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Robust stability of a class of stochastic functional differential equations with Markovian switching

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Abstract

Robust stability has been one of the most popular topics in the study of stochastic functional differential equations with Markovian switching (SFDEwMSs), including stochastic delay differential equations with Markovian switching (hybrid SDDEs). Most of the existing results on the robust stability require that the drift coefficient $\boldsymbol{f}$ and diffusion coefficient $\mathbf{g}$ of the stochastic system are either linear or nonlinear with linear growth condition. Recently Hu, Mao and Zhang (IEEE Trans. Autom. Control 58:2319-2332, 2013) obtained some new results for the robust stability of nonlinear hybrid SDDEs, requiring $\boldsymbol{x}^T\boldsymbol{f}$ and $|\mathbf{g}|^2$ to be bounded by polynomials with the same orders. However, there are many SFDEwMSs which do not satisfy the above requirement. Hence the existing criteria on the robust stability are not applicable and we see the necessity to develop some new criteria. Our aim in this paper is to establish some new criteria for the robust stability of a class of SFDEwMSs, where $\boldsymbol{x}^T\boldsymbol{f}$ and $|\mathbf{g}|^2$ are controlled by polynomials with different orders.

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1 Introduction

In general, time delays and system uncertainty are commonly encountered problems in deterministic or stochastic dynamical systems, which usually result in instability (see [2]). Hence the study of the stability is an interesting topic. Systems in many application fields do not only depend on the present state but also the past states. Stochastic functional differential equations (SFDEs), including stochastic delay differential equations (SDDEs), have been used to describe such systems. The stability theory of SFDEs has received lots of attention over the past years (see [3–8]). However, these systems may experience abrupt changes in their structure and parameters. So continuous-time Markovian chains have been introduced to cope with such situations. These systems can be described as stochastic functional differential equations with Markovian switching (SFDEwMSs) including stochastic delay differential equations with Markovian switching (SDDEwMSs or hybrid SDDEs), and many researchers have studied the stability theory of SFDEwMSs (see [9–17]).

Generally speaking, an n-dimensional SFDEwMS has the form

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}\big(\boldsymbol{x}_t, t, r(t)\big)\,\mathrm{d}t + \mathbf{g}\big(\boldsymbol{x}_t, t, r(t)\big)\,\mathrm{d}B(t), \tag{1}$$

on $t \geq 0$, where $\tau > 0$, $r(t)$ is a right-continuous Markovian chain with a finite state space $S = \{1, 2, \dots, N\}$ and $B(t)$ is an m-dimensional Brownian motion (the other notations used here will be explained in detail in Section 2). System (1) can be regarded as a stochastically perturbed system of the determined functional differential equation with Markovian switching (FDEwMS),

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}\big(\boldsymbol{x}_t, t, r(t)\big). \tag{2}$$

Then there is a natural problem: if the system (2) is asymptotically stable, how much stochastic perturbation can this system tolerate without losing the property of asymptotic stability? Such a kind of problem is known as the problem of robust stability, which has received a great deal of attention. For example, Mao [18] provided a sufficient condition such that the exponentially stable nonlinear system with a stochastic delay perturbation remains exponentially stable. Mao *et al.* [19] discussed the robust stability of uncertain linear/semilinear stochastic differential delay equations. Hu *et al.* [20] studied the robustness of exponential stability of stochastic functional differential system with infinite delay. Then Wu and Hu [21] investigated the robustness of exponential stability of the nonlinear functional differential system. There is also some other extensive literature about the robust stability of continuous differential systems and here we only mention [22, 23]. On the other hand, some results about the robust stability of differential systems with Markovian switching have been obtained over recent years. For example, Mao [24] discussed the robust stability of the stochastic delay interval system with Markovian switching. Yuan and Mao [10] investigated the controllability and robust stability for linear stochastic differential delay equations with Markovian switching. As to the further development, please see [1, 11, 25, 26], and the references therein.

In particular, we would like to mention the work of Hu *et al.* [1]. They studied the robust stability and robust boundedness for a stochastically perturbed system of the determined nonlinear delay differential equation having the form

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}_1\big(\boldsymbol{x}(t), \boldsymbol{x}(t-\tau), t, r(t)\big)\,\mathrm{d}t + \mathbf{g}_1\big(\boldsymbol{x}(t), \boldsymbol{x}(t-\tau), t, r(t)\big)\,\mathrm{d}B(t), \tag{3}$$

on $t \geq 0$, where $\tau > 0$, $\boldsymbol{f}_1 : R^n \times R^n \times R_+ \times S \to R^n$, $\mathbf{g}_1 : R^n \times R^n \times R_+ \times S \to R^{n\times m}$, $r(t)$ and $B(t)$ have the same definitions as with system (1). More precisely, their results were obtained under the condition that there exist nonnegative numbers $q > p \geq 2$, $\beta_{i1}, \beta_{i3}, \beta_{i4}, \beta_{i5}$ and a real number β_{i2} such that

$$\begin{aligned} &\boldsymbol{x}^T \boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{y}, t, i) + \frac{p-1}{2}\big|\mathbf{g}_1(\boldsymbol{x}, \boldsymbol{y}, t, i)\big|^2 \\ &\quad \leq \beta_{i1} + \beta_{i2}|\boldsymbol{x}|^2 + \beta_{i3}|\boldsymbol{y}|^2 - \beta_{i4}|\boldsymbol{x}|^{q-p+2} + \beta_{i5}|\boldsymbol{y}|^{q-p+2}, \end{aligned} \tag{4}$$

for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$.

It is natural to consider the robust stability of nonlinear SFDEwMSs. However, there are many SFDEwMSs which do not satisfy the conditions similar to (4). For example, let us investigate a generalized logistic differential system with Markovian switching,

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{x}(t)\biggl[c\bigl(r(t)\bigr) - a\bigl(r(t)\bigr)\bigl|\boldsymbol{x}(t)\bigr|^2 + b\bigl(r(t)\bigr)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,d\eta(\theta)\biggr], \tag{5}$$

where $r(t)$ is a right-continuous Markovian chain with a finite state space S, η is a probability measure on $[-\tau,0]$ satisfying $\int_{-\tau}^{0} d\eta(\theta) = 1$. When $\boldsymbol{x}(t)$ is given, by the well-known central limit theorem, we can estimate the parameter $c(r(t))$ as $\overline{c(r(t))} + \sqrt{\mathrm{Var}}\dot{B}(t)$ similar to [1], where $\dot{B}(t)$ is a white noise, $\overline{c(r(t))}$ is the average of the estimator of $c(r(t))$ and independent of $\boldsymbol{x}(t)$, Var is the variance. Noting that Var may depend on the present state $\boldsymbol{x}(t)$ (*e.g.* $\mathrm{Var} = (\rho(r(t))\boldsymbol{x}(t))^2$) or the past state $\boldsymbol{x}(t-\tau)$ (*e.g.* $\mathrm{Var} = (\rho(r(t))\boldsymbol{x}(t-\tau))^2$, due to time lag) for determined differential systems from [1], since $\boldsymbol{x}(t)$ of system (5) depends on any historical state on the interval $[t-\tau,t]$, it is often that we have to estimate the Var of the estimator of parameter $c(r(t))$ based on any historical state on the interval $[t-\tau,t]$ instead of the present state $\boldsymbol{x}(t)$ or the past state $\boldsymbol{x}(t-\tau)$. Similar to [1], the variance may have the form $\mathrm{Var} = [\rho(r(t))\int_{-\tau}^{0}|\boldsymbol{x}(t+\theta)|\,d\eta(\theta)]^2$, then system (5) becomes a SFDEwMS

$$\begin{aligned}\mathrm{d}\boldsymbol{x}(t) ={}& \boldsymbol{x}(t)\biggl[\overline{c\bigl(r(t)\bigr)} - a\bigl(r(t)\bigr)\bigl|\boldsymbol{x}(t)\bigr|^2 + b\bigl(r(t)\bigr)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,d\eta(\theta)\biggr]\,\mathrm{d}t \\ &+ \boldsymbol{x}(t)\bigl|\rho\bigl(r(t)\bigr)\bigr|\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|\,d\eta(\theta)\,\mathrm{d}B(t).\end{aligned} \tag{6}$$

By simple computation, we have

$$\begin{aligned}&\boldsymbol{x}^T\boldsymbol{f}(\boldsymbol{x}_t,t,i) + \frac{1}{2}\bigl|\mathbf{g}(\boldsymbol{x}_t,t,i)\bigr|^2 \\ &\quad= \bigl|\boldsymbol{x}(t)\bigr|^2\biggl[\overline{c(i)} - a(i)\bigl|\boldsymbol{x}(t)\bigr|^2 + b(i)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,d\eta(\theta)\biggr] \\ &\qquad+ \frac{1}{2}\rho^2(i)\biggl(\boldsymbol{x}(t)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|\,d\eta(\theta)\biggr)^2 \\ &\quad\le \overline{c(i)}\bigl|\boldsymbol{x}(t)\bigr|^2 - a(i)\bigl|\boldsymbol{x}(t)\bigr|^4 + \frac{1}{2}b(i)\bigl|\boldsymbol{x}(t)\bigr|^4 + \frac{1}{2}b(i)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^4\,d\eta(\theta) \\ &\qquad+ \frac{1}{2}\rho^2(i)\biggl[\frac{1}{2}\bigl|\boldsymbol{x}(t)\bigr|^2 + \frac{1}{2}\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,d\eta(\theta)\biggr]^2 \\ &\quad\le \overline{c(i)}\bigl|\boldsymbol{x}(t)\bigr|^2 - \biggl[a(i) - \frac{b(i)}{2} - \frac{\rho^2(i)}{4}\biggr]\bigl|\boldsymbol{x}(t)\bigr|^4 \\ &\qquad+ \biggl[\frac{b(i)}{2} + \frac{\rho^2(i)}{4}\biggr]\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^4\,d\eta(\theta),\end{aligned} \tag{7}$$

which can be seen as a trivial extension of (4) with $p = 2$ and $q = 4$.

Similar to [1], we can get some new conclusions about robust stability to deal with the above case. However, those new conclusions are trivial, which are not our main aim. In this paper, we mainly aim to consider the case of the variance with a lower order than

$[\rho(r(t))\int_{-\tau}^{0}|\boldsymbol{x}(t+\theta)|\,d\eta(\theta)]^2$, *e.g.* $\mathrm{Var}=[\rho(r(t))\int_{-\tau}^{0}|\boldsymbol{x}(t+\theta)|^{\varsigma}\,d\eta(\theta)]^2$, $\varsigma\in(0,1)$. For example, when taking $\varsigma=\frac{1}{3}$, system (5) becomes another SFDEwMS,

$$\begin{aligned}\mathrm{d}\boldsymbol{x}(t)&=\boldsymbol{x}(t)\left[\overline{c\bigl(r(t)\bigr)}-a\bigl(r(t)\bigr)\bigl|\boldsymbol{x}(t)\bigr|^2+b\bigl(r(t)\bigr)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,d\eta(\theta)\right]\mathrm{d}t\\&\quad+\boldsymbol{x}(t)\bigl|\rho\bigl(r(t)\bigr)\bigr|\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^{\frac{1}{3}}\,d\eta(\theta)\,\mathrm{d}B(t).\end{aligned}\tag{8}$$

Similar to (7), by simple computation, we have

$$\begin{aligned}&\boldsymbol{x}^T\boldsymbol{f}(\boldsymbol{x}_t,t,i)+\frac{1}{2}\bigl|\mathbf{g}(\boldsymbol{x}_t,t,i)\bigr|^2\\&\quad=\bigl|\boldsymbol{x}(t)\bigr|^2\left[\overline{c(i)}-a(i)\bigl|\boldsymbol{x}(t)\bigr|^2+b(i)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,d\eta(\theta)\right]\\&\qquad+\frac{1}{2}\rho^2(i)\left(\boldsymbol{x}(t)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^{\frac{1}{3}}\,d\eta(\theta)\right)^2\\&\quad\le\overline{c(i)}\bigl|\boldsymbol{x}(t)\bigr|^2-a(i)\bigl|\boldsymbol{x}(t)\bigr|^4+\frac{1}{2}b(i)\bigl|\boldsymbol{x}(t)\bigr|^4+\frac{1}{2}b(i)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^4\,d\eta(\theta)\\&\qquad+\rho^2(i)\left[\left(\frac{3}{4}\right)^2\bigl|\boldsymbol{x}(t)\bigr|^{\frac{8}{3}}+\left(\frac{1}{4}\right)^2\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^{\frac{8}{3}}\,d\eta(\theta)\right]\\&\quad\le\left(\frac{3}{4}\rho(i)\right)^2\bigl|\boldsymbol{x}(t)\bigr|^{\frac{8}{3}}+\left(\frac{1}{4}\rho(i)\right)^2\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^{\frac{8}{3}}\,d\eta(\theta)+\overline{c(i)}\bigl|\boldsymbol{x}(t)\bigr|^2\\&\qquad-\left[a(i)-\frac{1}{2}b(i)\right]\bigl|\boldsymbol{x}(t)\bigr|^4+\frac{1}{2}b(i)\int_{-\tau}^{0}\bigl|\boldsymbol{x}(t+\theta)\bigr|^4\,d\eta(\theta).\end{aligned}\tag{9}$$

Because of the existence of the terms $|\boldsymbol{x}(t)|^{\frac{8}{3}}$ and $\int_{-\tau}^{0}|\boldsymbol{x}(t+\theta)|^{\frac{8}{3}}\,d\eta(\theta)$ in (9), the technique used by Hu *et al.* [1] and other already existing results cannot be applied to system (8) directly. Hence, it is natural to ask the following questions:

(1) Is there a unique global solution, if a SFDEwMS satisfies some new conditions similar to (9)?
(2) If the answer to question (1) is yes, how much stochastic perturbation can a stable FDEwMS tolerate without losing the property of asymptotic stability?

Due to these questions, it is necessary to develop some new theory of the robust stability for SFDEwMSs with some new conditions similar to (9). So far as we know there is no such a kind of discussions for SFDEwMSs. Hence, our main aim in this paper is to establish some new results of the existence and uniqueness and the robust stability of the global solution for SFDEwMSs.

