the very useful examples of the q-symmetric derivatives of the next functions:

$$_x\widetilde{D}_q \overline{(x-a)}^{(\alpha)} = \overline{[\alpha]}_q \overline{\big(q^{-1}x-a\big)}^{(\alpha-1)}, \tag{2.7}$$

$$_x\widetilde{D}_q \overline{\big(a-q^{-1}x\big)}^{(\alpha)} = -\overline{[\alpha]}_q \overline{(a-x)}^{(\alpha-1)}. \tag{2.8}$$

Next, we consider the form of the multiple q-symmetric integration as follows:

$$\big(\widetilde{I}_q^n f\big)(x) = \int_0^x \widetilde{d}_q t \int_0^t \widetilde{d}_q t_{n-1} \int_0^{t_{n-1}} \widetilde{d}_q t_{n-2} \cdots \int_0^{t_2} \widetilde{d}_q t_1. \tag{2.9}$$

Theorem 1 *The form of the multiple q-symmetric integration* (2.9) *is equality to*

$$\big(\widetilde{I}_{q,0}^n f\big)(x) = \frac{1}{\overline{[n-1]}_q!} q^{\binom{n}{2}} \int_0^x \overline{(x-\tau)}^{(n-1)} f\big(q^{n-1}\tau\big)\,\widetilde{d}_q\tau, \tag{2.10}$$

where

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

Proof We prove (2.10) by using mathematical induction.
If $n = 1$, $(\widetilde{I}_{q,0}^1 f)(x) = \int_0^x f(\tau)\,\widetilde{d}_q\tau$.
If $n = 2$, we have

$$\begin{aligned}
\big(\widetilde{I}_{q,0}^2 f\big)(x) &= \int_0^x \int_0^s f(\tau)\,\widetilde{d}_q\tau\,\widetilde{d}_q s \\
&= x\big(1-q^2\big) \sum_{k=0}^{\infty} q^{2k} \int_0^{q^{2k+1}x} f(\tau)\,\widetilde{d}_q\tau \\
&= \big(x\big(1-q^2\big)\big)^2 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} q^{2k} q^{2k+2m+1} f\big(q^{2k+2m+2}x\big)
\end{aligned}$$

$$
\begin{aligned}
&= \left(x\left(1-q^2\right)\right)^2 \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} q^{2k} q^{2m+1} f\left(q^{2m+2}x\right) \\
&= \left(x\left(1-q^2\right)\right)^2 \sum_{m=0}^{\infty} \sum_{k=0}^{m} q^{2k} q^{2m+1} f\left(q^{2m+2}x\right) \\
&= \left(x\left(1-q^2\right)\right)^2 \sum_{m=0}^{\infty} \frac{1-q^{2m+2}}{1-q^2} q^{2m+1} f\left(q^{2m+2}x\right) \\
&= x\left(1-q^2\right) \sum_{m=0}^{\infty} \left(x-q^{2m+2}x\right) q^{2m+1} f\left(q^{2m+2}x\right) \\
&= \int_0^{qx} (x-\tau) f(\tau)\, \widetilde{d}_q \tau \\
&= q \int_0^{x} \overline{(x-\tau)}^{(1)} f(q\tau)\, \widetilde{d}_q \tau.
\end{aligned}
$$

We see that (2.10) holds.

Next, suppose that Theorem 1 holds for $n = k$. We consider the case $n = k + 1$.

By (2.5), we have

$$
\begin{aligned}
\left(\widetilde{I}_{q,0}^{k+1} f\right)(x) &= \widetilde{I}_{q,0}\left(\frac{1}{\overline{[k-1]}_q!} q^{\binom{k}{2}} \int_0^x \overline{(x-\tau)}^{(k-1)} f\left(q^{k-1}\tau\right) \widetilde{d}_q \tau\right) \\
&= \frac{1}{\overline{[k-1]}_q!} q^{\binom{k}{2}} \int_0^x \int_0^s \overline{(s-\tau)}^{(k-1)} f\left(q^{k-1}\tau\right) \widetilde{d}_q \tau\, \widetilde{d}_q s \\
&= \frac{1}{\overline{[k-1]}_q!} q^{\binom{k}{2}} x\left(1-q^2\right) \sum_{m=0}^{\infty} q^{2m} \int_0^{q^{2m+1}x} \overline{\left(q^{2m+1}x-\tau\right)}^{(k-1)} f\left(q^{k-1}\tau\right) \widetilde{d}_q \tau \\
&= \frac{1}{\overline{[k-1]}_q!} q^{\binom{k}{2}} \left(1-q^2\right)^2 x^{k+1} \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} q^{2l} q^{2m+1} \overline{\left(q^{2m+1}-q^{2l+2}\right)}^{(k-1)} f\left(q^{k-1} q^{2l+2} x\right) \\
&= \frac{1}{\overline{[k-1]}_q!} q^{\binom{k}{2}} \left(1-q^2\right)^2 x^{k+1} \sum_{l=0}^{\infty} q^{2l} f\left(q^{2l+k+1}x\right) \sum_{m=0}^{l} q^{2m+1} \overline{\left(q^{2m+1}-q^{2l+2}\right)}^{(k-1)}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\left(\widetilde{I}_{q,0}^{k+1} f\right)(x) &= \frac{1}{\overline{[k]}_q!} q^{\binom{k+1}{2}} \int_0^x \overline{(x-\tau)}^{(k)} f\left(q^k \tau\right) \widetilde{d}_q \tau \\
&= \frac{1}{\overline{[k]}_q!} q^{\binom{k+1}{2}} x\left(1-q^2\right) \sum_{l=0}^{\infty} q^{2l} \overline{\left(x-xq^{2l+1}\right)}^{(k)} f\left(q^{2l+k+1}x\right) \\
&= \frac{1}{\overline{[k]}_q!} q^{\binom{k+1}{2}} x^{k+1} \left(1-q^2\right) \sum_{l=0}^{\infty} q^{2l} \overline{\left(1-q^{2l+1}\right)}^{(k)} f\left(q^{2l+k+1}x\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left(1-q^2\right) q^{-k} \overline{[k]}_q \sum_{m=0}^{l} q^{2m+1} \overline{\left(q^{2m+1}-q^{2l+2}\right)}^{(k-1)} \\
&\quad = \left(1-q^2\right) q^{-k} \overline{[k]}_q \left(q \overline{\left(q-q^{2l+2}\right)}^{(k-1)} + q^3 \overline{\left(q^3-q^{2l+2}\right)}^{(k-1)} + \cdots\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{2l+1}\overline{\left(q^{2l+1} - q^{2l+2}\right)}^{(k-1)}\big) \\
&= \left(1-q^2\right)\overline{[k]}_q\big(\overline{\left(\left(1-q^{2l+1}\right)\right.}^{(k-1)} + q^{2k}\overline{\left(1-q^{2l-1}\right)}^{(k-1)} + \cdots + q^{2lk}\overline{(1-q)}^{(k-1)}\big) \\
&= \left(1-q^2\right)\overline{[k]}_q\big(q^{2lk}\overline{(1-q)}^{(k-1)} + q^{2(l-1)k}\overline{\left(1-q^3\right)}^{(k-1)} + \cdots \\
&\quad + q^{2k}\overline{\left(1-q^{2l-1}\right)}^{(k-1)} + \overline{\left(1-q^{2l+1}\right)}^{(k-1)}\big) \\
&= \left(1-q^2\right)\overline{[k]}_q\left(q^{2(l-1)k}\left[q^{2k}\prod_{i=0}^{k-2}\left(1-q^{2i+2}\right) + \prod_{i=0}^{k-2}\left(1-q^{2i+4}\right)\right] + q^{2(l-2)k}\overline{\left(1-q^5\right)}^{(k-1)} + \cdots\right. \\
&\quad \left. + \overline{\left(1-q^{2l+1}\right)}^{(k-1)}\right) \\
&= \left(1-q^2\right)\overline{[k]}_q\left(q^{2(l-1)k}\left(1-q^{2k+2}\right)\prod_{i=0}^{k-3}\left(1-q^{2i+4}\right) + q^{2(l-2)k}\overline{\left(1-q^5\right)}^{(k-1)} + \cdots\right. \\
&\quad \left. + \overline{\left(1-q^{2l+1}\right)}^{(k-1)}\right) \\
&= \left(1-q^2\right)\overline{[k]}_q\left(q^{2(l-2)k}\left[\left(1-q^{2k+2}\right)\prod_{i=0}^{k-3}\left(1-q^{2i+4}\right) + \prod_{i=0}^{k-2}\left(1-q^{2i+6}\right)\right]\right. \\
&\quad \left. + q^{2(l-3)k}\overline{\left(1-q^7\right)}^{(k-1)} + \cdots + \overline{\left(1-q^{2l+1}\right)}^{(k-1)}\right) \\
&= \left(1-q^2\right)\overline{[k]}_q\left(q^{2(l-2)k}\left(1-q^{2k+4}\right)\left(1-q^{2k+2}\right)\prod_{i=0}^{k-4}\left(1-q^{2i+6}\right)\right. \\
&\quad \left. + q^{2(l-3)k}\overline{\left(1-q^7\right)}^{(k-1)} + \cdots + \overline{\left(1-q^{2l+1}\right)}^{(k-1)}\right) \\
&= \cdots \\
&= \left(1-q^2\right)\overline{[k]}_q\left(1-q^{2l+2}\right)\left(1-q^{2l+4}\right)\cdots\left(1-q^{2k-2}\right)\left(1-q^{2k+2}\right)\cdots\left(1-q^{2l+2k}\right) \\
&= \overline{\left(1-q^{2l+1}\right)}^{(k)}.
\end{aligned}
$$