The organization of this paper is as follows. Some preliminaries are described in Section 2. The pth moment asymptotic stability and exponential stability of SFDEwMSs are proved in Section 3. Some sufficient conditions for the robust stability are established in Section 4. Some examples are given to illustrate our theorems in Section 5.

2 Preliminaries

Throughout this paper, let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\ge0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\ge0}$ satisfying the usual conditions. If A is a matrix or vector, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Let $B(t)$ be an m-dimensional Brownian motion defined on the probability space. Let $\tau > 0$ and let $C([-\tau,0];R^n)$ denote the family of all continuous R^n-valued functions φ on $[-\tau,0]$ with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0}|\varphi(\theta)|$. Let $C^b_{\mathscr{F}_0}([-\tau,0];R^n)$ be the family of all bounded, $\mathscr{F}_0$-measurable, $C([-\tau,0];R^n)$-valued, $\mathscr{F}_t$-adapted stochastic processes. Let η_j be probability measures on $[-\tau,0]$, which satisfy $\int_{-\tau}^0 d\eta_j(\theta) = 1$ ($j = 1,2,3,4$). Let $L^1(R_+;R_+)$ be the family of all functions $\xi : R_+ \to R_+$ such that $\int_0^{+\infty}\xi(t)\mathrm{d}t < \infty$. $\boldsymbol{x}(t)$ is a continuous R-valued stochastic process on $t \in [-\tau,\infty)$. We assume $\boldsymbol{x}_t = \{\boldsymbol{x}(t+\theta) : -\tau \le \theta \le 0\}$ for all $t \ge 0$, which is regarded as a $C([-\tau,0];R^n)$-valued stochastic process.

Let $r(t), t \ge 0$, be a right-continuous Markovian chain on the probability space taking values in a finite state space $S = \{1,2,\dots,N\}$, with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\bigl\{r(t+\Delta) = j|r(t) = i\bigr\} = \begin{cases}\gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j,\\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j,\end{cases}$$

where $\Delta > 0$. Here γ_{ij} is the transition rate from i to j, if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$.

Assume that Markovian chain $r(t)$ is independent of Brownian motion $B(t)$. It is well known that almost every sample path of $r(t)$ is right-continuous step function.

Consider an n-dimensional SFDE with Markovian switching of the form

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}\bigl(\boldsymbol{x}_t, t, r(t)\bigr)\,\mathrm{d}t + \mathbf{g}\bigl(\boldsymbol{x}_t, t, r(t)\bigr)\,\mathrm{d}B(t), \tag{10}$$

on $t \ge 0$ with initial data $\{\boldsymbol{x}(\theta) : -\tau \le \theta \le 0\} = \boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$, $i_0 \in S$, where

$$\boldsymbol{f} : C\bigl([-\tau,0];R^n\bigr) \times R_+ \times S \to R^n, \qquad \mathbf{g} : C\bigl([-\tau,0];R^n\bigr) \times R_+ \times S \to R^{n\times m}.$$

For the stability purpose, we furthermore assume that $\boldsymbol{f}(0,t,i) = 0$ and $\mathbf{g}(0,t,i) = 0$ for all $t \in R_+$, $i \in S$, so that system (10) admits a trivial solution $\boldsymbol{x}(t) = 0$. We also put forward the following standard local Lipschitz condition on the drift coefficient $\boldsymbol{f}$ and the diffusion coefficient $\mathbf{g}$.

Assumption 2.1 For each $k = 1,2,\dots$, there is a $c_k > 0$ such that

$$\bigl|\boldsymbol{f}(\boldsymbol{\varphi},t,i) - \boldsymbol{f}(\boldsymbol{\psi},t,i)\bigr| \vee \bigl|\mathbf{g}(\boldsymbol{\varphi},t,i) - \mathbf{g}(\boldsymbol{\psi},t,i)\bigr| \le c_k\|\boldsymbol{\varphi} - \boldsymbol{\psi}\|,$$

for all $t \in R_+$, $i \in S$ and $\boldsymbol{\varphi},\boldsymbol{\psi} \in C([-\tau,0];R^n)$ with $\|\boldsymbol{\varphi}\| \vee \|\boldsymbol{\psi}\| \le k$.

Motivated by system (8) in Section 1, we propose the following nonlinear growth condition on the drift coefficient $\boldsymbol{f}$ and the diffusion coefficient $\mathbf{g}$ naturally.

Assumption 2.2 There exist nonnegative numbers $\kappa_i, \bar{\kappa}_i, \underline{\kappa}_i, \lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i$, real numbers $\hat{\kappa}_i$, positive numbers $n_1 > 1, n_2 > 1$ satisfying $n_1 + 1 > 2n_2$, probability measures η_j on $[-\tau,0]$, $j = 1,2,3,4$, and bounded functions $\xi_{i1}(t), \xi_{i2}(t) \in L^1(R_+;R_+)$ such that

$$\begin{aligned}\boldsymbol{\varphi}(0)^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\varphi},t,i) \le{}& -\kappa_i\bigl|\boldsymbol{\varphi}(0)\bigr|^{n_1+1} + \bar{\kappa}_i\int_{-\tau}^0\bigl|\boldsymbol{\varphi}(\theta)\bigr|^{n_1+1}\,d\eta_1(\theta) - \hat{\kappa}_i\bigl|\boldsymbol{\varphi}(0)\bigr|^2\\ &+ \underline{\kappa}_i\int_{-\tau}^0\bigl|\boldsymbol{\varphi}(\theta)\bigr|^2\,d\eta_2(\theta) + \xi_{i1}(t),\end{aligned} \tag{11}$$

$$\begin{aligned}\big|\mathbf{g}(\boldsymbol{\varphi},t,i)\big| &\le \lambda_i\big|\boldsymbol{\varphi}(0)\big|^{n_2} + \bar{\lambda}_i\int_{-\tau}^{0}\big|\boldsymbol{\varphi}(\theta)\big|^{n_2}\,d\eta_3(\theta) + \hat{\lambda}_i\big|\boldsymbol{\varphi}(0)\big| + \underline{\lambda}_i\int_{-\tau}^{0}\big|\boldsymbol{\varphi}(\theta)\big|\,d\eta_4(\theta)\\ &\quad + \xi_{i2}(t),\end{aligned} \tag{12}$$

for all $i \in S$ and $\boldsymbol{\varphi} \in C([-\tau,0];R^n)$, $t \in R_+$.

Remark 1 Condition (11) imposed in this assumption is purely motivated by system (8) discussed in Section 1. It is natural that the last three terms in the right-hand side of (11) appear, because they are standard linear growth condition. We only need to explain how the terms $\int_{-\tau}^{0}|\boldsymbol{\varphi}(\theta)|^{n_1+1}\,d\eta_1(\theta)$ and $|\boldsymbol{\varphi}(0)|^{n_1+1}$ may appear. For example, taking $n_1 = 3$, condition (11) for system (8) becomes

$$\boldsymbol{x}^T\boldsymbol{f}(\boldsymbol{x}_t,t,i) \le -\left[a(i) - \frac{1}{2}b(i)\right]\big|\boldsymbol{x}(t)\big|^4 + \frac{1}{2}b(i)\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^4\,d\eta(\theta) + \overline{c(i)}\big|\boldsymbol{x}(t)\big|^2,$$

and we see how the terms $|\boldsymbol{x}(t)|^4$ and $\int_{-\tau}^{0}|\boldsymbol{x}(t+\theta)|^4\,d\eta(\theta)$ may appear naturally.

Remark 2 Similar to Remark 1, condition (12) imposed in this assumption is also purely motivated by system (8) discussed in Section 1. Similarly, we only need to explain how the terms $|\boldsymbol{\varphi}(0)|^{n_2}$ and $\int_{-\tau}^{0}|\boldsymbol{\varphi}(\theta)|^{n_2}\,d\eta_3(\theta)$ may appear. If setting $n_2 = \frac{4}{3}$, condition (12) for system (8) becomes

$$\big|\mathbf{g}(\boldsymbol{x}_t,t,i)\big| \le \frac{3}{4}\big|\rho(i)\big|\big|\boldsymbol{x}(t)\big|^{\frac{4}{3}} + \frac{1}{4}\big|\rho(i)\big|\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^{\frac{4}{3}}\,d\eta(\theta),$$

we see how the terms $|\boldsymbol{x}(t)|^{\frac{4}{3}}$ and $\int_{-\tau}^{0}|\boldsymbol{x}(t+\theta)|^{\frac{4}{3}}\,d\eta(\theta)$ may appear naturally.

Remark 3 Most of the existing results on the robust stability require that the drift coefficient $\boldsymbol{f}$ and diffusion coefficient $\mathbf{g}$ of the stochastic system are either linear or nonlinear with linear growth condition. Recently Hu, Mao and Zhang [1] obtained some new results for the robust stability of nonlinear hybrid SDDEs, requiring $\boldsymbol{x}^T\boldsymbol{f}$ and $|\mathbf{g}|^2$ to be bounded by polynomials with the same orders. However, there are many SFDEwMSs which do not satisfy this requirement, such as system (8) discussed in Section 1. Our aim in this paper is to establish some new criteria for the robust stability of a class of SFDEwMSs with Assumption 2.2, where $\boldsymbol{x}^T\boldsymbol{f}$ and $|\mathbf{g}|^2$ are controlled by polynomials with different orders.

Let $C^{2,1}(R^n \times [-\tau,+\infty) \times S; R_+)$ denote the family of all continuous nonnegative functions $V(x,t,i)$ on $R^n \times [-\tau,+\infty) \times S$ which are continuously twice differentiable in x and once differentiable in t. For each $V \in C^{2,1}(R^n \times [-\tau,+\infty) \times S; R_+)$, denote an operator $\mathscr{L}V$ from $C([-\tau,0];R^n) \times R_+ \times S$ to R by

$$\begin{aligned}\mathscr{L}V(\boldsymbol{\varphi},t,i) &= \sum_{j=1}^{N}\gamma_{ij}V\big(\boldsymbol{\varphi}(0),t,j\big) + V_t\big(\boldsymbol{\varphi}(0),t,i\big) + V_x\big(\boldsymbol{\varphi}(0),t,i\big)\boldsymbol{f}(\boldsymbol{\varphi},t,i)\\ &\quad + \frac{1}{2}\operatorname{trace}\big[\mathbf{g}^T(\boldsymbol{\varphi},t,i)V_{xx}\big(\boldsymbol{\varphi}(0),t,i\big)\mathbf{g}(\boldsymbol{\varphi},t,i)\big],\end{aligned} \tag{13}$$

where $V_t(\boldsymbol{x},t,i) = \frac{\partial V(\boldsymbol{x},t,i)}{\partial t}$, $V_{xx}(\boldsymbol{x},t,i) = (\frac{\partial^2 V(\boldsymbol{x},t,i)}{\partial x_i x_j})_{n\times n}$, $V_x(\boldsymbol{x},t,i) = (\frac{\partial V(\boldsymbol{x},t,i)}{\partial x_1},\ldots,\frac{\partial V(\boldsymbol{x},t,i)}{\partial x_n})$. For the convenience of the reader we cite the generalized Itô formula (see [11]): if $V \in C^{2,1}(R^n \times$

$[-\tau, +\infty) \times S; R_+)$, then for any $t \geq 0$,

$$\begin{aligned} V(\boldsymbol{x}(t), t, r(t)) = {} & V(\boldsymbol{x}(0), 0, r(0)) + \int_0^t \mathscr{L}V(\boldsymbol{x}_s, s, r(s))\,\mathrm{d}s + \int_0^t V_x(\boldsymbol{x}(s), s, r(s)) \\ & \cdot \mathbf{g}(\boldsymbol{x}_s, s, r(s))\,\mathrm{d}B(s) + \int_0^t \int_R V(\boldsymbol{x}(s), s, i_0 + \overline{h}(r(s-), l)) \\ & - V(\boldsymbol{x}(s), s, r(s))\mu(\mathrm{d}s, \mathrm{d}l), \end{aligned} \tag{14}$$

where $\mu(\mathrm{d}s, \mathrm{d}l) = \nu(\mathrm{d}s, \mathrm{d}l) - m(\mathrm{d}l)$ is a martingale measure.

To obtain our main theorems, we present a number of lemmas which are essential to the proofs.

Lemma 2.1 (*cf.* [27] Barbalat lemma) *Let $f(t)$ be uniformly continuous on $[0, \infty)$, and $f(t) \in L^1(R_+; R_+)$, then $\lim_{t\to\infty} f(t) = 0$.*

Lemma 2.2 (*cf.* [27]) *Let $f(t)$ be a bounded function on $[0, \infty)$, and $f(t) \in L^1(R_+; R_+)$, then $\int_0^{+\infty} f^\alpha(t)\,\mathrm{d}t < \infty$ holds for any $\alpha \geq 1$.*

Lemma 2.3 (*cf.* [14]) *Let $a, b, q > 0, b \geq q, \alpha > \beta > 0$ and the following condition holds:*

$$\frac{a}{b} > (\alpha - \beta)\beta^{\frac{\beta}{\alpha-\beta}}\alpha^{\frac{-\alpha}{\alpha-\beta}},$$

then there exists $\bar{a} \in (0, a)$ such that $a + bt^\alpha - qt^\beta > \bar{a}$, for all $t \geq 0$.

Lemma 2.4 (*cf.* [28]) *Assume $\alpha, \beta > 0$. If $h \in C(R^n; R)$ satisfies $\limsup_{|t|\to\infty}(h(t)/|t|^\alpha) = 0$, then there exists a constant H satisfying $\sup_{t\in R^n}\{-\beta|t|^\alpha + h(t)\} < H$.*

3 Asymptotic stability of SFDEwMSs

In this section, we aim to discuss the asymptotic stability of SFDEwMSs using M-matrices. First we adopt the traditional notation by letting $Z^{N\times N} = \{(b_{ij})_{N\times N} : b_{ij} \leq 0, i \neq j\}$. For more detailed information please see [11].

Lemma 3.1 *If $B \in Z^{N\times N}$, then the following statements are equivalent:*

(1) *B is a nonsingular M-matrix;*
(2) *B is semipositive, that is, there exists $x \gg 0$ in R^N such that $Bx \gg 0$;*
(3) *all the leading principal minors of B are positive.*

Because of the existence of nonlinear growth condition (Assumption 2.2), the classical theory cannot be used directly to system (10). So it is necessary to establish the following result of the existence and uniqueness and the asymptotic boundedness.

Lemma 3.2 *If Assumptions 2.1, 2.2 hold and $\kappa_i > \bar{\kappa}_i$ for all $i \in S$, then for any number $p \geq 0$, any initial data $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ and $i_0 \in S$, there is a unique global solution $\boldsymbol{x}(t, \boldsymbol{\zeta}, i_0)$ of system (10) on $t \geq -\tau$, and there exists a constant $M_p > 0$ such that $\sup_{-\tau\leq t<+\infty} E|\boldsymbol{x}(t, \boldsymbol{\zeta}, i_0)|^p \leq M_p$.*

Proof For any given initial data $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$ and $i_0 \in S$, by Theorem 3.16 in [11], p.91, Assumption 2.1 and conditions $\boldsymbol{f}(0,t,i) = 0$ and $\mathbf{g}(0,t,i) = 0$ ($t \in R_+$, $i \in S$) guarantee a unique maximal local solution $\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)$ to system (10) on $t \in [-\tau,\sigma_\infty)$, where σ_∞ is the explosion time. Let $k_0 > 0$ be sufficiently large satisfying $\|\boldsymbol{\zeta}\| < k_0$. For each integer $k \geq k_0$, define the stopping time $\tau_k = \inf\{t \in [0,\sigma_\infty) : |x(t,\boldsymbol{\zeta},i_0)| \geq k\}$. Obviously, τ_k is increasing about k. Let $\tau_\infty = \lim_{t\to\infty} \tau_k$, so $\tau_\infty \leq \sigma_\infty$ a.s. If we obtain $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. For the sake of simplicity, write $\boldsymbol{x}(t) = \boldsymbol{x}(t,\boldsymbol{\zeta},i_0)$. Applying the generalized Itô formula to $V(\boldsymbol{x},t,i) = |\boldsymbol{x}|^2$, we find

$$\begin{aligned}
&\mathscr{L}V\big(\boldsymbol{x}(t),t,i\big) \\
&\quad = 2\boldsymbol{x}^T(t)\boldsymbol{f}(\boldsymbol{x}_t,t,i) + \big|\mathbf{g}(\boldsymbol{x}_t,t,i)\big|^2 \\
&\quad \leq 2\bigg(-\kappa_i\big|\boldsymbol{x}(t)\big|^{n_1+1} + \bar{\kappa}_i\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^{n_1+1}\,d\eta_1(\theta) - \hat{\kappa}_i\big|\boldsymbol{x}(t)\big|^2 \\
&\qquad + \underline{\kappa}_i\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^2\,d\eta_2(\theta) + \xi_{i1}(t)\bigg) + 5\bigg(\gamma_i^2\big|\boldsymbol{x}(t)\big|^{2n_2} + \bar{\gamma}_i^2\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^{2n_2}\,d\eta_3(\theta) \\
&\qquad + \hat{\gamma}_i^2\big|\boldsymbol{x}(t)\big|^2 + \underline{\gamma}_i^2\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^2\,d\eta_4(\theta) + \xi_{i2}^2(t)\bigg) \\
&\quad \leq -2(\kappa_i - \bar{\kappa}_i)\big|\boldsymbol{x}(t)\big|^{n_1+1} + 5\big(\gamma_i^2 + \bar{\gamma}_i^2\big)\big|\boldsymbol{x}(t)\big|^{2n_2} + \big(5\big(\hat{\gamma}_i^2 + \underline{\gamma}_i^2\big) - 2(\hat{\kappa}_i - \underline{\kappa}_i)\big)\big|\boldsymbol{x}(t)\big|^2 \\
&\qquad + 2\bar{\kappa}_i\tilde{J}_1 + 2\underline{\kappa}_i\tilde{J}_2 + 5\bar{\gamma}_i^2\tilde{J}_3 + 5\underline{\gamma}_i^2\tilde{J}_4 + 2\xi_{i1}(t) + 5\xi_{i2}^2(t),
\end{aligned}$$

where $\tilde{J}_1 = \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|^{n_1+1}\,\mathrm{d}\eta_1(\theta) - |\boldsymbol{x}(t)|^{n_1+1}, \tilde{J}_2 = \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|^2\,\mathrm{d}\eta_2(\theta) - |\boldsymbol{x}(t)|^2, \tilde{J}_3 = \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|^{2n_2}\,\mathrm{d}\eta_3(\theta) - |\boldsymbol{x}(t)|^{2n_2}, \tilde{J}_4 = \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|^2\,\mathrm{d}\eta_4(\theta) - |\boldsymbol{x}(t)|^2$.

Noting $\kappa_i > \bar{\kappa}_i, n_1 + 1 > 2n_2 \geq 2$ and $|\boldsymbol{x}(t)| \geq 0$ for any $t \geq 0$, by Lemma 2.4, $R_{1i}(|\boldsymbol{x}(t)|) = -2(\kappa_i - \bar{\kappa}_i)|\boldsymbol{x}(t)|^{n_1+1} + 5(\gamma_i^2 + \bar{\gamma}_i^2)|\boldsymbol{x}(t)|^{2n_2} + (5(\hat{\gamma}_i^2 + \underline{\gamma}_i^2) - 2(\hat{\kappa}_i - \underline{\kappa}_i))|\boldsymbol{x}(t)|^2$, as a function of $|\boldsymbol{x}(t)|$, has a positive upper-boundedness, *i.e.*, there is a positive constant $\tilde{R}_{1i}$ such that

$$\begin{aligned}
R_{1i}\big(\big|\boldsymbol{x}(t)\big|\big) &= -2(\kappa_i - \bar{\kappa}_i)\big|\boldsymbol{x}(t)\big|^{n_1+1} + 5\big(\gamma_i^2 + \bar{\gamma}_i^2\big)\big|\boldsymbol{x}(t)\big|^{2n_2} \\
&\quad + \big(5\big(\hat{\gamma}_i^2 + \underline{\gamma}_i^2\big) - 2(\hat{\kappa}_i - \underline{\kappa}_i)\big)\big|\boldsymbol{x}(t)\big|^2 \\
&\leq \tilde{R}_{1i}.
\end{aligned} \tag{15}$$

(This technique has been used by many researchers, for example [28].)

From Lemma 2.2, we have $\int_0^{+\infty}(5\xi_{i2}^2(s) + 2\xi_{i1}(s))\,\mathrm{d}s < \infty$. In view of the fact

$$\int_0^{t\wedge\tau_k}\tilde{J}_i\,\mathrm{d}s = \int_0^{t\wedge\tau_k}\bigg(\int_{-\tau}^0\big|\boldsymbol{x}(s+\theta)\big|^{w_i'}\,\mathrm{d}\eta_i(\theta) - \big|\boldsymbol{x}(s)\big|^{w_i'}\bigg)\,\mathrm{d}s \leq \int_{-\tau}^0\big|\boldsymbol{x}(s)\big|^{w_i'}\,\mathrm{d}s,$$

for $w_1' = n_1 + 1, w_3' = 2n_2, w_2' = w_4' = 2$, we find that, for $t \geq 0$,

$$\begin{aligned}
&E\big|\boldsymbol{x}(t\wedge\tau_k)\big|^2 \\
&\quad = E\big|\boldsymbol{x}(0)\big|^2 + E\int_0^{t\wedge\tau_k}\big[2\boldsymbol{x}^T(s)\boldsymbol{f}(\boldsymbol{x}_s,s,i) + \big|\mathbf{g}(\boldsymbol{x}_s,s,i)\big]\big|^2\,\mathrm{d}s \\
&\quad \leq E\big|\boldsymbol{x}(0)\big|^2 + E\int_0^{t\wedge\tau_k}\big[\tilde{R}_{1i} + 2\bar{\kappa}_i\tilde{J}_1 + 2\underline{\kappa}_i\tilde{J}_2 + 5\bar{\gamma}_i^2\tilde{J}_3 + 5\underline{\gamma}_i^2\tilde{J}_4 + 2\xi_{i1}(s) + 5\xi_{i2}^2(s)\big]\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le E\big|\mathbf{x}(0)\big|^2 + \tilde{R}_{1i}E(t \wedge \tau_k) + 2\bar{\kappa}_i \int_{-\tau}^{0} \big|\mathbf{x}(s)\big|^{n_1+1}\,\mathrm{d}s + 2\underline{\kappa}_i \int_{-\tau}^{0} \big|\mathbf{x}(s)\big|^2\,\mathrm{d}s \\
&\quad + 5\bar{\gamma}_i^2 \int_{-\tau}^{0} \big|\mathbf{x}(s)\big|^{2n_2}\,\mathrm{d}s + 5\underline{\gamma}_i^2 \int_{-\tau}^{0} \big|\mathbf{x}(s)\big|^2\,\mathrm{d}s + \int_{0}^{\infty} \big[2\xi_{i1}(s) + 5\xi_{i2}^2(s)\big]\,\mathrm{d}s \\
&\le \bar{R}_{1i} + \tilde{R}_{1i}t \\
&\le \bar{R}_1 + \tilde{R}_1 t,
\end{aligned}
$$

where $\bar{R}_{1i} = E|\mathbf{x}(0)|^2 + 2\bar{\kappa}_i \int_{-\tau}^{0} |\mathbf{x}(s)|^{n_1+1}\,\mathrm{d}s + 2\underline{\kappa}_i \int_{-\tau}^{0} |\mathbf{x}(s)|^2\,\mathrm{d}s + 5\bar{\gamma}_i^2 \int_{-\tau}^{0} |\mathbf{x}(s)|^{2n_2}\,\mathrm{d}s + 5\underline{\gamma}_i^2 \int_{-\tau}^{0} |\mathbf{x}(s)|^2\,\mathrm{d}s + \int_{0}^{\infty} [2\xi_{i1}(s) + 5\xi_{i2}^2(s)]\,\mathrm{d}s$, $\bar{R}_1 = \max_{i\in S} \bar{R}_{1i}, \tilde{R}_1 = \max_{i\in S} \tilde{R}_{1i}$. Noting that

$$
E\big|\mathbf{x}(t \wedge \tau_k)\big|^2 \ge E\big(\big|\mathbf{x}(t \wedge \tau_k)\big|^2 I_{\{\tau_k \le t\}}\big) \ge k^2 P\{\tau_k \le t\},
$$

we get

$$
P\{\tau_\infty \le t\} = \lim_{k\to\infty} P\{\tau_k \le t\} \le \lim_{k\to\infty} \frac{\bar{R}_1 + \tilde{R}_1 t}{k^2} = 0.
$$

Since t is arbitrary, we must have $\tau_\infty = \infty$ a.s. Hence, $\sigma_\infty = \infty$ a.s., namely, the unique maximal local solution $\mathbf{x}(t)$ on $t \in [-\tau, \sigma_\infty)$ becomes the unique global solution on $t \in [-\tau, \infty)$.

Next, we prove the asymptotic boundedness of the global solution $\mathbf{x}(t)$.

It is easy to see that there exists at least a sufficiently small positive constant ε satisfying $\kappa_i > \bar{\kappa}_i e^{\varepsilon\tau}$, $i \in S$. So by the continuity, define $\varepsilon'' = \sup\{\varepsilon > 0 : \kappa_i > \bar{\kappa}_i e^{\varepsilon\tau}\}$. For any $p \ge 2$, applying the generalized Itô formula to $V(\mathbf{x}, t, i) = e^{\varepsilon t}|\mathbf{x}|^p$, $\varepsilon \in (0, \varepsilon'']$, we find

$$
\begin{aligned}
&\mathscr{L}V\big(\mathbf{x}(t), t, i\big) \\
&\quad = e^{\varepsilon t}\big(\mathscr{L}\big|\mathbf{x}(t)\big|^p + \varepsilon\big|\mathbf{x}(t)\big|^p\big) \\
&\quad \le e^{\varepsilon t}\bigg[\frac{p}{2}\big|\mathbf{x}(t)\big|^{p-2}\big(2\mathbf{x}^T(t)\mathbf{f}(\mathbf{x}_t, t, i) + (p-1)\big|\mathbf{g}(\mathbf{x}_t, t, i)\big|^2\big) + \varepsilon\big|\mathbf{x}(t)\big|^p\bigg] \\
&\quad \le e^{\varepsilon t}\frac{p}{2}\big|\mathbf{x}(t)\big|^{p-2}\bigg[2\bigg(-\kappa_i\big|\mathbf{x}(t)\big|^{n_1+1} + \bar{\kappa}_i \int_{-\tau}^{0} \big|\mathbf{x}(t+\theta)\big|^{n_1+1}\,d\eta_1(\theta) - \hat{\kappa}_i\big|\mathbf{x}(t)\big|^2 \\
&\qquad + \underline{\kappa}_i \int_{-\tau}^{0} \big|\mathbf{x}(t+\theta)\big|^2\,d\eta_2(\theta) + \xi_{i1}(t)\bigg) + 5(p-1)\bigg(\gamma_i^2\big|\mathbf{x}(t)\big|^{2n_2} + \hat{\gamma}_i^2\big|\mathbf{x}(t)\big|^2 \\
&\qquad + \bar{\gamma}_i^2 \int_{-\tau}^{0} \big|\mathbf{x}(t+\theta)\big|^{2n_2}\,d\eta_3(\theta) + \underline{\gamma}_i^2 \int_{-\tau}^{0} \big|\mathbf{x}(t+\theta)\big|^2\,d\eta_4(\theta) + \xi_{i2}^2(t)\bigg) + \frac{2}{p}\varepsilon\big|\mathbf{x}(t)\big|^2\bigg] \\
&\quad \le \frac{p}{2}e^{\varepsilon t}\big[-2\big(\kappa_i - \bar{\kappa}_i e^{\varepsilon\tau}\big)\big|\mathbf{x}(t)\big|^{p+n_1-1} + o\big(\big|\mathbf{x}(t)\big|^{p+n_1-1}\big)\big] + 2\underline{\kappa}_i e^{\varepsilon t} J_2 \\
&\qquad + 5(p-1)\underline{\gamma}_i^2 e^{\varepsilon t} J_4 + \frac{p}{2}5(p-1)\bar{\gamma}_i^2 \frac{2n_2}{p+2n_2-2} e^{\varepsilon t} J_3 + p\bar{\kappa}_i \frac{n_1+1}{p+n_1-1} e^{\varepsilon t} J_1 \\
&\qquad + 2e^{\varepsilon t}\xi_{i1}^{\frac{p}{2}}(t) + 5(p-1)e^{\varepsilon t}\xi_{i2}^{p}(t),
\end{aligned}
$$

where $J_1 = \int_{-\tau}^{0} |\mathbf{x}(t+\theta)|^{p+n_1-1}\,\mathrm{d}\eta_1(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^{p+n_1-1}$, $J_2 = \int_{-\tau}^{0} |\mathbf{x}(t+\theta)|^{p}\,\mathrm{d}\eta_2(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^{p}$, $J_3 = \int_{-\tau}^{0} |\mathbf{x}(t+\theta)|^{p+2n_2-2}\,\mathrm{d}\eta_3(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^{p+2n_2-2}$, $J_4 = \int_{-\tau}^{0} |\mathbf{x}(t+\theta)|^{p}\,\mathrm{d}\eta_4(\theta) - e^{\varepsilon\tau}|\mathbf{x}(t)|^{p}$.