We may see that (2.10) holds when $n = k + 1$. □

3 The $\widetilde{I}^{\alpha}_{q,0}$ operator

We now introduce the fractional q-symmetric integral operator,

$$\left(\widetilde{I}^{\alpha}_{q,0}f\right)(x) = \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^x \overline{(x-\tau)}^{(\alpha-1)}f\left(q^{\alpha-1}\tau\right)\widetilde{d}_q\tau \quad \left(\alpha \in \mathbb{R}^+\right), \tag{3.1}$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \quad (k \in \mathbb{N}).$$

To prove the semigroup property of the fractional q-symmetric integral, we need Lemma 1.

Lemma 1 [12] *For $\mu, \alpha, \beta \in \mathbb{R}^+$, the following identity is valid*:

$$\sum_{n=0}^{\infty} \frac{(1-\mu q^{1-n})^{(\alpha-1)}(1-\mu q^{1+n})^{(\beta-1)}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}} q^{\alpha n} = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}},$$

where

$$(a-b)^{(\alpha)} = a^{\alpha} \frac{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^i)}{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^{\alpha+i})} \quad (a, b \in \mathbb{R}, a \neq 0).$$

Theorem 2 *Let $\alpha, \beta \in \mathbb{R}^+$. The fractional q-symmetric integration has the following semigroup property*:

$$\big(\widetilde{I}_{q,0}^{\alpha}\widetilde{I}_{q,0}^{\beta} f\big)(x) = \big(\widetilde{I}_{q,0}^{\alpha+\beta} f\big)(x).$$

Proof By (3.1) and (2.3), we have

$$\begin{aligned}
\big(\widetilde{I}_{q,0}^{\alpha}\widetilde{I}_{q,0}^{\beta} f\big)(x) &= \widetilde{I}_{q,0}^{\alpha}\left(\frac{q^{\binom{\beta}{2}}}{\widetilde{\Gamma}_q(\beta)} \int_0^x \overline{(x-\tau)}^{(\beta-1)} f\big(q^{\beta-1}\tau\big)\, \widetilde{d}_q\tau\right) \\
&= \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\beta)} \int_0^x \overline{(x-s)}^{(\alpha-1)} \int_0^{q^{\alpha-1}s} \overline{\big(q^{\alpha-1}s-\tau\big)}^{(\beta-1)} f\big(q^{\beta-1}\tau\big)\, \widetilde{d}_q\tau\, \widetilde{d}_q s \\
&= \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\beta)} x\big(1-q^2\big) \sum_{k=0}^{\infty} q^{2k} \overline{\big(x-q^{2k+1}x\big)}^{(\alpha-1)} \\
&\quad \times \int_0^{q^{\alpha+2k}x} \overline{\big(q^{2k+\alpha}x-\tau\big)}^{(\beta-1)} f\big(q^{\beta-1}\tau\big)\, \widetilde{d}_q\tau \\
&= \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}}}{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\beta)} \big(1-q^2\big)^2 x^{\alpha+\beta} \\
&\quad \times \sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty} q^{2k} q^{2k+2m} q^{\alpha\beta}\right) q^{2m} \overline{\big(q^{2k}-q^{2k+2m+1}\big)}^{(\beta-1)} \overline{\big(1-q^{2k+1}\big)}^{(\alpha-1)} \\
&\quad \times f\big(q^{\alpha+\beta+2k+2m}x\big) \\
&= \frac{q^{\binom{\alpha}{2}+\binom{\beta}{2}+\alpha\beta}}{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\beta)} \big(1-q^2\big)^2 x^{\alpha+\beta} \\
&\quad \times \sum_{k=0}^{\infty}\sum_{m=k}^{\infty} q^{2k} q^{2m} \overline{\big(q^{2k}-q^{2m+1}x\big)}^{(\beta-1)} \overline{\big(1-q^{2k+1}\big)}^{(\alpha-1)} f\big(q^{\alpha+\beta+2m}x\big) \\
&= \frac{q^{\binom{\alpha+\beta}{2}}}{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\beta)} \big(1-q^2\big)^2 x^{\alpha+\beta} \\
&\quad \times \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} q^{2k} \overline{\big(q^{2k}-q^{2m+1}\big)}^{(\beta-1)} \overline{\big(1-q^{2k+1}\big)}^{(\alpha-1)}\right) q^{2m} f\big(q^{\alpha+\beta+2m}x\big) \\
&= \frac{q^{\binom{\alpha+\beta}{2}}}{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\beta)} \big(1-q^2\big)^2 x^{\alpha+\beta} \\
&\quad \times \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} q^{2k\beta} \overline{\big(1-q^{2m-2k+1}\big)}^{(\beta-1)} \overline{\big(1-q^{2k+1}\big)}^{(\alpha-1)}\right) q^{2m} f\big(q^{\alpha+\beta+2m}x\big).
\end{aligned}$$

Next, we denote $q^2 = \overline{q}$, then

$$\begin{aligned}\overline{\left(1-q^{2m-2k+1}\right)}^{(\beta-1)} &= \frac{\prod_{i=0}^{\infty}(1-q^{2m-2k+1}q^{2i+1})}{\prod_{i=0}^{\infty}(1-q^{2m-2k+1}q^{2(i+\beta-1)+1})} \\ &= \frac{\prod_{i=0}^{\infty}(1-\overline{q}^{m-k+i+1})}{\prod_{i=0}^{\infty}(1-\overline{q}^{m-k+i+\beta})} \\ &= \left(1-\overline{q}^{m-k+1}\right)^{(\beta-1)}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Gamma}_q(\alpha) &= \overline{(1-q)}^{(\alpha-1)}\left(1-q^2\right)^{1-\alpha} \\ &= \left(1-q^2\right)^{1-\alpha}\frac{\prod_{i=0}^{\infty}(1-q^{2i+2})}{\prod_{i=0}^{\infty}(1-q^{2(i+\alpha-1)+2})} \\ &= (1-\overline{q})^{1-\alpha}\frac{\prod_{i=0}^{\infty}(1-\overline{q}^{i+1})}{\prod_{i=0}^{\infty}(1-\overline{q}^{i+\alpha})} \\ &= (1-\overline{q})^{1-\alpha}(1-\overline{q})^{(\alpha-1)}.\end{aligned}$$

Using Lemma 1, we obtain

$$\begin{aligned}&\sum_{k=0}^{m} q^{2k\beta}\overline{\left(1-q^{2m-2k+1}\right)}^{(\beta-1)}\overline{\left(1-q^{2k+1}\right)}^{(\alpha-1)} \\ &\quad= \sum_{k=0}^{\infty} q^{2k\beta}\overline{\left(1-q^{2m-2k+1}\right)}^{(\beta-1)}\overline{\left(1-q^{2k+1}\right)}^{(\alpha-1)} \\ &\quad= \sum_{k=0}^{\infty} \overline{q}^{k\beta}\left(1-\overline{q}^{m-k+1}\right)^{(\beta-1)}\left(1-\overline{q}^{k+1}\right)^{(\alpha-1)} \\ &\quad= (1-\overline{q})^{(\alpha-1)}(1-\overline{q})^{(\beta-1)}\sum_{k=0}^{\infty}\frac{\overline{q}^{k\beta}(1-\overline{q}^{m-k+1})^{(\beta-1)}(1-\overline{q}^{k+1})^{(\alpha-1)}}{(1-\overline{q})^{(\alpha-1)}(1-\overline{q})^{(\beta-1)}} \\ &\quad= (1-\overline{q})^{(\alpha-1)}(1-\overline{q})^{(\beta-1)}\frac{(1-\overline{q}^{m+1})^{(\alpha+\beta-1)}}{(1-\overline{q})^{(\alpha+\beta-1)}}.\end{aligned}$$

Thus

$$\begin{aligned}\left(\widetilde{I}_{q,0}^{\alpha}\widetilde{I}_{q,0}^{\beta}f\right)(x) &= \frac{q^{\binom{\alpha+\beta}{2}}}{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\beta)}\left(1-q^2\right)^2 x^{\alpha+\beta} \\ &\quad\times\sum_{m=0}^{\infty} q^{2m}(1-\overline{q})^{(\beta-1)}(1-\overline{q})^{(\alpha-1)}\frac{(1-\overline{q}^{m+1})^{(\alpha+\beta-1)}}{(1-\overline{q})^{(\alpha+\beta-1)}}f\left(q^{\alpha+\beta+2m}x\right) \\ &= \frac{q^{\binom{\alpha+\beta}{2}}}{\widetilde{\Gamma}_q(\alpha+\beta)}\left(1-q^2\right)x^{\alpha+\beta}\sum_{m=0}^{\infty} q^{2m}\left(1-\overline{q}^{m+1}\right)^{(\alpha+\beta-1)}f\left(q^{\alpha+\beta+2m}x\right) \\ &= \frac{q^{\binom{\alpha+\beta}{2}}}{\widetilde{\Gamma}_q(\alpha+\beta)}x\left(1-q^2\right)\sum_{m=0}^{\infty} q^{2m}\left(x-\overline{q}^{m+1}x\right)^{(\alpha+\beta-1)}f\left(q^{\alpha+\beta+2m}x\right)\end{aligned}$$

$$= \frac{q^{\binom{\alpha+\beta}{2}}}{\widetilde{\Gamma}_q(\alpha+\beta)} \int_0^x \overline{(x-\tau)}^{(\alpha+\beta-1)} f\big(q^{\alpha+\beta-1}\tau\big)\,\widetilde{d}_q\tau$$

$$= \big(\widetilde{I}_{q,0}^{\alpha+\beta} f\big)(x). \qquad \square$$

Theorem 3 *For $\alpha \in \mathbb{R}^+$, the following identity is valid*:

$$\big(\widetilde{I}_{q,0}^{\alpha} f\big)(x) = \big(\widetilde{I}_{q,0}^{\alpha+1} \widetilde{D}_q f\big)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(\alpha+1)} q^{\binom{\alpha}{2}} x^{\alpha}.$$

Proof Using the q-symmetric integration by parts and (2.8), we obtain

$$\begin{aligned}
\big(\widetilde{I}_{q,0}^{\alpha} f\big)(x) &= \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^x \overline{(x-\tau)}^{(\alpha-1)} f\big(q^{\alpha-1}\tau\big)\,\widetilde{d}_q\tau \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha)\overline{[\alpha]}_q} q^{\binom{\alpha}{2}} \left(-\int_0^x f\big(q^{\alpha-1}\tau\big)\,\widetilde{d}_q \overline{\big(x-q^{-1}\tau\big)}^{(\alpha)} \right) \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha+1)} q^{\binom{\alpha}{2}} \left(-\overline{\big(x-q^{-1}\tau\big)}^{(\alpha)} f\big(q^{\alpha}\tau\big)\Big|_0^x + \int_0^x q^{\alpha} \overline{(x-\tau)}^{(\alpha)} \widetilde{D}_q f\big(q^{\alpha}\tau\big)\,\widetilde{d}_q\tau \right) \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha+1)} q^{\binom{\alpha}{2}} \left(-\overline{\big(x-q^{-1}x\big)}^{(\alpha)} f\big(q^{\alpha}x\big) \right. \\
&\quad \left. + x^{\alpha} f(0) + \int_0^x q^{\alpha} \overline{(x-\tau)}^{(\alpha)} \widetilde{D}_q f\big(q^{\alpha}\tau\big)\,\widetilde{d}_q\tau \right) \\
&= \big(\widetilde{I}_{q,0}^{\alpha+1} \widetilde{D}_q f\big)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(\alpha+1)} q^{\binom{\alpha}{2}} x^{\alpha}. \qquad \square
\end{aligned}$$

4 The fractional q-symmetric derivative of Riemann-Liouville type

We define the fractional q-symmetric derivative of Riemann-Liouville type of a function $f(x)$ by

$$\big(\widetilde{D}_{q,0}^{\alpha} f\big)(x) = \begin{cases} (\widetilde{I}_{q,0}^{-\alpha} f)(x), & \alpha < 0, \\ f(x), & \alpha = 0, \\ (D_q^{[\alpha]} \widetilde{I}_{q,0}^{[\alpha]-\alpha} f)(x), & \alpha > 0. \end{cases} \tag{4.1}$$

Here $[\alpha]$ denotes the smallest integer greater than or equal to α.

Theorem 4 *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid*:

$$\big(\widetilde{D}_q \widetilde{D}_{q,0}^{\alpha} f\big)(x) = \big(\widetilde{D}_{q,0}^{\alpha+1} f\big)(x).$$

Proof We consider three cases. For $\alpha \leq -1$, according to Theorem 3 and (2.6), we have

$$\begin{aligned}
\big(\widetilde{D}_q \widetilde{D}_{q,0}^{\alpha} f\big)(x) &= \big(\widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha} f\big)(x) \\
&= \big(\widetilde{D}_q \widetilde{I}_{q,0}^{1-\alpha-1} f\big)(x) \\
&= \big(\widetilde{I}_{q,0}^{-(\alpha+1)} f\big)(x) \\
&= \big(\widetilde{D}_{q,0}^{(\alpha+1)} f\big)(x).
\end{aligned}$$

In the case $-1 < \alpha < 0$, we obtain

$$\bigl(\widetilde{D}_q\widetilde{D}^{\alpha}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}_q\widetilde{I}^{-\alpha}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}_q\widetilde{I}^{1-(\alpha+1)}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}^{\alpha+1}_{q,0}f\bigr)(x).$$

For $\alpha > 0$, we get

$$\bigl(\widetilde{D}_q\widetilde{D}^{\alpha}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}_q\widetilde{D}^{[\alpha]}_{q}\widetilde{I}^{[\alpha]-\alpha}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}^{1+\alpha}_{q,0}f\bigr)(x).$$ □

Theorem 5 *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid*:

$$\bigl(\widetilde{D}_q\widetilde{D}^{\alpha}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}^{\alpha}_{q,0}\widetilde{D}_qf\bigr)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1}.$$

Proof Let us consider two cases. For $\alpha < 0$,

$$\begin{aligned}
\bigl(\widetilde{D}_q\widetilde{D}^{\alpha}_{q,0}f\bigr)(x) &= \bigl(\widetilde{D}_q\widetilde{I}^{-\alpha}_{q,0}f\bigr)(x)\\
&= \widetilde{D}_q\widetilde{I}^{-\alpha}_{q,0}\bigl[(\widetilde{I}_{q,0}\widetilde{D}_qf)(x) + f(0)\bigr]\\
&= \bigl(\widetilde{D}_q\widetilde{I}^{-\alpha}_{q,0}\widetilde{I}_{q,0}\widetilde{D}_qf\bigr)(x) + f(0)\bigl(\widetilde{D}_q\widetilde{I}^{-\alpha}_{q,0}1\bigr)\\
&= \bigl(\widetilde{D}_q\widetilde{I}^{-\alpha+1}_{q,0}\widetilde{D}_qf\bigr)(x) + f(0)q^{\binom{-\alpha}{2}}\frac{1}{\widetilde{\Gamma}_q(-\alpha)\overline{[-\alpha]}_q}\widetilde{D}_qx^{-\alpha}\\
&= \bigl(\widetilde{D}^{\alpha}_{q,0}\widetilde{D}_qf\bigr)(x) + f(0)q^{\binom{-\alpha}{2}}q^{-(\alpha+1)}\frac{1}{\widetilde{\Gamma}_q(-\alpha)}x^{-\alpha-1}\\
&= \bigl(\widetilde{D}^{\alpha}_{q,0}\widetilde{D}_qf\bigr)(x) + f(0)q^{\binom{-(\alpha+1)}{2}}\frac{1}{\widetilde{\Gamma}_q(-\alpha)}x^{-\alpha-1}.
\end{aligned}$$