Noting that $\kappa_i > \bar{\kappa}_i e^{\varepsilon\tau}$ and $|\mathbf{x}(t)| \ge 0$ for any $t \ge 0$, by Lemma 2.4 and the same technique as (15), $\bar{G}_i(|\mathbf{x}(t)|) = -2(\kappa_i - \bar{\kappa}_i e^{\varepsilon\tau})|\mathbf{x}(t)|^{p+n_1-1} + o(|\mathbf{x}(t)|^{p+n_1-1})$, as a function of $|\mathbf{x}(t)|$, has a positive upper-boundedness, *i.e.*, there is a positive constant $\underline{Q}_i$ such that

$$\bar{\bar{G}}_i\big(\big|\boldsymbol{x}(t)\big|\big) = -2\big(\kappa - \bar{\kappa}e^{\varepsilon\tau}\big)\big|\boldsymbol{x}(t)\big|^{p+n_1-1} + o\big(\big|\boldsymbol{x}(t)\big|^{p+n_1-1}\big) \le \underline{Q}_i. \tag{16}$$

And in view of the fact that

$$\int_0^t e^{\varepsilon s} J_i \,\mathrm{d}s = \int_0^t e^{\varepsilon s}\left(\int_{-\tau}^0 \big|\boldsymbol{x}(s+\theta)\big|^{w_i}\,\mathrm{d}\eta_i(\theta) - e^{\varepsilon\tau}\big|\boldsymbol{x}(s)\big|^{w_i}\right)\mathrm{d}s \le e^{\varepsilon\tau}\int_{-\tau}^0 \big|\boldsymbol{x}(s)\big|^{w_i}\,\mathrm{d}s,$$

for $w_1 = p + n_1 - 1, w_3 = p + 2n_2 - 2, w_2 = w_4 = p$, we find that, for $t \ge 0$,

$$\begin{aligned}
&Ee^{\varepsilon t}\big|\boldsymbol{x}(t)\big|^p \\
&\quad\le E\big|\boldsymbol{x}(0)\big|^p + \frac{p}{2}\underline{Q}_i\int_0^t e^{\varepsilon s}\,\mathrm{d}s + p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}e^{\varepsilon\tau}\int_{-\tau}^0\big|\boldsymbol{x}(s)\big|^{p+n_1-1}\,\mathrm{d}s \\
&\qquad + 2\underline{\kappa}_i e^{\varepsilon\tau}\int_{-\tau}^0\big|\boldsymbol{x}(s)\big|^p\,\mathrm{d}s + \frac{p}{2}5(p-1)\bar{\gamma}_i^2\frac{2n_2}{p+2n_2-2}e^{\varepsilon\tau}\int_{-\tau}^0\big|\boldsymbol{x}(s)\big|^{p+2n_2-2}\,\mathrm{d}s \\
&\qquad + 5(p-1)\underline{\gamma}_i^2 e^{\varepsilon\tau}\int_{-\tau}^0\big|\boldsymbol{x}(s)\big|^p\,\mathrm{d}s + \int_0^t e^{\varepsilon s}\big[2\xi_{i1}^{\frac{p}{2}}(s) + 5(p-1)\xi_{i2}^p(s)\big]\,\mathrm{d}s.
\end{aligned} \tag{17}$$

By virtue of the boundedness of $\xi_{i1}(t), \xi_{i2}(t)$, there is a constant $\Psi > 0$ such that $\xi_{i1}(t) \vee \xi_{i2}(t) \le \Psi$, which implies that

$$Ee^{\varepsilon t}\big|\boldsymbol{x}(t)\big|^p \le c + \left[2\Psi^{\frac{p}{2}} + 5(p-1)\Psi^p + \frac{p}{2}\underline{Q}\right]\int_0^t e^{\varepsilon s}\,\mathrm{d}s,$$

where $c = \max_{i\in S}\{p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}e^{\varepsilon\tau}\int_{-\tau}^0 |\boldsymbol{x}(s)|^{p+n_1-1}\,\mathrm{d}s + 2\underline{\kappa}_i e^{\varepsilon\tau}\int_{-\tau}^0 |\boldsymbol{x}(s)|^p\,\mathrm{d}s + \frac{p}{2}5(p-1)\bar{\gamma}_i^2\frac{2n_2}{p+2n_2-2}\times e^{\varepsilon\tau}\int_{-\tau}^0 |\boldsymbol{x}(s)|^{p+2n_2-2}\,\mathrm{d}s + 5(p-1)\underline{\gamma}_i^2 e^{\varepsilon\tau}\int_{-\tau}^0 |\boldsymbol{x}(s)|^p\,\mathrm{d}s + E|\boldsymbol{x}(0)|^p\}$, $\underline{Q} = \max_{i\in S}\underline{Q}_i$. This implies

$$E\big|\boldsymbol{x}(t)\big|^p \le ce^{-\varepsilon t} + \frac{2\Psi^{\frac{p}{2}} + 5(p-1)\Psi^p + \frac{p}{2}\underline{Q}}{\varepsilon}\big(1 - e^{-\varepsilon t}\big). \tag{18}$$

From the boundedness of the initial data $\boldsymbol{\zeta}$, we claim that for any $p \ge 2$, there exists a constant $M_p > 0$ such that $\sup_{-\tau\le t<+\infty} E|\boldsymbol{x}(t)|^p \le M_p$. When $p \in (0,2)$, using Hölder's inequality, we claim $\sup_{-\tau\le t<\infty} E|\boldsymbol{x}(t)|^p \le (\sup_{-\tau\le t<\infty} E|\boldsymbol{x}(t)|^2)^{\frac{p}{2}} \le M_2^{\frac{p}{2}}$. □

To study the asymptotic stability, we need to provide the following lemma.

Lemma 3.3 *If Assumptions* 2.1, 2.2 *hold and* $\kappa_i > \bar{\kappa}_i$ *for all* $i \in S$, *then* $E|\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)|^p$ *is uniformly continuous on* $[0,\infty)$, *which holds for any* $p \ge 2$.

Proof From Lemma 3.2, there is a unique global solution $\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)$. For simplicity, write $\boldsymbol{x}(t) = \boldsymbol{x}(t,\boldsymbol{\zeta},i_0)$. Using the generalized Itô formula, we compute that

$$\begin{aligned}
E\big|\boldsymbol{x}(t)\big|^p - E\big|\boldsymbol{x}(s)\big|^p = E\int_s^t \frac{p}{2}\big|\boldsymbol{x}(u)\big|^{p-2}&(2\boldsymbol{x}^T(u)\boldsymbol{f}\big(\boldsymbol{x}_u,u,r(u)\big) \\
&+ (p-1)\big|\mathbf{g}\big(\boldsymbol{x}_u,u,r(u)\big)\big|^2\,du.
\end{aligned} \tag{19}$$

From Lemma 3.2, we also know that for number $p \geq 2$, $\sup_{-\tau \leq t < \infty} E|\boldsymbol{x}(t)|^p \leq M_p$. Then we see that, for any $0 < s < t < \infty$,

$$
\begin{aligned}
&\bigl|E\bigl|\boldsymbol{x}(t)\bigr|^p - E\bigl|\boldsymbol{x}(s)\bigr|^p\bigr| \\
&\quad \leq \frac{p}{2} E \int_s^t \bigl[\bigl|\boldsymbol{x}(u)\bigr|^{p-2}\bigl(\bigl|2\boldsymbol{x}^T(u)\mathbf{f}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr| + (p-1)\bigl|\mathbf{g}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr|^2\bigr)\bigr]\, du \\
&\quad \leq \frac{p}{2} \int_s^t E\bigl(2\bigl|\boldsymbol{x}(u)\bigr|^{p-2}\bigl|\boldsymbol{x}^T(u)\mathbf{f}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr| \\
&\qquad + (p-1)\bigl|\boldsymbol{x}(u)\bigr|^{p-2}\bigl|\mathbf{g}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr|^2\bigr)\, du.
\end{aligned}
\tag{20}
$$

From Assumption 2.2, letting $\kappa = \max_{i\in S} \kappa_i, \bar{\kappa} = \max_{i\in S} \bar{\kappa}_i, \hat{\kappa} = \max_{i\in S} |\hat{\kappa}_i|, \underline{\kappa} = \max_{i\in S} \underline{\kappa}_i$, $\lambda = \max_{i\in S} \lambda_i, \bar{\lambda} = \max_{i\in S} \bar{\lambda}_i, \hat{\lambda} = \max_{i\in S} \hat{\lambda}_i, \underline{\lambda} = \max_{i\in S} \underline{\lambda}_i$, and $\max_{i\in S}[\xi_{i1}(t) \vee \xi_{i2}(t)] \leq \Psi$, we see that

$$
\begin{aligned}
&\bigl|\boldsymbol{x}(u)\bigr|^{p-2}\bigl|\boldsymbol{x}^T(u)\mathbf{f}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr| \\
&\quad \leq \frac{p-2}{p-1}\bigl|\boldsymbol{x}(u)\bigr|^{p-1} + \frac{1}{p-1}\bigl|\boldsymbol{x}^T(u)\mathbf{f}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr|^{p-1} \\
&\quad \leq \frac{p-2}{p-1}\bigl|\boldsymbol{x}(u)\bigr|^{p-1} + \frac{5^{p-2}}{p-1}\biggl[\bar{\kappa}^{p-1}\biggl(\int_{-\tau}^0 \bigl|\boldsymbol{x}(u+\theta)\bigr|^{(n_1+1)}\, d\eta_1(\theta)\biggr)^{p-1} + \hat{\kappa}^{p-1}\bigl|\boldsymbol{x}(u)\bigr|^{2(p-1)} \\
&\qquad + \kappa^{p-1}\bigl|\boldsymbol{x}(u)\bigr|^{(n_1+1)(p-1)} + \underline{\kappa}^{p-1}\biggl(\int_{-\tau}^0 \bigl|\boldsymbol{x}(u+\theta)\bigr|^2\, d\eta_2(\theta)\biggr)^{p-1} + \Psi^{p-1}\biggr], \\
&\bigl|\boldsymbol{x}(u)\bigr|^{p-2}\bigl|\mathbf{g}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr|^2 \\
&\quad \leq \frac{p-2}{p}\bigl|\boldsymbol{x}(u)\bigr|^p + \frac{2}{p}\bigl|\mathbf{g}\bigl(\boldsymbol{x}_u, u, r(u)\bigr)\bigr|^p \\
&\quad \leq \frac{p-2}{p}\bigl|\boldsymbol{x}(u)\bigr|^p + \frac{2}{p}5^{p-1}\biggl[\lambda^p\bigl|\boldsymbol{x}(u)\bigr|^{pn_2} + \bar{\lambda}^p\biggl(\int_{-\tau}^0 \bigl|\boldsymbol{x}(u+\theta)\bigr|^{n_2}\, d\eta_3(\theta)\biggr)^p + \hat{\lambda}^p\bigl|\boldsymbol{x}(u)\bigr|^p \\
&\qquad + \underline{\lambda}^p\biggl(\int_{-\tau}^0 \bigl|\boldsymbol{x}(u+\theta)\bigr|\, d\eta_4(\theta)\biggr)^p + \Psi^p\biggr].
\end{aligned}
\tag{21}
$$

Substituting (21) into (20), we find

$$
\begin{aligned}
&\bigl|E\bigl|\boldsymbol{x}(t)\bigr|^p - E\bigl|\boldsymbol{x}(s)\bigr|^p\bigr| \\
&\quad \leq \frac{p}{2}\biggl[2\frac{p-2}{p-1}M_{p-1} + 2\frac{5^{p-2}}{p-1}\bigl(\kappa^{p-1}M_{(n_1+1)(p-1)} + \bar{\kappa}^{p-1}M_{(n_1+1)(p-1)} + \hat{\kappa}^{p-1}M_{2(p-1)} \\
&\qquad + \underline{\kappa}^{p-1}M_{2(p-1)} + \Psi^{p-1}\bigr) + \frac{p-2}{p}(p-1)M_p + \frac{2}{p}5^{p-1}(p-1)\bigl(\lambda^p M_{pn_2} + \bar{\lambda}^p M_{pn_2} \\
&\qquad + \hat{\lambda}^p M_p + \underline{\lambda}^p M_p + \Psi^p\bigr)\biggr](t-s),
\end{aligned}
$$

which means $E|\boldsymbol{x}(t)|^p$ is uniformly continuous, for $p \geq 2$. □

Next, we propose one of our main theorems. The following theorem establishes a new sufficient condition for the pth moment asymptotic stability.