If $\alpha > 0$, there exists $l \in \mathbb{N}_0$, such that $\alpha \in (l, l+1)$. Then applying a similar procedure, we get

$$\begin{aligned}
\bigl(\widetilde{D}_q\widetilde{D}^{\alpha}_{q,0}f\bigr)(x) &= \widetilde{D}_q\widetilde{D}^{l+1}_{q}\widetilde{I}^{l+1-\alpha}_{q,0}f)(x)\\
&= \widetilde{D}^{l+2}_{q}\widetilde{I}^{l+1-\alpha}_{q,0}\bigl[(\widetilde{I}_{q,0}\widetilde{D}_qf)(x) + f(0)\bigr]\\
&= \bigl(\widetilde{D}^{l+2}_{q}\widetilde{I}^{l+2-\alpha}_{q,0}\widetilde{D}_qf\bigr)(x) + f(0)\frac{q^{\binom{l+1-\alpha}{2}}}{\widetilde{\Gamma}_q(l+1-\alpha)}\left(\widetilde{D}^{l+2}_{q}\int_0^x\overline{(x-\tau)}^{(l-\alpha)}\widetilde{d}_q\tau\right)\\
&= \bigl(\widetilde{D}^{l+2}_{q}\widetilde{I}^{l+2-\alpha}_{q,0}\widetilde{D}_qf\bigr)(x) + f(0)\frac{q^{\binom{l+1-\alpha}{2}}}{\widetilde{\Gamma}_q(l+2-\alpha)}\bigl(\widetilde{D}^{l+2}_{q}x^{l+1-\alpha}\bigr)\\
&= \bigl(\widetilde{D}^{\alpha}_{q,0}\widetilde{D}_qf\bigr)(x) + f(0)q^{\binom{-(\alpha+1)}{2}}\frac{1}{\widetilde{\Gamma}_q(-\alpha)}x^{-\alpha-1}.
\end{aligned}$$ □

Theorem 6 *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid*:

$$\bigl(\widetilde{D}^{\alpha}_{q,0}\widetilde{I}^{\alpha}_{q,0}f\bigr)(x) = f(x).$$

Proof

$$\bigl(\widetilde{D}^{\alpha}_{q,0}\widetilde{I}^{\alpha}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}^{[\alpha]}_{q}\widetilde{I}^{[\alpha]-\alpha}_{q,0}\widetilde{I}^{\alpha}_{q,0}f\bigr)(x) = \bigl(\widetilde{D}^{[\alpha]}_{q}\widetilde{I}^{[\alpha]}_{q,0}f\bigr)(x) = f(x).$$ □

Theorem 7 *Let* $\alpha \in (0,1)$. *Then*

$$\left(\widetilde{I}^{\alpha}_{q,0}\widetilde{D}^{\alpha}_{q,0}f\right)(x) = f(x) + Kx^{\alpha-1}.$$

Proof Let $A(x) = (\widetilde{I}^{\alpha}_{q,0}\widetilde{D}^{\alpha}_{q,0}f)(x) - f(x)$.

Apply $\widetilde{D}^{\alpha}_{q,0}$ to both sides of the above expression, and using Theorem 6, we get

$$\left(\widetilde{D}^{\alpha}_{q,0}\right)A(x) = \left(\widetilde{D}^{\alpha}_{q,0}\widetilde{I}^{\alpha}_{q,0}\widetilde{D}^{\alpha}_{q,0}f\right)(x) - \left(\widetilde{D}^{\alpha}_{q,0}f\right)(x) = \left(\widetilde{D}^{\alpha}_{q,0}f\right)(x) - \left(\widetilde{D}^{\alpha}_{q,0}f\right)(x) = 0.$$

On the other hand,

$$\begin{aligned}
&\int_0^x \overline{(x-\tau)}^{(-\alpha)}\left(q^{-\alpha}\tau\right)^{\alpha-1}\widetilde{d}_q\tau \\
&\quad = x\left(1-q^2\right)\sum_{m=0}^{\infty} q^{2m}\overline{\left(x-q^{2m+1}x\right)}^{(-\alpha)}\left(q^{-\alpha}q^{2m+1}x\right)^{\alpha-1} \\
&\quad = \left(1-q^2\right)\sum_{m=0}^{\infty} q^{2m}\overline{\left(1-q^{2m+1}\right)}^{(-\alpha)}\left(q^{-\alpha}q^{2m+1}\right)^{\alpha-1}.
\end{aligned}$$

Using the above form and according to (3.1), (4.1), we obtain

$$\widetilde{D}^{\alpha}_{q,0}x^{\alpha-1} = \widetilde{D}_q\widetilde{I}^{1-\alpha}_{q,0}x^{\alpha-1} = \widetilde{D}_q q^{\binom{1-\alpha}{2}}\frac{1}{\widetilde{\Gamma}_q(1-\alpha)}\int_0^x \overline{(x-\tau)}^{(-\alpha)}\left(q^{-\alpha}\tau\right)^{\alpha-1}\widetilde{d}_q\tau = 0.$$

Hence $A(x) = Kx^{\alpha-1}$. □

Theorem 8 *Let* $\alpha \in (N-1,N]$. *Then for some constants* $c_i \in \mathbb{R}$, $i = 1,2,\ldots,N$, *the following equality holds*:

$$\left(\widetilde{I}^{\alpha}_{q,0}\widetilde{D}^{\alpha}_{q,0}f\right)(x) = f(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + \cdots + c_Nx^{\alpha-N}. \tag{4.2}$$

Proof By Theorem 3 and Theorem 7, we have

$$\begin{aligned}
\left(\widetilde{I}^{\alpha}_{q,0}\widetilde{D}^{\alpha}_{q,0}f\right)(x) &= \left(\widetilde{I}^{\alpha}_{q,0}\widetilde{D}^{N}_{q}\widetilde{I}^{N-\alpha}_{q,0}f\right)(x) \\
&= \left(\widetilde{I}^{\alpha-1}_{q,0}\widetilde{D}^{N-1}_{q}\widetilde{I}^{N-\alpha}_{q,0}f\right)(x) - q^{\binom{\alpha-1}{2}}\frac{\widetilde{D}^{N-1}_{q}\widetilde{I}^{N-\alpha}_{q,0}f(0)}{\widetilde{\Gamma}_q(\alpha)}x^{\alpha-1} \\
&= \left(\widetilde{I}^{\alpha-2}_{q,0}\widetilde{D}^{N-2}_{q}\widetilde{I}^{N-\alpha}_{q,0}f\right)(x) - q^{\binom{\alpha-2}{2}}\frac{\widetilde{D}^{N-2}_{q}\widetilde{I}^{N-\alpha}_{q,0}f(0)}{\widetilde{\Gamma}_q(\alpha-1)}x^{\alpha-2} \\
&\quad - q^{\binom{\alpha-1}{2}}\frac{\widetilde{D}^{N-1}_{q}\widetilde{I}^{N-\alpha}_{q,0}f(0)}{\widetilde{\Gamma}_q(\alpha)}x^{\alpha-1} \\
&= \cdots \\
&= \left(\widetilde{I}^{\alpha-N+1}_{q,0}\widetilde{D}^{\alpha-N+1}_{q,0}f\right)(x) - q^{\binom{\alpha-N+1}{2}}\frac{\widetilde{D}_{q}\widetilde{I}^{N-\alpha}_{q,0}f(0)}{\widetilde{\Gamma}_q(\alpha-N+2)}x^{\alpha-N+1} - \cdots \\
&\quad - q^{\binom{\alpha-1}{2}}\frac{\widetilde{D}^{N-1}_{q}\widetilde{I}^{N-\alpha}_{q,0}f(0)}{\widetilde{\Gamma}_q(\alpha)}x^{\alpha-1} \\
&= f(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + \cdots + c_Nx^{\alpha-N}.
\end{aligned}$$

□

5 The fractional q-symmetric derivative of Caputo type

If we change the order of operators, we can introduce another type of fractional q-derivative.