Theorem 3.1 *Let Assumptions* 2.1, 2.2 *hold and* $\frac{2(\kappa_i-\overline{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} > 1$ *for all* $i \in S$. *Set*

$$\begin{aligned}
&\mathscr{B}(p) = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_N) - \frac{2}{p}\Gamma, \\
&\mathscr{A} = \left\{p \geq 2 : \mathscr{B}(p) \text{ is a nonsingular } M\text{-matrix}, p < \min_{i\in S}\left\{\frac{2(\kappa_i - \overline{\kappa}_i)}{(\lambda_i + \bar{\lambda}_i)^2} + 1\right\}\right\},
\end{aligned}$$

where $\Delta_i = 2(\hat{\kappa}_i - \underline{\kappa}_i) - 2L(\kappa_i - \overline{\kappa}_i) - \frac{2(\kappa_i-\overline{\kappa}_i)(\hat{\lambda}_i+\underline{\lambda}_i)^2(p-1)}{2(\kappa_i-\overline{\kappa}_i)-(\lambda_i+\bar{\lambda}_i)^2(p-1)}$, $L = (n_1 - 2n_2 + 1)(2n_2 - 2)^{\frac{2n_2-2}{n_1-2n_2+1}}(n_1 - 1)^{\frac{1-n_1}{n_1-2n_2+1}}$. *If* $\mathscr{A} \neq \emptyset$ *and* $p_0 = \sup\{p : p \in \mathscr{A}\} > 2$, *then for any* $p \in (0, p_0)$, *any initial data* $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ *and* $i_0 \in S$, *there is a unique global solution* $\boldsymbol{x}(t, \boldsymbol{\zeta}, i_0)$ *of system* (10) *on* $t \geq -\tau$, *and the solution is pth moment asymptotically stable, i.e.*,

$$\lim_{t\to\infty} E\bigl|\boldsymbol{x}(t, \boldsymbol{\zeta}, i_0)\bigr|^p = 0. \tag{22}$$

Proof The existence and uniqueness of the global solution follow from Lemma 3.2. For simplicity, write $\boldsymbol{x}(t) = \boldsymbol{x}(t, \boldsymbol{\zeta}, i_0)$. Since $\mathscr{A} \neq \emptyset$, there exists $p > 2$ such that $\mathscr{B}(p)$ is a nonsingular M-matrix and $p < \min_{i\in S}\{\frac{2(\kappa_i-\overline{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} + 1\}$. From Lemma 3.1, there exists $c_p = (c_{p1}, c_{p2}, \ldots, c_{pN}) \gg 0$ such that $\mathscr{B}(p)c_p \gg 0$. Applying the generalized Itô formula to $V(\boldsymbol{x}, t, i) = c_{pi}|\boldsymbol{x}|^p$, we find

$$\begin{aligned}
&\mathscr{L}V\bigl(\boldsymbol{x}(t), t, i\bigr) \\
&\quad\leq \frac{p}{2}c_{pi}\bigl|\boldsymbol{x}(t)\bigr|^{p-2}\bigl(2\boldsymbol{x}^T(t)\mathbf{f}\bigl(\boldsymbol{x}_t, t, r(t)\bigr) + (p-1)\bigl|\mathbf{g}\bigl(\boldsymbol{x}_t, t, r(t)\bigr)\bigr|^2\bigr) + \sum_{j=1}^N \gamma_{ij}c_{pj}\bigl|\boldsymbol{x}(t)\bigr|^p \\
&\quad\leq \frac{p}{2}c_{pi}\bigl|\boldsymbol{x}(t)\bigr|^{p-2}\biggl[-2\kappa_i\bigl|\boldsymbol{x}(t)\bigr|^{n_1+1} + 2\bar{\kappa}_i\int_{-\tau}^0\bigl|\boldsymbol{x}(t+\theta)\bigr|^{n_1+1}\,\mathrm{d}\eta_1(\theta) - 2\hat{\kappa}_i\bigl|\boldsymbol{x}(t)\bigr|^2 \\
&\qquad + 2\underline{\kappa}_i\int_{-\tau}^0\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,\mathrm{d}\eta_2(\theta) + \frac{p-1}{\rho_{i1}}\biggl(\frac{\lambda_i^2}{\delta_{i1}}\bigl|\boldsymbol{x}(t)\bigr|^{2n_2} + \frac{\bar{\lambda}_i^2}{1-\delta_{i1}}\int_{-\tau}^0\bigl|\boldsymbol{x}(t+\theta)\bigr|^{2n_2}\,\mathrm{d}\eta_3(\theta)\biggr) \\
&\qquad + \frac{p-1}{\rho_{i2}(1-\rho_{i1})}\biggl(\frac{\hat{\lambda}_i^2}{\delta_{i2}}\bigl|\boldsymbol{x}(t)\bigr|^2 + \frac{\underline{\lambda}_i^2}{1-\delta_{i2}}\int_{-\tau}^0\bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,\mathrm{d}\eta_4(\theta)\biggr) + 2\xi_{i1}(t) \\
&\qquad + \frac{p-1}{(1-\rho_{i1})(1-\rho_{i2})}\xi_{i2}^2(t)\biggr] + \sum_{j=1}^N \gamma_{ij}c_{pj}\bigl|\boldsymbol{x}(t)\bigr|^p \\
&\quad\leq \frac{p}{2}c_{pi}\biggl[-2(\kappa_i - \bar{\kappa}_i)\bigl|\boldsymbol{x}(t)\bigr|^{p+n_1-1} - \biggl(2\hat{\kappa}_i - 2\underline{\kappa}_i - \frac{p-1}{\rho_{i2}(1-\rho_{i1})}\biggl(\frac{\hat{\lambda}_i^2}{\delta_{i2}} + \frac{\underline{\lambda}_i^2}{1-\delta_{i2}}\biggr) \\
&\qquad - 2\frac{p-2}{p}\varepsilon_{i1}^p - \frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\frac{p-2}{p}\varepsilon_{i2}^p\biggr)\bigl|\boldsymbol{x}(t)\bigr|^p + \frac{p-1}{\rho_{i1}}\biggl(\frac{\lambda_i^2}{\delta_{i1}} + \frac{\bar{\lambda}_i^2}{1-\delta_{i1}}\biggr) \\
&\qquad \cdot\bigl|\boldsymbol{x}(t)\bigr|^{p+2n_2-2}\biggr] + \sum_{j=1}^N \gamma_{ij}c_{pj}\bigl|\boldsymbol{x}(t)\bigr|^p + pc_{pi}\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}J_1 + 2c_{pi}\underline{\kappa}_iJ_2 \\
&\qquad + c_{pi}\frac{p-1}{\rho_{i2}(1-\rho_{i1})}\frac{\underline{\lambda}_i^2}{1-\delta_{i2}}J_4 + \frac{p}{2}c_{pi}\frac{p-1}{\rho_{i1}}\frac{\bar{\lambda}_i^2}{1-\delta_{i1}}\frac{2n_2}{p+2n_2-2}J_3 \\
&\qquad + c_{pi}\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}\xi_{i2}^p(t) + 2c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}\xi_{i1}^{\frac{p}{2}}(t),
\end{aligned} \tag{23}$$

where $\rho_{i1}, \rho_{i2}, \delta_{i1}, \delta_{i2} \in (0,1)$, $J_1 = \int_{-\tau}^{0} |\boldsymbol{x}(t+\theta)|^{p+n_1-1}\,\mathrm{d}\eta_1(\theta) - |\boldsymbol{x}(t)|^{p+n_1-1}$, $J_2 = \int_{-\tau}^{0} |\boldsymbol{x}(t+\theta)|^{p}\,\mathrm{d}\eta_2(\theta) - |\boldsymbol{x}(t)|^{p}$, $J_3 = \int_{-\tau}^{0} |\boldsymbol{x}(t+\theta)|^{p+2n_2-2}\,\mathrm{d}\eta_3(\theta) - |\boldsymbol{x}(t)|^{p+2n_2-2}$, $J_4 = \int_{-\tau}^{0} |\boldsymbol{x}(t+\theta)|^{p}\,\mathrm{d}\eta_4(\theta) - |\boldsymbol{x}(t)|^{p}$. Substituting $\delta_{i1} = \frac{\lambda_i}{\lambda_i+\bar{\lambda}_i}$, $\delta_{i2} = \frac{\hat{\lambda}_i}{\hat{\lambda}_i+\underline{\lambda}_i}$ into (23) yields

$$
\begin{aligned}
&\mathscr{L}V\bigl(\boldsymbol{x}(t),t,i\bigr) \\
&\quad\le \frac{p}{2}\Biggl[-2c_{pi}(\kappa_i-\bar{\kappa}_i)\bigl|\boldsymbol{x}(t)\bigr|^{p+n_1-1} + c_{pi}\frac{(p-1)(\lambda_i+\bar{\lambda}_i)^2}{\rho_{i1}}\bigl|\boldsymbol{x}(t)\bigr|^{p+2n_2-2} \\
&\qquad - \Biggl(c_{pi}\biggl(2\hat{\kappa}_i - 2\underline{\kappa}_i - \frac{(p-1)(\hat{\lambda}_i+\underline{\lambda}_i)^2}{\rho_{i2}(1-\rho_{i1})} - 2\frac{p-2}{p}\varepsilon_{i1}^{p} - \frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\frac{p-2}{p}\varepsilon_{i2}^{p}\biggr) \\
&\qquad - \frac{2}{p}\sum_{j=1}^{N}\gamma_{ij}c_{pj}\Biggr)\bigl|\boldsymbol{x}(t)\bigr|^{p}\Biggr] + 2c_{pi}\underline{\kappa}_i J_2 + \frac{p}{2}c_{pi}\frac{(p-1)(\bar{\lambda}_i+\lambda_i)\bar{\lambda}_i}{\rho_{i1}}\frac{2n_2}{p+2n_2-2}J_3 \\
&\qquad + c_{pi}p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}J_1 + c_{pi}\frac{(p-1)(\underline{\lambda}_i+\hat{\lambda}_i)\underline{\lambda}_i}{\rho_{i2}(1-\rho_{i1})}J_4 + 2c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}\xi_{i1}^{\frac{p}{2}}(t) \\
&\qquad + c_{pi}\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}\xi_{i2}^{p}(t).
\end{aligned}
\tag{24}
$$

Let $G_i(|\boldsymbol{x}(t)|) = (c_{pi}(2(\hat{\kappa}_i-\underline{\kappa}_i) - \frac{(\hat{\lambda}_i+\underline{\lambda}_i)^2}{1-\rho_{i1}}(p-1)) - \frac{2}{p}\sum_{j=1}^{N}\gamma_{ij}c_{pj}) + 2c_{pi}(\kappa_i-\bar{\kappa}_i)|\boldsymbol{x}(t)|^{n_1-1} - \frac{p-1}{\rho_{i1}}c_{pi}(\lambda_i+\bar{\lambda}_i)^2|\boldsymbol{x}(t)|^{2n_2-2}$. Note $\mathscr{B}(p)c_p \gg 0$ can be rewritten in a component-wise form: $c_{pi}(2(\hat{\kappa}_i-\underline{\kappa}_i) - 2L(\kappa_i-\overline{\kappa}_i) - \frac{2(\kappa_i-\overline{\kappa}_i)(\hat{\lambda}_i+\underline{\lambda}_i)^2(p-1)}{2(\kappa_i-\overline{\kappa}_i)-(\lambda_i+\overline{\lambda}_i)^2(p-1)}) - \frac{2}{p}\sum_{j=1}^{N}\gamma_{ij}c_{pj} > 0$. From the definition of $\mathscr{A}$, we have $2(\kappa_i-\overline{\kappa}_i) > (\lambda_i+\overline{\lambda}_i)^2(p-1), i \in S$. Then for any $p \in [2,p_0)$, $\frac{(p-1)(\lambda_i+\bar{\lambda}_i)^2}{2(\kappa_i-\bar{\kappa}_i)} + \frac{c_{pi}(p-1)(\hat{\lambda}_i+\underline{\lambda}_i)^2}{c_{pi}(2(\hat{\kappa}_i-\underline{\kappa}_i)-2L(\kappa_i-\bar{\kappa}_i))-\frac{2}{p}\sum_{j=1}^{N}\gamma_{ij}c_{pj}} < 1$, and further there exists a $w_{i0} > 1$ such that $\frac{c_{pi}(p-1)(\hat{\lambda}_i+\underline{\lambda}_i)^2}{c_{pi}(2(\hat{\kappa}_i-\underline{\kappa}_i)-2L(\kappa_i-\bar{\kappa}_i))-\frac{2}{p}\sum_{j=1}^{N}\gamma_{ij}c_{pj}} + \frac{(p-1)(\lambda_i+\bar{\lambda}_i)^2}{2(\kappa_i-\bar{\kappa}_i)}w_{i0} = 1$. Set $\rho_i(w) = \frac{(p-1)(\lambda_i+\bar{\lambda}_i)^2}{2(\kappa_i-\bar{\kappa}_i)}w_i$, $w_i \in (1,w_{i0})$, and $\rho_i(w)$ satisfies $2(\kappa_i-\bar{\kappa}_i) \ge \frac{p-1}{\rho_i(w)}(\lambda_i+\bar{\lambda}_i)^2$, $2c_{pi}(\hat{\kappa}_i-\underline{\kappa}_i) - c_{pi}\frac{(\hat{\lambda}_i+\underline{\lambda}_i)^2}{1-\rho_i(w)}(p-1) - \frac{2}{p}\sum_{j=1}^{N}\gamma_{ij}c_{pj} > 2c_{pi}(\kappa_i-\bar{\kappa}_i)L$. By using Lemma 2.3, we see that there exists a constant $\bar{a}_i > 0$ such that $\inf_{t\ge 0} G_i(|\boldsymbol{x}(t)|) > \bar{a}_i$. Then choose ρ_{i2} sufficiently close to 1, and choose $\varepsilon_{i1}, \varepsilon_{i2}$ sufficiently small such that $(c_{pi}(2\hat{\kappa}_i - 2\underline{\kappa}_i - \frac{(p-1)(\hat{\lambda}_i+\underline{\lambda}_i)^2}{\rho_{i2}(1-\rho_{i1})} - 2\frac{p-2}{p}\varepsilon_{i1}^{p} - \frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\frac{p-2}{p}\varepsilon_{i2}^{p}) - \frac{2}{p}\sum_{j=1}^{N}\gamma_{ij}c_{pj}) + 2c_{pi}(\kappa_i-\bar{\kappa}_i)|\boldsymbol{x}(t)|^{n_1-1} - c_{pi}\frac{(p-1)(\lambda_i+\bar{\lambda}_i)^2}{\rho_{i1}}|\boldsymbol{x}(t)|^{2n_2-2} > \bar{a}_i$, for all $t \ge 0$. Then (24) can be rewritten as

$$
\begin{aligned}
\mathscr{L}V\bigl(\boldsymbol{x}(t),t,i\bigr) \le{}& -\frac{p}{2}\bar{a}_i\bigl|\boldsymbol{x}(t)\bigr|^{p} + c_{pi}p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}J_1 + 2c_{pi}\underline{\kappa}_i J_2 + c_{pi}\frac{p}{2}\frac{(p-1)(\bar{\lambda}_i+\lambda_i)\bar{\lambda}_i}{\rho_{i1}} \\
&\cdot\frac{2n_2}{p+2n_2-2}J_3 + c_{pi}\frac{(p-1)(\underline{\lambda}_i+\hat{\lambda}_i)\underline{\lambda}_i}{1-\rho_{i1}}J_4 + 2c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}\xi_{i1}^{\frac{p}{2}}(t) \\
&+ c_{pi}\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}\xi_{i2}^{p}(t).
\end{aligned}
$$