The fractional q-symmetric derivative of Caputo type is

$$\left({}^c\widetilde{D}_{q,0}^{\alpha}f\right)(x) = \begin{cases} (\widetilde{I}_{q,0}^{-\alpha}f)(x), & \alpha < 0, \\ f(x), & \alpha = 0, \\ (\widetilde{I}_{q,0}^{[\alpha]-\alpha}D_q^{[\alpha]}f)(x), & \alpha > 0. \end{cases} \tag{5.1}$$

Here $[\alpha]$ denotes the smallest integer greater than or equal to α.

Theorem 9 *For* $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, *and* $x > 0$, *the following is valid*:

$$\left({}^c\widetilde{D}_{q,0}^{\alpha+1}f\right)(x) - \left({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f\right)(x) = \begin{cases} \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1}, & \alpha \leq -1, \\ 0, & \alpha > -1. \end{cases} \tag{5.2}$$

Proof Clearly, (5.2) holds for $\alpha = -1$. Next, we will consider three cases.

(i) $\alpha < -1$, according to (5.1), (2.8), and (2.6), we have

$$\begin{aligned} \left({}^c\widetilde{D}_{q,0}^{\alpha+1}f\right)(x) &= \left(\widetilde{I}_{q,0}^{-\alpha-1}f\right)(x) \\ &= \widetilde{I}_{q,0}^{-\alpha-1}\left(\widetilde{I}_{q,0}\widetilde{D}_q f(x) + f(0)\right) \\ &= \left(\widetilde{I}_{q,0}^{-\alpha}\widetilde{D}_q f\right)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1} \\ &= \left({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f\right)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1}. \end{aligned}$$

(ii) $-1 < \alpha \leq 0$, we obtain

$$\left({}^c\widetilde{D}_{q,0}^{\alpha+1}f\right)(x) = \left(\widetilde{I}_{q,0}^{1-(\alpha+1)}\widetilde{D}_q f\right)(x) = \left(\widetilde{I}_{q,0}^{-\alpha}\widetilde{D}_q f\right)(x) = \left({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f\right)(x).$$

(iii) $\alpha > 0$, we assume $\alpha = n + \varepsilon, n \in \mathbb{N}_0, 0 < \varepsilon < 1$, then $\alpha + 1 \in (n+1, n+2)$, so we obtain

$$\left({}^c\widetilde{D}_{q,0}^{\alpha+1}f\right)(x) = \left(\widetilde{I}_{q,0}^{1-\varepsilon}\widetilde{D}_q^{n+2}f\right)(x) = \left(\widetilde{I}_{q,0}^{1-\varepsilon}\widetilde{D}_q^{n+1}\widetilde{D}_q f\right)(x) = \left({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f\right)(x). \qquad \square$$

Theorem 10 *For* $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, *and* $x > 0$, *the following is valid*:

$$\left(\widetilde{D}_q{}^c\widetilde{D}_{q,0}^{\alpha}f\right)(x) - \left({}^c\widetilde{D}_{q,0}^{\alpha+1}f\right)(x) = \begin{cases} 0, & \alpha \leq -1, \\ \frac{\widetilde{D}_q^{[\alpha]}f(0)}{\widetilde{\Gamma}_q([\alpha]-\alpha)}q^{\binom{[\alpha]-(\alpha+1)}{2}}x^{-\alpha-1}, & \alpha > -1. \end{cases} \tag{5.3}$$

Proof We will consider two cases.

(i) $\alpha < 0$, using Theorem 3, (5.1), (2.6), and (2.8), we obtain

$$\begin{aligned} \left(\widetilde{D}_q{}^c\widetilde{D}_{q,0}^{\alpha}f\right)(x) &= \left(\widetilde{D}_q\widetilde{I}_{q,0}^{-\alpha}f\right)(x) \\ &= \left(\widetilde{D}_q\widetilde{I}_{q,0}^{-\alpha+1}\widetilde{D}_q f\right)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha+1)}q^{\binom{-\alpha}{2}}\widetilde{D}_q\left(x^{-\alpha}\right) \end{aligned}$$

$$= \left({}^{c}\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_{q}f\right)(x) + \frac{f(0)}{\widetilde{\Gamma}_{q}(-\alpha+1)}q^{\binom{-\alpha}{2}}q^{1+\alpha}\overline{[-\alpha]}_{q}x^{-\alpha-1}$$

$$= \left({}^{c}\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_{q}f\right)(x) + \frac{f(0)}{\widetilde{\Gamma}_{q}(-\alpha)}q^{\binom{-(\alpha+1)}{2}}x^{-\alpha-1}.$$

By Theorem 9, the required equalities are valid both for $\alpha \leq -1$ and $-1 < \alpha < 0$.

(ii) $\alpha > 0$, we assume $\alpha = n+\varepsilon, n \in \mathbb{N}_0, 0 < \varepsilon < 1$, then $\alpha + 1 \in (n+1, n+2)$, by Theorem 3, (5.1), (2.6), and (2.8), we obtain

$$\begin{aligned}
\left(\widetilde{D}_{q}{}^{c}\widetilde{D}_{q,0}^{\alpha}f\right)(x) &= \left(\widetilde{D}_{q}\widetilde{I}_{q,0}^{1-\varepsilon}\widetilde{D}_{q}^{n+1}f\right)(x) \\
&= \left(\widetilde{D}_{q}\widetilde{I}_{q,0}^{2-\varepsilon}\widetilde{D}_{q}^{n+2}f\right)(x) + \frac{\widetilde{D}_{q}^{n+1}f(0)}{\widetilde{\Gamma}_{q}(2-\varepsilon)}q^{\binom{1-\varepsilon}{2}}\widetilde{D}_{q}\left(x^{1-\varepsilon}\right) \\
&= \left({}^{c}\widetilde{D}_{q,0}^{\alpha+1}f\right)(x) + \frac{\widetilde{D}_{q}^{n+1}f(0)}{\widetilde{\Gamma}_{q}(2-\varepsilon)}q^{\binom{1-\varepsilon}{2}}q^{\varepsilon}\overline{[1-\varepsilon]}_{q}x^{-\varepsilon} \\
&= \left({}^{c}\widetilde{D}_{q,0}^{\alpha+1}f\right)(x) + \frac{\widetilde{D}_{q}^{n+1}f(0)}{\widetilde{\Gamma}_{q}(n+1-\alpha)}q^{\binom{n-\alpha}{2}}x^{n-\alpha}.
\end{aligned}$$

□

Theorem 11 *Let $\alpha \in (N-1, N]$. Then for some constants $c_i \in \mathbb{R}, i = 0, 1, \ldots, N-1$, the following equality holds*:

$$\left(\widetilde{I}_{q,0}^{\alpha}{}^{c}\widetilde{D}_{q,0}^{\alpha}f\right)(x) = f(x) + c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1}x^{N-1}.$$

Proof By (5.1), (2.6), and (2.7), we have

$$\begin{aligned}
\left(\widetilde{I}_{q,0}^{\alpha}{}^{c}\widetilde{D}_{q,0}^{\alpha}f\right)(x) &= \left(\widetilde{I}_{q,0}^{\alpha}\widetilde{I}_{q,0}^{N-\alpha}\widetilde{D}_{q}^{N}f\right)(x) \\
&= \left(\widetilde{I}_{q,0}^{N}\widetilde{D}_{q}^{N}f\right)(x) \\
&= \widetilde{I}_{q,0}^{N-1}\left(\left(\widetilde{D}_{q}^{N-1}f\right)(x) - \left(\widetilde{D}_{q}^{N-1}f\right)(0)\right) \\
&= \left(\widetilde{I}_{q,0}^{N-1}\widetilde{D}_{q}^{N-1}f\right)(x) - \frac{q^{\binom{N-1}{2}}\left(\widetilde{D}_{q}^{N-1}f\right)(0)}{\overline{[N-1]}_{q}!}x^{N-1} \\
&= \left(\widetilde{I}_{q,0}^{N-2}\widetilde{D}_{q}^{N-2}f\right)(x) - \frac{q^{\binom{N-2}{2}}\left(\widetilde{D}_{q}^{N-2}f\right)(0)}{\overline{[N-2]}_{q}!}x^{N-2} - \frac{q^{\binom{N-1}{2}}\left(\widetilde{D}_{q}^{N-1}f\right)(0)}{\overline{[N-1]}_{q}!}x^{N-1} \\
&= \cdots \\
&= f(x) - \sum_{k=0}^{N-1}\frac{q^{\binom{k}{2}}\left(\widetilde{D}_{q}^{k}f\right)(0)}{\overline{[k]}_{q}!}x^{k}.
\end{aligned}$$

□

6 The application

In this section, we deal with the following nonlocal q-symmetric integral boundary value problem of nonlinear fractional q-symmetric derivatives equations:

$$\left(\widetilde{D}_{q,0}^{\alpha}u\right)(t) + f\left(q^{-\alpha}t, u\left(q^{-\alpha}t\right)\right) = 0, \quad t \in \left(0, q^{\alpha}\right), \tag{6.1}$$

$$u(0) = 0, \qquad u(1) = \mu\left(\widetilde{I}_{q,0}^{\beta}u\right)(\eta), \tag{6.2}$$

where $q \in (0,1), 1 < \alpha \leq 2, 0 < \beta \leq 2, 0 < \eta < 1$, and $\mu > 0$ is a parameter, $\widetilde{D}_{q,0}^{\alpha}$ is the q-symmetric derivative of Riemann-Liouville type of order $\alpha, f : [0,1] \times \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ is continuous.

We give the corresponding Green's function of the boundary value problem and its properties. By using the Krasnoselskii fix point theorem, an existence result of positive solutions to the above boundary value problem is enunciated.

For convenience, we need some preliminaries.

Let $q^2 = \overline{q}$, then by (2.2), we have

$$\begin{aligned}\widetilde{\Gamma}_q(x) &= \frac{\prod_{i=0}^{\infty}(1-q^{2i+2})}{\prod_{i=0}^{\infty}(1-q^{2(i+x-1)+2})}\left(1-q^2\right)^{1-x} \\ &= (1-\overline{q})^{(x-1)}(1-\overline{q})^{1-x} = \Gamma_{\overline{q}}(x) \quad \bigl(x \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}\bigr).\end{aligned} \tag{6.3}$$

The basic q-integrals are defined by

$$\begin{aligned}(I_{q,0}f)(t) &= \int_0^x f(t)\,d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f\bigl(xq^k\bigr), \\ (I_{q,a}f)(t) &= \int_a^x f(t)\,d_qt = \int_0^x f(t)\,d_qt - \int_0^a f(t)\,d_qt.\end{aligned} \tag{6.4}$$

Definition 1 ([23] (q-Beta function)) For any $x, y > 0$, $B_q(x,y) = \int_0^1 t^{(x-1)}(1-qt)^{(y-1)}\,d_qt$. Recall that

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

Therefore,

$$B_{\overline{q}}(x,y) = \frac{\Gamma_{\overline{q}}(x)\Gamma_{\overline{q}}(y)}{\Gamma_{\overline{q}}(x+y)} = \frac{\widetilde{\Gamma}_q(x)\widetilde{\Gamma}_q(y)}{\widetilde{\Gamma}_q(x+y)}. \tag{6.5}$$

Lemma 2 *For* $\lambda \in (-1,\infty)$, *the following is valid*:

$$\begin{aligned}&\text{(i)}\quad \widetilde{I}_{q,0}^{\alpha}x^{\lambda} = \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+\alpha+1)}q^{\binom{\alpha}{2}+\lambda\alpha}x^{\lambda+\alpha} \quad \bigl(\alpha \in \mathbb{R}^{+}\bigr), \\ &\text{(ii)}\quad \widetilde{D}_{q,0}^{\alpha}x^{\lambda} = \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda-\alpha+1)}q^{\binom{-\alpha}{2}-\lambda\alpha}x^{\lambda-\alpha} \quad \bigl(\alpha \in \mathbb{R}^{+}, \lambda-\alpha+1 \neq 0\bigr).\end{aligned}$$

Proof (i) For $\lambda \neq 0$, according to (3.1), we have

$$\begin{aligned}\widetilde{I}_{q,0}^{\alpha}x^{\lambda} &= \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^x \overline{(x-\tau)}^{(\alpha-1)}\bigl(q^{\alpha-1}\tau\bigr)^{\lambda}\,\widetilde{d}_q\tau \\ &= \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}x^{\alpha+\lambda}q^{\lambda(\alpha-1)}\int_0^x \overline{\left(1-\frac{\tau}{x}\right)}^{(\alpha-1)}\left(\frac{\tau}{x}\right)^{\lambda}\widetilde{d}_q\left(\frac{\tau}{x}\right) \\ &= \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}x^{\alpha+\lambda}q^{\lambda(\alpha-1)}\int_0^1 \overline{(1-s)}^{(\alpha-1)}s^{\lambda}\,\widetilde{d}_qs.\end{aligned}$$

Let $q^2 = \overline{q}$, by (2.3), (6.3), (6.4), and Definition 1, we get

$$\begin{aligned}\int_0^1 \overline{(1-s)}^{(\alpha-1)} s^\lambda \,\widetilde{d}_q s &= \bigl(1-q^2\bigr)\sum_{k=0}^{\infty} q^{2k}\overline{\bigl(1-q^{2k+1}\bigr)}^{(\alpha-1)} q^{\lambda(2k+1)} \\ &= q^\lambda\bigl(1-q^2\bigr)\sum_{k=0}^{\infty} q^{2k}\prod_{i=0}^{\infty}\frac{1-q^{2(k+i+1)}}{1-q^{2(k+i+\alpha)}} q^{2\lambda k} \\ &= \overline{q}^{\frac{\lambda}{2}}(1-\overline{q})\sum_{k=0}^{\infty}\overline{q}^k\prod_{i=0}^{\infty}\frac{1-\overline{q}^{k+i+1}}{1-\overline{q}^{k+i+\alpha}}\overline{q}^{\lambda k} \\ &= \overline{q}^{\frac{\lambda}{2}}(1-\overline{q})\sum_{k=0}^{\infty}\overline{q}^k\bigl(1-\overline{q}^{k+1}\bigr)^{(\alpha-1)}\overline{q}^{\lambda k} \\ &= \overline{q}^{\frac{\lambda}{2}}\int_0^1 (1-\overline{q}x)^{(\alpha-1)}x^\lambda \,d_{\overline{q}}x \\ &= \overline{q}^{\frac{\lambda}{2}} B_{\overline{q}}(\lambda+1,\alpha) \\ &= q^\lambda\frac{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+\alpha+1)}.\end{aligned}$$

Hence, we obtain the required formula for $\widetilde{I}_{q,0}^{\alpha}x^{(\lambda)}$ when $\lambda \neq 0$.