In view of the fact $\int_0^t J_i\,\mathrm{d}s = \int_0^t(\int_{-\tau}^{0}|\boldsymbol{x}(s+\theta)|^{w_i}\,\mathrm{d}\eta_i(\theta) - |\boldsymbol{x}(s)|^{w_i})\,\mathrm{d}s \le \int_{-\tau}^{0}|\boldsymbol{x}(s)|^{w_i}\,\mathrm{d}s$, for $w_1 = p+n_1-1, w_3 = p+2n_2-2, w_2 = w_4 = p$, respectively. Setting $\bar{a} = \min_{i\in S}\frac{p\bar{a}_i}{2}$, we have

$$
\begin{aligned}
&V\bigl(\boldsymbol{x}(t),t,r(t)\bigr) \\
&\quad\le V\bigl(\boldsymbol{x}(0),0,r(0)\bigr) - \bar{a}\int_0^t\bigl|\boldsymbol{x}(s)\bigr|^{p}\,\mathrm{d}s + \sum_{i=1}^{N}\Biggl[c_{pi}p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}\int_{-\tau}^{0}\bigl|\boldsymbol{x}(s)\bigr|^{p+n_1-1}\,\mathrm{d}s
\end{aligned}
$$

$$+ c_{pi}\frac{p}{2}\frac{(p-1)(\bar{\lambda}_i+\lambda_i)\bar{\lambda}_i}{\rho_{i1}}\frac{2n_2}{p+2n_2-2}\int_{-\tau}^{0}\big|\boldsymbol{x}(s)\big|^{p+2n_2-2}\,\mathrm{d}s + c_{pi}\bigg(2\underline{\kappa}_i + \frac{(p-1)(\underline{\lambda}_i+\hat{\lambda}_i)\underline{\lambda}_i}{1-\rho_{i1}}\bigg)\int_{-\tau}^{0}\big|\boldsymbol{x}(s)\big|^{p}\,\mathrm{d}s + \int_0^t\bigg[c_{pi}\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}\xi_{i2}^{p}(s) + 2c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}\xi_{i1}^{\frac{p}{2}}(s)\bigg]\,\mathrm{d}s\bigg] + M(t), \tag{25}$$

where $M(t) = \int_0^t V_x(\boldsymbol{x}(s),s,r(s))\mathbf{g}(\boldsymbol{x}_s,s,r(s))\,\mathrm{d}B(s) + \int_0^t\int_R V(\boldsymbol{x}(s),s,i_0+\overline{h}(r(s-),l)) - V(\boldsymbol{x}(s),s,r(s))\mu(\mathrm{d}s,\mathrm{d}l)$ is a continuous local martingale with $M(0)=0$ a.s. Taking the expectations on both sides of (25), we have

$$\begin{aligned}
&EV\big(\boldsymbol{x}(t),t,r(t)\big)\\
&\quad\le EV\big(\boldsymbol{x}(0),0,r(0)\big) - \bar{a}\int_0^t E\big|\boldsymbol{x}(s)\big|^p\,\mathrm{d}s + \sum_{i=1}^{N}\bigg[c_{pi}p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}\int_{-\tau}^{0}E\big|\boldsymbol{x}(s)\big|^{p+n_1-1}\,\mathrm{d}s\\
&\qquad + c_{pi}\frac{p}{2}\frac{(p-1)(\bar{\lambda}_i+\lambda_i)\bar{\lambda}_i}{\rho_{i1}}\frac{2n_2}{p+2n_2-2}\int_{-\tau}^{0}E\big|\boldsymbol{x}(s)\big|^{p+2n_2-2}\,\mathrm{d}s + c_{pi}\bigg(2\underline{\kappa}_i\\
&\qquad + \frac{(p-1)(\underline{\lambda}_i+\hat{\lambda}_i)\underline{\lambda}_i}{1-\rho_{i1}}\bigg)\int_{-\tau}^{0}E\big|\boldsymbol{x}(s)\big|^{p}\,\mathrm{d}s + \int_0^t\bigg[c_{pi}\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}\xi_{i2}^{p}(s)\\
&\qquad + 2c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}\xi_{i1}^{\frac{p}{2}}(s)\bigg]\,\mathrm{d}s\bigg].
\end{aligned}$$

It follows from Lemma 2.2 that $\int_0^{+\infty}\xi_{i1}^{\frac{p}{2}}(s)\,\mathrm{d}s<\infty$, $\int_0^{+\infty}\xi_{i2}^{p}(s)\,\mathrm{d}s<\infty$. This implies $\int_0^\infty E|\boldsymbol{x}(s)|^p\,\mathrm{d}s<\infty$. Combining with Lemma 3.3, we claim that $E|\boldsymbol{x}(t)|^p$ is uniformly continuous on $[0,\infty)$. From Lemma 2.1 we claim that $\lim_{t\to\infty}E|\boldsymbol{x}(t)|^p=0$.

From the definition of p_0, we claim that $\mathscr{B}(p_0-\varepsilon)$ is a nonsingular M-matrix for any sufficiently small $\varepsilon>0$. If this is not true, we can find a sequence $\{\varepsilon_l\}_{1\le l}$ such that $\mathscr{B}(p_0-\varepsilon_l)$ are not nonsingular M-matrix for every $l\ge1$. According to the equivalent condition (3) in Lemma 3.1, there exists $1\le k(l)\le N$ such that the $k(l)$th leading principal minor of $\mathscr{B}(p_0-\varepsilon_l)$ is equal to zero. Since all the leading principal minors of $\mathscr{B}(p)$ are rational functions of p, there must have finite singular points and zero points at most, which is a contradiction. So our claim holds.

For any $p\in(0,p_0)$, we can find sufficiently small $\varepsilon>0$ such that $p<p_0-\varepsilon$ and $\mathscr{B}(p_0-\varepsilon)$ is a nonsingular M-matrix. By using the Hölder inequality, we get the required result (22). □

To study the exponential stability, we slightly modify Assumption 2.2. That is, in the case of $\xi_{i1}(t)=\kappa_{i0}e^{-\alpha_{i1}t}$, $\xi_{i2}(t)=\lambda_{i0}e^{-\alpha_{i2}t}$ in Assumption 2.2, we have the following theorem for the exponential stability.

Theorem 3.2 *If Assumptions* 2.1, 2.2 *hold,* $\xi_{i1}(t)=\kappa_{i0}e^{-\alpha_{i1}t}$, $\xi_{i2}(t)=\lambda_{i0}e^{-\alpha_{i2}t}$ *in Assumption* 2.2, $\frac{2(\kappa_i-\overline{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2}>1$ *for all* $i\in S$, *and there exists* $2\le p<\min_{i\in S}\{\frac{2(\kappa_i-\overline{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2}+1\}$ *such that* $\mathscr{B}(p)$ *is a nonsingular* M*-matrix, then for any initial data* $\boldsymbol{\zeta}\in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$, $i_0\in S$, *there is a unique global solution* $\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)$ *of system* (10) *on* $t\ge-\tau$, *and the solution is almost surely* *exponentially stable and rth moment exponentially stable, for* $r \in (0, p]$, *i.e.*,

$$\limsup_{t\to\infty} \frac{\log(|\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)|)}{t} \le -\frac{\varepsilon_p}{p} \quad a.s., \qquad \limsup_{t\to\infty} \frac{\log(E|\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)|^r)}{t} \le -\frac{r}{p}\varepsilon_p, \tag{26}$$

where $\kappa_{i0} \ge 0, \lambda_{i0} \ge 0, \alpha_{i1} > 0, \alpha_{i2} > 0$ *for all* $i \in S$, $\mathscr{B}(p)$ *is the same as defined in Theorem* 3.1, *and* ε_p *is a positive constant which depends on p but not on* ζ.

Proof Since $\mathscr{B}(p)$ is a nonsingular M-matrix, using the same technique as applied in the proof of Theorem 3.1, we claim that there exists at least a sufficiently small positive constant $\varepsilon > 0$ such that $\mathscr{B}'(p,\varepsilon) = \mathrm{diag}(\Delta_1', \Delta_2', \dots, \Delta_N') - \frac{2}{p}\Gamma - \frac{2\varepsilon}{p}I$ is still a nonsingular M-matrix and $\varepsilon < \min_{i\in S}\{\frac{p\alpha_{i1}}{2}, p\alpha_{i2}\}$, where $\Delta_i' = 2(\hat{\kappa}_i - \underline{\kappa}_i e^{\varepsilon\tau}) - 2L(\kappa_i - \overline{\kappa}_i e^{\varepsilon\tau}) - \frac{2(\kappa_i - \overline{\kappa}_i e^{\varepsilon\tau})(\hat{\lambda}_i + \underline{\lambda}_i)(\hat{\lambda}_i + \underline{\lambda}_i e^{\varepsilon\tau})(p-1)}{2(\kappa_i - \overline{\kappa}_i e^{\varepsilon\tau}) - (\lambda_i + \bar{\lambda}_i)(\lambda_i + \bar{\lambda}_i e^{\varepsilon\tau})(p-1)}$. So we can define ε_p as $\varepsilon_p = \sup\{\varepsilon > 0 : \varepsilon \in \mathscr{B}\}$, where $\mathscr{B} = \{\varepsilon : \mathscr{B}'(p,\varepsilon)$ is a nonsingular M-matrix, $0 < \varepsilon < \min_{i\in S}\{\frac{p\alpha_{i1}}{2}, p\alpha_{i2}\}\}$. And there exists $c_p = (c_{p1}, c_{p2}, \dots, c_{pN}) \gg 0$ such that $\mathscr{B}'(p,\varepsilon)c_p \gg 0$, for $\varepsilon \in \mathscr{B}$.

From the condition $\frac{2(\kappa_i - \overline{\kappa}_i)}{(\lambda_i + \bar{\lambda}_i)^2} > 1$, we get $\kappa_i - \bar{\kappa}_i > 0$ for all $i \in S$, so the existence and uniqueness of the solution follow from Lemma 3.2. Therefore, we only need to prove other assertions here. For simplicity, write $\boldsymbol{x}(t) = \boldsymbol{x}(t,\zeta,i_0)$. Applying the generalized Itô formula to $V(\boldsymbol{x},t,i) = c_{pi}e^{\varepsilon t}|\boldsymbol{x}|^p$, $\varepsilon \in \mathscr{B}$, we find