If $\lambda = 0$, then using (2.8), we have

$$\begin{aligned}\bigl(\widetilde{I}_{q,0}^{\alpha}\mathbf{1}\bigr)(x) &= \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^x \overline{(x-\tau)}^{(\alpha-1)}\,\widetilde{d}_q\tau \\ &= \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^x \frac{\widetilde{D}_q\overline{(x-q^{-1}\tau)}^{(\alpha)}}{-\overline{[\alpha]}_q}\,\widetilde{d}_q\tau \\ &= \frac{-1}{\widetilde{\Gamma}_q(\alpha+1)}q^{\binom{\alpha}{2}}\int_0^x \widetilde{D}_q\overline{\bigl(x-q^{-1}\tau\bigr)}^{(\alpha)}\,\widetilde{d}_q\tau \\ &= \frac{1}{\widetilde{\Gamma}_q(\alpha+1)}q^{\binom{\alpha}{2}}x^\alpha.\end{aligned}$$

(ii) By (i) and (4.1) and (2.7), we get

$$\begin{aligned}\widetilde{D}_{q,0}^{\alpha}\bigl(x^\lambda\bigr) &= \widetilde{D}_q^{[\alpha]}\widetilde{I}_{q,0}^{[\alpha]-\alpha}\bigl(x^\lambda\bigr) \\ &= \widetilde{D}_q^{[\alpha]}\left(\frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}q^{\binom{[\alpha]-\alpha}{2}}q^{\lambda([\alpha]-\alpha)}x^{\lambda+[\alpha]-\alpha}\right) \\ &= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}q^{\binom{[\alpha]-\alpha}{2}}q^{\lambda([\alpha]-\alpha)}\widetilde{D}_q^{[\alpha]}\bigl(x^{\lambda+[\alpha]-\alpha}\bigr) \\ &= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}q^{\binom{[\alpha]-\alpha}{2}}q^{\lambda([\alpha]-\alpha)}q^{-\binom{[\alpha]+1}{2}-[\alpha](\lambda-\alpha-1)} \\ &\quad\times\frac{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}{\widetilde{\Gamma}_q(\lambda-\alpha+1)}x^{\lambda-\alpha} \\ &= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda-\alpha+1)}q^{\binom{-\alpha}{2}-\lambda\alpha}x^{\lambda-\alpha}.\end{aligned}$$

□

Lemma 3 *Let* $M = \widetilde{\Gamma}_q(\alpha+\beta) - \mu\eta^{\alpha+\beta-1}q^{\binom{\beta}{2}+(\alpha-1)\beta}\widetilde{\Gamma}_q(\alpha) > 0$. *Then, for a given* $y \in C[0,1]$, *the unique solution of the boundary value problem*

$$\big(\widetilde{D}_{q,0}^{\alpha}u\big)(t) + y\big(q^{-\alpha}t\big) = 0, \quad t \in \big(0,q^{\alpha}\big), 1 < \alpha \leq 2, \tag{6.6}$$

subject to the boundary condition

$$u(0) = 0, \qquad u(1) = \mu\widetilde{I}_{q,0}^{\beta}u(\eta), \quad 0 < \beta \leq 2, 0 < \eta < 1, \tag{6.7}$$

is given by

$$u(t) = \int_0^1 G(t,s)y\big(q^{-1}s\big)\,\widetilde{d}_q s, \quad t \in [0,1], \tag{6.8}$$

where

$$G(t,s) = g(t,s) + \frac{\mu t^{\alpha-1}}{M}H(\eta,s), \tag{6.9}$$

$$g(t,s) = \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\begin{cases} t^{\alpha-1}\overline{(1-s)}^{(\alpha-1)} - \overline{(t-s)}^{(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}\overline{(1-s)}^{(\alpha-1)}, & 0 \leq t < s \leq 1, \end{cases} \tag{6.10}$$

$$H(\eta,s) = \eta^{\alpha+\beta-1}q^{\binom{\alpha+\beta}{2}-\beta}\begin{cases} \overline{(1-s)}^{(\alpha-1)} - \overline{(1-\eta^{-1}q^{-\beta}s)}^{(\alpha+\beta-1)}, & 0 \leq s \leq \eta q^{\beta}, \\ \overline{(1-s)}^{(\alpha-1)}, & \eta q^{\beta} < s \leq 1. \end{cases} \tag{6.11}$$

Proof In view of Theorem 8, we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^t \overline{(t-s)}^{(\alpha-1)}y\big(q^{-1}s\big)\,\widetilde{d}_q s, \quad t \in [0,1], \tag{6.12}$$

for some constant $c_1, c_2 \in \mathbb{R}$. Since $u(0) = 0$, we have $c_2 = 0$.

Using Lemma 2, Theorem 2, we have

$$\begin{aligned}\big(I_{q,0}^{\beta}u\big)(t) &= c_1 I_{q,0}^{\beta}t^{\alpha-1} - \big(I_{q,0}^{\alpha+\beta}y\big)\big(q^{-\alpha}t\big) \\ &= c_1\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\beta}{2}+(\alpha-1)\beta}t^{\alpha+\beta-1} \\ &\quad - \frac{1}{\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\int_0^t \overline{(t-s)}^{(\alpha+\beta-1)}y\big(q^{\beta-1}s\big)\,\widetilde{d}_q s.\end{aligned}$$

From the boundary condition $u(1) = \mu\widetilde{I}_{q,0}^{\beta}u(\eta)$, we get

$$\begin{aligned}c_1 = \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{M}&\bigg(\frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^1 \overline{(1-s)}^{(\alpha-1)}y\big(q^{-1}s\big)\,\widetilde{d}_q s \\ &- \frac{\mu}{\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\int_0^{\eta}\overline{(\eta-s)}^{(\alpha+\beta-1)}y\big(q^{\beta-1}s\big)\,\widetilde{d}_q s\bigg).\end{aligned}$$

Hence

$$
\begin{aligned}
u(t) &= \frac{t^{\alpha-1}\widetilde{\Gamma}_q(\alpha+\beta)}{M}\biggl(\frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^1 \overline{(1-s)}^{(\alpha-1)}y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs \\
&\quad - \frac{\mu}{\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\int_0^\eta \overline{(\eta-s)}^{(\alpha+\beta-1)}y\bigl(q^{\beta-1}s\bigr)\,\widetilde{d}_qs\biggr) \\
&\quad - \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^t \overline{(t-s)}^{(\alpha-1)}y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs \\
&= \frac{t^{\alpha-1}}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^1 \overline{(1-s)}^{(\alpha-1)}y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs \\
&\quad - \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^t \overline{(t-s)}^{(\alpha-1)}y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs \\
&\quad + \frac{\mu t^{\alpha-1}\eta^{\alpha+\beta-1}q^{\binom{\alpha+\beta}{2}-\beta}}{M}\int_0^1 \overline{(1-s)}^{(\alpha-1)}y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs \\
&\quad - \frac{\mu t^{\alpha-1}}{M}q^{\binom{\alpha+\beta}{2}}\int_0^\eta \overline{(\eta-s)}^{(\alpha+\beta-1)}y\bigl(q^{\beta-1}s\bigr)\,\widetilde{d}_qs \\
&= \int_0^1 g(t,s)y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs + \frac{\mu t^{\alpha-1}}{M}\int_0^1 H(\eta,s)y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs \\
&= \int_0^1 G(t,s)y\bigl(q^{-1}s\bigr)\,\widetilde{d}_qs. \qquad \square
\end{aligned}
$$

According to the property of being non-increasing of $\overline{(t-s)}^{(\alpha)}$ on s, we may easily obtain Lemma 4 and Lemma 5 as follows.

Lemma 4 *The functions $g(t,s)$ and $H(\eta,s)$ satisfy the following properties*:

(i) $g(t,s)\ge 0, g(t,s)\le g(s,s), 0\le t,s\le 1$.

(ii) $H(\eta,s)\ge 0, 0\le s\le 1$.

Lemma 5 *The function $G(t,s)$ satisfies the following properties*:

(i) *G is a continuous function and $G(t,s)\ge 0, (t,s)\in[0,1]\times[0,1]$.*

(ii) *There exists a positive function $\rho\in C((0,1),(0,+\infty))$ such that* $\max_{0\le t\le 1}G(t,s)\le g(s,s)+\frac{\mu}{M}H(\eta,s) =: \rho(s), s\in(0,1)$.

Lemma 6 (Krasnoselskii) *Let $\mathbb{E}$ be a Banach space, and let $P\subset\mathbb{E}$ be a cone. Assume Ω_1,Ω_2 are open subsets of $\mathbb{E}$ with $\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T: P\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow P$ be a completely continuous operator such that*

$$\|Tu\|\ge\|u\|,\quad u\in P\cap\partial\Omega_1,\quad \textit{and}\quad \|Tu\|\le\|u\|,\quad u\in P\cap\partial\Omega_2.$$

Then T has at least one fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Let $X=C[0,1]$ be a Banach space endowed with norm $\|u\|_X=\max_{0\le t\le 1}|u(t)|$. Define the cone $P\subset\{u\in X: u(t)\ge 0, 0\le t\le 1\}$.

Define the operator $T: P\longrightarrow X$ as follows:

$$(Tu)(t)=\int_0^1 G(t,s)f\bigl(q^{-1}s,u\bigl(q^{-1}s\bigr)\bigr)\,\widetilde{d}_qs. \tag{6.13}$$

It follows from the non-negativeness and continuity of G and f that the operator $T : P \longrightarrow X$ satisfies $T(P) \subset P$ and is completely continuous.