$$\begin{aligned}
&\mathscr{L}V\big(\boldsymbol{x}(t),t,i\big) \\
&\quad\le c_{pi}e^{\varepsilon t}\bigg[\varepsilon\big|\boldsymbol{x}(t)\big|^p + \frac{p}{2}\big|\boldsymbol{x}(t)\big|^{p-2}\bigg(-2\kappa_i\big|\boldsymbol{x}(t)\big|^{n_1+1} + 2\bar{\kappa}_i\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^{n_1+1}\,\mathrm{d}\eta_1(\theta) \\
&\qquad - 2\hat{\kappa}_i\big|\boldsymbol{x}(t)\big|^2 + 2\underline{\kappa}_i\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^2\,\mathrm{d}\eta_2(\theta) + \frac{p-1}{\rho_{i1}}\bigg(\frac{\lambda_i^2}{\delta_{i1}}\big|\boldsymbol{x}(t)\big|^{2n_2} + \frac{\bar{\lambda}_i^2}{1-\delta_{i1}} \\
&\qquad \cdot\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^{2n_2}\,\mathrm{d}\eta_3(\theta)\bigg) + \frac{p-1}{\rho_{i2}(1-\rho_{i1})}\bigg(\frac{\underline{\lambda}_i^2}{1-\delta_{i2}}\int_{-\tau}^0\big|\boldsymbol{x}(t+\theta)\big|^2\,\mathrm{d}\eta_4(\theta) \\
&\qquad + \frac{\hat{\lambda}_i^2}{\delta_{i2}}\big|\boldsymbol{x}(t)\big|^2\bigg) + \frac{p-1}{(1-\rho_{i1})(1-\rho_{i2})}\lambda_{i0}^2e^{-2\alpha_{i2}t} + 2\kappa_{i0}e^{-\alpha_{i1}t}\bigg)\bigg] + \sum_{j=1}^N\gamma_{ij}c_{pj}e^{\varepsilon t}\big|\boldsymbol{x}(t)\big|^p \\
&\quad\le \frac{p}{2}c_{pi}e^{\varepsilon t}\bigg[-2\big(\kappa_i - \bar{\kappa}_ie^{\varepsilon\tau}\big)\big|\boldsymbol{x}(t)\big|^{p+n_1-1} - \bigg(2\hat{\kappa}_i - 2\underline{\kappa}_ie^{\varepsilon\tau} - \frac{p-1}{\rho_{i2}(1-\rho_{i1})}\bigg(\frac{\hat{\lambda}_i^2}{\delta_{i2}} \\
&\qquad + \frac{\underline{\lambda}_i^2}{1-\delta_{i2}}e^{\varepsilon\tau}\bigg) - 2\frac{p-2}{p}\varepsilon_{i1}^p - \frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\frac{p-2}{p}\varepsilon_{i2}^p - \frac{2}{p}\varepsilon\bigg)\big|\boldsymbol{x}(t)\big|^p + \frac{p-1}{\rho_{i1}}\bigg(\frac{\lambda_i^2}{\delta_{i1}} \\
&\qquad + \frac{\bar{\lambda}_i^2}{1-\delta_{i1}}e^{\varepsilon\tau}\bigg)\big|\boldsymbol{x}(t)\big|^{p+2n_2-2}\bigg] + 2\underline{\kappa}_ic_{pi}e^{\varepsilon t}\bar{J}_2 + \frac{p-1}{\rho_{i2}(1-\rho_{i1})}\frac{\underline{\lambda}_i^2}{1-\delta_{i2}}c_{pi}e^{\varepsilon t}\bar{J}_4 \\
&\qquad + \frac{p}{2}\frac{p-1}{\rho_{i1}}\frac{\bar{\lambda}_i^2}{1-\delta_{i1}}\frac{2n_2}{p+2n_2-2}c_{pi}e^{\varepsilon t}\bar{J}_3 + p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}c_{pi}e^{\varepsilon t}\bar{J}_1 + 2\kappa_{i0}^{\frac{p}{2}}c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}} \\
&\qquad \cdot e^{(\varepsilon-\frac{p}{2}\alpha_{i1})t} + \frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\lambda_{i0}^pc_{pi}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}e^{(\varepsilon-p\alpha_{i2})t} + \sum_{j=1}^N\gamma_{ij}c_{pj}e^{\varepsilon t}\big|\boldsymbol{x}(t)\big|^p,
\end{aligned}\tag{27}$$

where $\rho_{i1}, \rho_{i2}, \delta_{i1}, \delta_{i2} \in (0,1)$, $\bar{J}_1 = \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|^{p+n_1-1}\mathrm{d}\eta_1(\theta) - e^{\varepsilon\tau}|\boldsymbol{x}(t)|^{p+n_1-1}$, $\bar{J}_2 = \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|^p\,\mathrm{d}\eta_2(\theta) - e^{\varepsilon\tau}|\boldsymbol{x}(t)|^p$, $\bar{J}_3 = \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|^{p+2n_2-2}\,\mathrm{d}\eta_3(\theta) - e^{\varepsilon\tau}|\boldsymbol{x}(t)|^{p+2n_2-2}$, $\bar{J}_4 = \int_{-\tau}^0 |\boldsymbol{x}(t +$

$\theta)|^p\,\mathrm{d}\eta_4(\theta) - e^{\varepsilon\tau}|\boldsymbol{x}(t)|^p$. Substituting $\delta_{i1} = \frac{\lambda_i}{\lambda_i+\bar{\lambda}_i}$, $\delta_{i2} = \frac{\hat{\lambda}_i}{\hat{\lambda}_i+\underline{\lambda}_i}$ into (27), we have

$$\begin{aligned}
&\mathscr{L}V\bigl(\boldsymbol{x}(t),t,i\bigr)\\
&\quad\le \frac{p}{2}e^{\varepsilon t}\Biggl[-2c_{pi}\bigl(\kappa_i-\bar{\kappa}_ie^{\varepsilon\tau}\bigr)\bigl|\boldsymbol{x}(t)\bigr|^{p+n_1-1}-\Biggl(c_{pi}\Biggl(2\hat{\kappa}_i-2\underline{\kappa}_ie^{\varepsilon\tau}-\frac{p-1}{\rho_{i2}(1-\rho_{i1})}(\hat{\lambda}_i+\underline{\lambda}_i)\\
&\qquad\cdot\bigl(\hat{\lambda}_i+\underline{\lambda}_ie^{\varepsilon\tau}\bigr)-2\frac{p-2}{p}\varepsilon_{i1}^p-\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\frac{p-2}{p}\varepsilon_{i2}^p-\frac{2}{p}\varepsilon\Biggr)-\frac{2}{p}\sum_{j=1}^N\gamma_{ij}c_{pj}\Biggr)\bigl|\boldsymbol{x}(t)\bigr|^p\\
&\qquad+c_{pi}\frac{p-1}{\rho_{i1}}(\lambda_i+\bar{\lambda}_i)\bigl(\lambda_i+\bar{\lambda}_ie^{\varepsilon\tau}\bigr)\bigl|\boldsymbol{x}(t)\bigr|^{p+2n_2-2}\Biggr]+2\underline{\kappa}_ic_{pi}e^{\varepsilon t}\bar{J}_2+p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}c_{pi}e^{\varepsilon t}\bar{J}_1\\
&\qquad+\frac{p-1}{\rho_{i2}(1-\rho_{i1})}\underline{\lambda}_i(\hat{\lambda}_i+\underline{\lambda}_i)c_{pi}e^{\varepsilon t}\bar{J}_4+\frac{p}{2}\frac{p-1}{\rho_{i1}}\bar{\lambda}_i(\lambda_i+\bar{\lambda}_i)\frac{2n_2}{p+2n_2-2}c_{pi}e^{\varepsilon t}\bar{J}_3\\
&\qquad+2\kappa_{i0}^{\frac{p}{2}}c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}e^{(\varepsilon-\frac{p}{2}\alpha_{i1})t}+\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\lambda_{i0}^pc_{pi}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}e^{(\varepsilon-p\alpha_{i2})t}.
\end{aligned}\tag{28}$$

Let $H_i(|\boldsymbol{x}(t)|) = (c_{pi}(2\hat{\kappa}_i - 2\underline{\kappa}_ie^{\varepsilon\tau} - \frac{p-1}{1-\rho_{i1}}(\hat{\lambda}_i+\underline{\lambda}_i)(\hat{\lambda}_i+\underline{\lambda}_ie^{\varepsilon\tau}) - \frac{2}{p}\varepsilon) - \frac{2}{p}\sum_{j=1}^N\gamma_{ij}c_{pj}) + 2c_{pi}(\kappa_i - \bar{\kappa}_ie^{\varepsilon\tau})|\boldsymbol{x}(t)|^{n_1-1} - c_{pi}\frac{p-1}{\rho_{i1}}(\lambda_i+\bar{\lambda}_i)(\lambda_i+\bar{\lambda}_ie^{\varepsilon\tau})|\boldsymbol{x}(t)|^{2n_2-2}$. Note that $\mathscr{B}'(p,\varepsilon)c_p \gg 0$ can be rewritten in a component-wise form: $c_{pi}(2(\hat{\kappa}_i-\underline{\kappa}_ie^{\varepsilon\tau})-2L(\kappa_i-\overline{\kappa}_ie^{\varepsilon\tau})-\frac{2(\kappa_i-\overline{\kappa}_ie^{\varepsilon\tau})(\hat{\lambda}_i+\underline{\lambda}_i)(\hat{\lambda}_i+\underline{\lambda}_ie^{\varepsilon\tau})(p-1)}{2(\kappa_i-\overline{\kappa}_ie^{\varepsilon\tau})-(\lambda_i+\bar{\lambda}_i)(\lambda_i+\bar{\lambda}_ie^{\varepsilon\tau})(p-1)} - \frac{2}{p}\varepsilon) - \frac{2}{p}\sum_{j=1}^N\gamma_{ij}c_{pj} > 0$. Using the same technique as applied in the proof of Theorem 3.1, we claim that there exists an $\bar{\bar{a}}_i > 0$ such that $\inf_{t\ge0}H_i(|\boldsymbol{x}(t)|) > \bar{\bar{a}}_i$. Then choose ρ_{i2} sufficiently close to 1, and choose $\varepsilon_{i1}, \varepsilon_{i2}$ sufficiently small such that $(c_{pi}(2\hat{\kappa}_i - 2\underline{\kappa}_ie^{\varepsilon\tau} - \frac{p-1}{\rho_{i2}(1-\rho_{i1})}(\hat{\lambda}_i + \underline{\lambda}_i)(\hat{\lambda}_i+\underline{\lambda}_ie^{\varepsilon\tau}) - 2\frac{p-2}{p}\varepsilon_{i1}^p - \frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\frac{p-2}{p}\varepsilon_{i2}^p - \frac{2}{p}\varepsilon) - \frac{2}{p}\sum_{j=1}^N\gamma_{ij}c_{pj}) + 2c_{pi}(\kappa_i - \bar{\kappa}_ie^{\varepsilon\tau})|\boldsymbol{x}(t)|^{n_1-1} - c_{pi}\frac{p-1}{\rho_{i1}}(\lambda_i+\bar{\lambda}_i)(\lambda_i+\bar{\lambda}_ie^{\varepsilon\tau})|\boldsymbol{x}(t)|^{2n_2-2} > \bar{\bar{a}}_i$, for all $t\ge0$. Then (28) can be rewritten as

$$\begin{aligned}
&\mathscr{L}V\bigl(\boldsymbol{x}(t),t,i\bigr)\\
&\quad\le -\frac{p}{2}\bar{\bar{a}}_ie^{\varepsilon t}\bigl|\boldsymbol{x}(t)\bigr|^p+p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}c_{pi}e^{\varepsilon t}\bar{J}_1+2\underline{\kappa}_ic_{pi}e^{\varepsilon t}\bar{J}_2\\
&\qquad+\frac{p-1}{\rho_{i2}(1-\rho_{i1})}\underline{\lambda}_i(\hat{\lambda}_i+\underline{\lambda}_i)c_{pi}e^{\varepsilon t}\bar{J}_4+\frac{p}{2}\frac{p-1}{\rho_{i1}}\bar{\lambda}_i(\lambda_i+\bar{\lambda}_i)\frac{2n_2}{p+2n_2-2}c_{pi}e^{\varepsilon t}\bar{J}_3\\
&\qquad+2\kappa_{i0}^{\frac{p}{2}}c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}e^{(\varepsilon-\frac{p}{2}\alpha_{i1})t}+\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\lambda_{i0}^pc_{pi}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}e^{(\varepsilon-p\alpha_{i2})t}.
\end{aligned}$$

In view of the fact that $\int_0^t e^{\varepsilon s}\bar{J}_i\,\mathrm{d}s = \int_0^t e^{\varepsilon s}(\int_{-\tau}^0|\boldsymbol{x}(s+\theta)|^{w_i}\,\mathrm{d}\eta_i(\theta) - e^{\varepsilon\tau}|\boldsymbol{x}(s)|^{w_i})\,\mathrm{d}s \le e^{\varepsilon\tau}\times\int_{-\tau}^0|\boldsymbol{x}(s)|^{w_i}\,\mathrm{d}s$, for $w_1 = p+n_1-1$, $w_3 = p+2n_2-2$, $w_2 = w_4 = p$, respectively. Setting $\bar{\bar{a}} = \min_{i\in S}\frac{p\bar{\bar{a}}_i}{2}$, we have

$$\begin{aligned}
&V\bigl(\boldsymbol{x}(t),t,r(t)\bigr)\\
&\quad= V\bigl(\boldsymbol{x}(0),0,r(0)\bigr)+\int_0^t\mathscr{L}V\bigl(\boldsymbol{x}(t),t,r(t)\bigr)\,\mathrm{d}s+M_1(t)\\
&\quad\le V\bigl(\boldsymbol{x}(0),0,r(0)\bigr)-\bar{\bar{a}}\int_0^te^{\varepsilon s}\bigl|\boldsymbol{x}(s)\bigr|^p\,\mathrm{d}s+\sum_{i=1}^N\Biggl[p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}c_{pi}e^{\epsilon\tau}\int_{-\tau}^0\bigl|\boldsymbol{x}(s)\bigr|^{p+n_1-1}\,\mathrm{d}s\\
&\qquad+\frac{p}{2}\frac{p-1}{\rho_{i1}}\bar{\lambda}_i(\lambda_i+\bar{\lambda}_i)\frac{2n_2}{p+2n_2-2}c_{pi}e^{\epsilon\tau}\int_{-\tau}^0\bigl|\boldsymbol{x}(s)\bigr|^{p+2n_2-2}\,\mathrm{d}s+\Biggl(2\underline{\kappa}_ic_{pi}
\end{aligned}$$

$$+\frac{p-1}{\rho_{i2}(1-\rho_{i1})}\underline{\lambda}_i(\hat{\lambda}_i+\underline{\lambda}_i)c_{pi}\Big)e^{\epsilon\tau}\int_{-\tau}^{0}\big|\boldsymbol{x}(s)\big|^p\,\mathrm{d}s+2\kappa_{i0}^{\frac{p}{2}}c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}\frac{1}{(\frac{p}{2}\alpha_{i1}-\varepsilon)}$$

$$+\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\lambda_{i0}^{p}c_{pi}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}\frac{1}{(p\alpha_{i2}-\varepsilon)}\Bigg]+M_1(t), \tag{29}$$

where $M_1(t)=\int_0^t V_x(\boldsymbol{x}(s),s,r(s))\mathbf{g}(\boldsymbol{x}_s,s,r(s))\,\mathrm{d}B(s)+\int_0^t\int_R V(\boldsymbol{x}(s),s,i_0+\overline{h}(r(s-),l))-V(\boldsymbol{x}(s),s,r(s))\mu(\mathrm{d}s,\mathrm{d}l)$ is a local martingale with the initial value $M_1(0)=0$. Applying the nonnegative semi-martingale convergence theorem (see [29]), we obtain $\limsup_{t\to\infty}e^{\varepsilon t}|\boldsymbol{x}(t)|^p<\infty$ a.s. Hence, there exists a finite positive random variable $\bar{\zeta}$ such that

$$\sup_{0\le t<\infty}e^{\varepsilon t}\big|\boldsymbol{x}(t)\big|^p<\bar{\zeta}\quad\text{a.s.} \tag{30}$$

So we claim that $\limsup_{t\to\infty}\frac{\log(|\boldsymbol{x}(t)|)}{t}\le-\frac{\varepsilon}{p}$ a.s. Then as $\varepsilon\to\varepsilon_p$, we get the previous part of assertion (26).