Theorem 12 *Let $f(t,u)$ be a nonnegative continuous function on $[0,1] \times \mathbb{R}^+$. In addition, we assume that*:

(H_1) *There exists a positive constant r_1 such that*

$$f(t,u) \geq \kappa r_1, \quad \textit{for } (t,u) \in [\tau_1,\tau_2] \times [0,r_1],$$

where $\tau_1 = q^{m_3}, \tau_2 = q^{m_4}$ with $m_3, m_4 \in \mathbb{N}_0, m_3 > m_4 > 0$, and

$$\kappa \geq \left(q^{-1} \int_{q\tau_1}^{q\tau_2} \left(g(s,s) + \frac{\mu s^{\alpha-1}}{M} H(\eta,s) \right) \widetilde{d}_q s \right)^{-1}.$$

(H_2) *There exists a positive constant r_2 with $r_2 > r_1$ such that*

$$f(t,u) \leq L r_2, \quad \textit{for } (t,u) \in [0,1] \times [0,r_2],$$

where

$$L = \left(\int_0^1 \left(\frac{q^{\binom{\alpha}{2}} \overline{(1-s)}^{(\alpha-1)}}{\widetilde{\Gamma}_q(\alpha)} + \frac{\mu}{M} H(\eta,s) \right) \widetilde{d}_q s \right)^{-1}.$$

Then the boundary value problem (6.1), (6.2) *has at least one positive solution u_0 satisfying* $0 < r_1 \leq \|u_0\|_X \leq r_2$.

Proof By Lemma 4, we obtain $\max_{0\leq t\leq 1} g(t,s) = g(s,s)$. Let $\Omega_1 = \{u \in X : \|u\|_X < r_1\}$. For any $u \in X \cap \partial\Omega_1$, according to ($H_1$), we have

$$\begin{aligned}
\|Tu\|_X &= \max_{0\leq t\leq 1} \left(\int_0^1 g(t,s) f\left(q^{-1}s, u\left(q^{-1}s\right)\right) \widetilde{d}_q s \right. \\
&\quad \left. + \int_0^1 \frac{\mu t^{\alpha-1}}{M} H(\eta,s) f\left(q^{-1}s, u\left(q^{-1}s\right)\right) \widetilde{d}_q s \right) \\
&\geq \int_0^1 \left[g(s,s) + \frac{\mu (s)^{\alpha-1}}{M} H(\eta,s) \right] f\left(q^{-1}s, u\left(q^{-1}s\right)\right) \widetilde{d}_q s \\
&= \left(1-q^2\right) \sum_{k=0}^{\infty} q^{2k} \left[g\left(q^{2k+1}, q^{2k+1}\right) + \frac{\mu (q^{2k+1})^{\alpha-1}}{M} H\left(\eta, q^{2k+1}\right) \right] f\left(q^{2k}, u\left(q^{2k}\right)\right) \\
&= \int_0^1 \left[g(qs,qs) + \frac{\mu (sq)^{\alpha-1}}{M} H(\eta,qs) \right] f\left(s,u(s)\right) d_{q^2} s \\
&\geq \kappa r_1 \int_{\tau_1}^{\tau_2} \left[g(qs,qs) + \frac{\mu (qs)^{\alpha-1}}{M} H(\eta,qs) \right] d_{q^2} s \\
&= q^{-1} \kappa r_1 \int_{q\tau_1}^{q\tau_2} \left[g(s,s) + \frac{\mu s^{\alpha-1}}{M} H(\eta,s) \right] d_{q^2} s \\
&= r_1 = \|u\|_X.
\end{aligned}$$

Let $\Omega_2 = \{u \in X : \|u\|_X < r_2\}$. For any $u \in X \cap \partial\Omega_2$, by ($H_2$), we have

$$\begin{aligned} \|Tu\|_X &= \max_{0\leq t\leq 1} \int_0^1 G(t,s) f\bigl(q^{-1}s, u\bigl(q^{-1}s\bigr)\bigr)\,\widetilde{d}_q s \\ &\leq L r_2 \int_0^1 \rho(s)\,\widetilde{d}_q s \\ &\leq \|u\|_X = r_2. \end{aligned}$$

Now, an application of Lemma 6 concludes the proof. □

It is hoped that our work will provide motivation for further results for fractional q-symmetric quantum calculus.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CH, MS and YJ worked together in the derivation of the mathematical results. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the referee for invaluable comments and insightful suggestions. This work was supported by NSFC project (No. 11161049, 11361066).

Received: 6 March 2016 Accepted: 16 August 2016 Published online: 30 August 2016

References
1. Jackson, FH: On q-functions and a certain difference operator. Trans. R. Soc. Edinb. **46**, 253-281 (1908)
2. Jackson, FH: On q-definite integrals. Q. J. Pure Appl. Math. **41**, 193-203 (1910)
3. Strominger, A: Information in black hole radiation. Phys. Rev. Lett. **71**, 3743-3746 (1993)
4. Youm, D: q-Deformed conformal quantum mechanics. Phys. Rev. D **62**, 095009 (2000)
5. Lavagno, A, Swamy, PN: q-Deformed structures and nonextensive statistics: a comparative study. Physica A **305**(1-2), 310-315 (2002)
6. Ernst, T: The History of q-calculus and New Method. (Licentiate Thesis), U.U.D.M. Report (2000) http://math.uu.se/thomas/Lics.pdf
7. Al-Salam, WA: q-Analogues of Cauchy's formula. Proc. Am. Math. Soc. **17**, 182-184 (1952-1953)
8. Al-Salam, WA, Verma, A: A fractional Leibniz q-formula. Pac. J. Math. **60**, 1-9 (1975)
9. Al-Salam, WA: Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. **15**, 135-140 (1969)
10. Agrawal, RP: Certain fractional q-integrals and q-derivatives. Proc. Camb. Philos. Soc. **66**, 365-370 (1969)
11. Atici, FM, Eloe, PW: Fractional q-calculus on a time scale. J. Nonlinear Math. Phys. **14**(3), 333-344 (2007)
12. Rajković, PM, Marinković, SD, Stanković, MS: Fractional integrals and derivatives in q-calculus. Appl. Anal. Discrete Math. **1**(1), 311-323 (2007)
13. Abdeljawad, T, Benli, B, Baleanu, D: A generalized q-Mittag-Leffler function by q-Caputo fractional linear equations. Abstr. Appl. Anal. **2012**, 546062 (2012)
14. Abdeljawad, T, Baleanu, D: Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function. Commun. Nonlinear Sci. Numer. Simul. **16**, 4682-4688 (2011)
15. Baleanu, D, Agarwal, P: Certain inequalities involving the fractional q-integral operators. Abstr. Appl. Anal. **2014**, 371274 (2014)
16. Zhang, L, Baleanu, D, Wang, G: Nonlocal boundary value problem for nonlinear impulsive $q(k)$-integrodifference equation. Abstr. Appl. Anal. **2014**, 478185 (2014)
17. Ferreira, RAC: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. **61**(2), 367-373 (2011)
18. Ferreira, RAC: Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2010**, 70 (2010)
19. El-Shahed, M, Al-Askar, FM: Positive solutions for boundary value problem of nonlinear fractional q-difference equation. ISRN Math. Anal. **2011**, 385459 (2011)
20. Ahmad, B, Alsaedi, A, Ntouyas, SK: A study of second-order q-difference equations with boundary conditions. Adv. Differ. Equ. **2012**, 35 (2012)
21. Ahmad, B, Nieto, JJ: On nonlocal boundary value problems of nonlinear q-difference equations. Adv. Differ. Equ. **2012**, 81 (2012)
22. Zhou, W-X, Liu, H-Z: Existence solutions for boundary value problem of nonlinear fractional q-difference equations. Adv. Differ. Equ. **2013**, 113 (2013)
23. Ahmad, B, Nieto, JJ, Alsaedi, A, Al-Hutami, H: Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. J. Franklin Inst. **351**, 2890-2909 (2014)
24. Lavagno, A: Basic-deformed quantum mechanics. Rep. Math. Phys. **64**, 79-88 (2009)
25. Brito Da Cruz, AMC, Martins, N: The q-symmetric variational calculus. Comput. Math. Appl. **64**, 2241-2250 (2012)