Taking the expectations on both sides of (29), we have $EV(\boldsymbol{x}(t),t,r(t))\le c_1'$, where $c_1'=EV(\boldsymbol{x}(0),0,r(0))+\sum_{i=1}^N[p\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}c_{pi}e^{\epsilon\tau}\int_{-\tau}^0E|\boldsymbol{x}(s)|^{p+n_1-1}\,\mathrm{d}s+\frac{p}{2}\frac{p-1}{\rho_{i1}}\bar{\lambda}_i(\lambda_i+\bar{\lambda}_i)\frac{2n_2}{p+2n_2-2}c_{pi}e^{\epsilon\tau}\times\int_{-\tau}^0E|\boldsymbol{x}(s)|^{p+2n_2-2}\,\mathrm{d}s+(2\underline{\kappa}_ic_{pi}+\frac{p-1}{\rho_{i2}(1-\rho_{i1})}\underline{\lambda}_i(\hat{\lambda}_i+\underline{\lambda}_i)c_{pi})e^{\epsilon\tau}\int_{-\tau}^0E|\boldsymbol{x}(s)|^p\mathrm{d}s+2\kappa_{i0}^{\frac{p}{2}}c_{pi}\varepsilon_{i1}^{-\frac{p(p-2)}{2}}\times\frac{1}{(\frac{p}{2}\alpha_{i1}-\varepsilon)}+\frac{p-1}{1-\rho_{i1}}\frac{1}{1-\rho_{i2}}\lambda_{i0}^pc_{pi}\varepsilon_{i2}^{-\frac{p(p-2)}{2}}\frac{1}{(p\alpha_{i2}-\varepsilon)}]$. This implies

$$E\big|\boldsymbol{x}(t)\big|^p\le\frac{c_1'}{\min_{i\in S}c_{pi}}e^{-\varepsilon t},\quad\forall t>0, \tag{31}$$

which yields the assertion $\limsup_{t\to\infty}\frac{\log(E|\boldsymbol{x}(t)|^p)}{t}\le-\varepsilon$. Then we claim that $\limsup_{t\to\infty}\frac{\log(E|\boldsymbol{x}(t)|^p)}{t}\le-\varepsilon_p$, as $\varepsilon\to\varepsilon_p$. For any $r\in(0,p)$, using the Hölder inequality we find

$$E\big|\boldsymbol{x}(t)\big|^r\le\big(E\big|\boldsymbol{x}(t)\big|^p\big)^{\frac{r}{p}}\le\left(\frac{c_1'}{\min_{i\in S}c_{pi}}\right)^{\frac{r}{p}}e^{-\frac{r}{p}\varepsilon_pt}, \tag{32}$$

which yields the latter part of assertion (26). □

4 Robust stability

Now we discuss our main problem of the robust stability. Consider an n-dimensional stable FDEwMS

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=\boldsymbol{f}\big(\boldsymbol{x}_t,t,r(t)\big) \tag{33}$$

on $t\ge0$, where $\boldsymbol{f}$ satisfies growth condition (11). Hence, the robust stability in this section is to give the conditions on the parameters $\lambda_i,\bar{\lambda}_i,\hat{\lambda}_i,\underline{\lambda}_i$ such that the stable system (33) with a nonlinear stochastic perturbation $\mathbf{g}(\boldsymbol{x}_t,t,r(t))\dot{B}(t)$ satisfying growth condition (12)

$$\mathrm{d}\boldsymbol{x}(t)=\boldsymbol{f}\big(\boldsymbol{x}_t,t,r(t)\big)\,\mathrm{d}t+\mathbf{g}\big(\boldsymbol{x}_t,t,r(t)\big)\,\mathrm{d}B(t) \tag{34}$$

remains stable. Further we assume that $\boldsymbol{f}$ and $\mathbf{g}$ satisfy the local Lipschitz condition (Assumption 2.1).

From Theorem 3.1, we have the robustness result of the moment asymptotical stability directly.

Theorem 4.1 *If there exists a series of nonnegative numbers* $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i$ *such that* $\frac{2(\kappa_i-\bar{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} > 1$ *for all* $i \in S$ *and* $p_0 > 2$ *exists, then for any* $p \in (0, p_0)$, *any initial data* $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ *and* $i_0 \in S$, *there is a unique global solution of the stochastically perturbed system* (34) *on* $t \geq -\tau$ *and the solution is pth moment asymptotically stable, where* p_0 *is the same as defined in Theorem* 3.1.

Proof Since there exists a series of nonnegative numbers $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i$ such that $\frac{2(\kappa_i-\bar{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} > 1$ for all $i \in S$ and $p_0 > 2$ exists, all conditions of Theorem 3.1 are satisfied for system (34). Hence, by Theorem 3.1, all conclusions of Theorem 4.1 hold. □

Obviously, Theorem 4.1 is a simple application of Theorem 3.1. Similarly, we get the robustness result of exponential stability directly from Theorem 3.2.

Theorem 4.2 *If* $\xi_{i1}(t) = \kappa_{i0}e^{-\alpha_{i1}t}$ *in condition* (11), *and there exist a series of nonnegative numbers* $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i, \lambda_{i0}, p$ *and positive numbers* α_{i2} *such that* $\frac{2(\kappa_i-\bar{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} > 1$ *for all* $i \in S$, $\xi_{i2}(t) = \lambda_{i0}e^{-\alpha_{i2}t}$ *in condition* (12), $2 \leq p < \min_{i\in S}\{\frac{2(\kappa_i-\bar{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} + 1\}$ *and* $\mathscr{B}(p)$ *is a nonsingular M-matrix, then for any initial data* $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ *and* $i_0 \in S$, *there is a unique global solution* $\boldsymbol{x}(t, \boldsymbol{\zeta}, i_0)$ *of the stochastically perturbed system* (34) *on* $t \geq -\tau$ *and the solution is almost surely exponentially stable and rth moment exponentially stable, for* $r \in (0, p]$, *where* $\kappa_{i0} \geq 0, \alpha_{i1} > 0$ *for all* $i \in S$, $\mathscr{B}(p)$ *is the same as defined in Theorem* 3.1, *and* ε_p *is a positive constant which depends on p but not on* ζ.

Proof Since $\xi_{i1}(t) = \kappa_{i0}e^{-\alpha_{i1}t}$ in condition (11) holds and there exist a series of nonnegative numbers $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i, \lambda_{i0}, p$ and positive numbers α_{i2} such that $\frac{2(\kappa_i-\bar{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} > 1$ for all $i \in S$, $\xi_{i2}(t) = \lambda_{i0}e^{-\alpha_{i2}t}$ in condition (12), $2 \leq p < \min_{i\in S}\{\frac{2(\kappa_i-\bar{\kappa}_i)}{(\lambda_i+\bar{\lambda}_i)^2} + 1\}$ and $\mathscr{B}(p)$ is a nonsingular M-matrix, all conditions of Theorem 3.2 are satisfied for system (34). Hence, by Theorem 3.2, all conclusions of Theorem 4.2 hold. □

Remark 4 Obviously, both Theorems 4.1 and 4.2 above require one to choose appropriate parameters $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i$ such that $\mathscr{B}(p)$ is a nonsingular M-matrix. Compared with [1], our requirement is more general, but it cannot easily be checked. Hence, it is necessary to propose some easily verifiable conditions for the robust stability in the following part.

Next, we replace the requirement of $\mathscr{B}(p)$ with the following assumption to continue to use the property of M-matrices.

Assumption 4.1 Let $q \geq 2$ and $\bar{\mathscr{B}}(q) = \operatorname{diag}(\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_N) - \frac{(5q-3)(q-2)}{q}I_N - \frac{2}{q}\Gamma$ is a nonsingular M-matrix, where $\bar{\Delta}_i = 2(\hat{\kappa}_i - \underline{\kappa}_i)$, I_N is the identity matrix.

Under Assumption 4.1 and due to Lemma 3.1 we see that

$$c_q = (c_{q1}, c_{q2}, \ldots, c_{qN})^T = \bar{\mathscr{B}}(q)^{-1}\vec{1}_N \gg 0, \tag{35}$$

where $\vec{1}_N = (1, 1, \ldots, 1)^T_N$.

Next, we propose another new easily verifiable condition for the pth moment asymptotic stability of the solution of system (34).

Theorem 4.3 *If Assumptions* 2.1, 2.2, 4.1 *hold*, $\kappa_i > \overline{\kappa}_i$ *for all* $i \in S$, *and the following condition* (36) *holds*:

$$\left(\frac{5\Theta_1}{n_1 - 1}\right)^{\frac{n_1-1}{n_1-2n_2+1}} \left(\frac{n_2 - 1}{\Theta_2}\right)^{\frac{2n_2-2}{n_1-2n_2+1}} (n_1 - 2n_2 + 1) + 5\Theta_3 < 1, \tag{36}$$

then for any $p \in (0, q]$, *any initial data* $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ *and* $i_0 \in S$, *there is a unique global solution* $\mathbf{x}(t, \boldsymbol{\zeta}, i_0)$ *of system* (34) *on* $t \geq -\tau$ *and the solution is pth moment asymptotically stable, i.e.*,

$$\lim_{t\to\infty} E\left|\mathbf{x}(t, \boldsymbol{\zeta}, i_0)\right|^p = 0, \tag{37}$$

where $\Theta_1 = \max_{i\in S} c_{qi}(q-1)(\lambda_i^2 + \overline{\lambda}_i^2)$, $\Theta_2 = \min_{i\in S} c_{qi}(\kappa_i - \overline{\kappa}_i)$, $\Theta_3 = \max_{i\in S} c_{qi}(q-1)(\hat{\lambda}_i^2 + \underline{\lambda}_i^2)$.

Proof We give an outline focusing on the innovative parts of the proof, and the details of the proof are similar to those of Theorem 3.1.

Applying the generalized Itô formula to $V(\mathbf{x}, t, i) = c_{qi}|\mathbf{x}|^q$, we find

$$\begin{aligned}
&\mathscr{L}V\big(\mathbf{x}(t), t, i\big) \\
&\quad\leq \frac{q}{2} c_{qi}\left|\mathbf{x}(t)\right|^{q-2}\big(2\mathbf{x}^T(t)\mathbf{f}\big(\mathbf{x}_t, t, r(t)\big) + (q-1)\left|\mathbf{g}\big(\mathbf{x}_t, t, r(t)\big)\right|^2\big) + \sum_{j=1}^N \gamma_{ij}c_{qj}\left|\mathbf{x}(t)\right|^q \\
&\quad\leq \frac{q}{2} c_{qi}\left|\mathbf{x}(t)\right|^{q-2}\bigg[-2\kappa_i\left|\mathbf{x}(t)\right|^{n_1+1} + 2\bar{\kappa}_i\int_{-\tau}^0 \left|\mathbf{x}(t+\theta)\right|^{n_1+1}\,\mathrm{d}\eta_1(\theta) - 2\hat{\kappa}_i\left|\mathbf{x}(t)\right|^2 \\
&\qquad + 2\underline{\kappa}_i\int_{-\tau}^0\left|\mathbf{x}(t+\theta)\right|^2\,\mathrm{d}\eta_2(\theta) + 2\xi_{i1}(t) + 5(q-1)\bigg(\lambda_i^2\left|\mathbf{x}(t)\right|^{2n_2} + \hat{\lambda}_i^2\left|\mathbf{x}(t)\right|^2 \\
&\qquad + \bar{\lambda}_i^2\int_{-\tau}^0\left|\mathbf{x}(t+\theta)\right|^{2n_2}\,\mathrm{d}\eta_3(\theta) + \underline{\lambda}_i^2\int_{-\tau}^0\left|\mathbf{x}(t+\theta)\right|^2\,\mathrm{d}\eta_4(\theta) + \xi_{i2}^2(t)\bigg)\bigg] \\
&\qquad + \sum_{j=1}^N \gamma_{ij}c_{qj}\left|\mathbf{x}(t)\right|^q \\
&\quad\leq \frac{q}{2}\bigg[-2c_{qi}(\kappa_i - \bar{\kappa}_i)\left|\mathbf{x}(t)\right|^{q+n_1-1} + 5c_{qi}(q-1)\big(\lambda_i^2 + \bar{\lambda}_i^2\big)\left|\mathbf{x}(t)\right|^{q+2n_2-2} - \big(c_{qi}(2\hat{\kappa}_i \\
&\qquad - 2\underline{\kappa}_i - 5(q-1)\big(\hat{\lambda}_i^2 + \underline{\lambda}_i^2\big) - \frac{(5q-3)(q-2)}{q}\big) - \frac{2}{q}\sum_{j=1}^N \gamma_{ij}c_{qj}\big)\left|\mathbf{x}(t)\right|^q\bigg] + 2c_{qi}\underline{\kappa}_i J_2 \\
&\qquad + qc_{qi}\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}J_1 + 5c_{qi}(q-1)\underline{\lambda}_i^2 J_4 + \frac{q}{2}c_{qi}5(q-1)\bar{\lambda}_i^2\frac{2n_2}{q+2n_2-2}J_3 \\
&\qquad + 2c_{qi}\xi_{i1}^{\frac{q}{2}}(t) + 5c_{qi}(q-1)\xi_{i2}^q(t),
\end{aligned} \tag{38}$$

where J_1, J_2, J_3, J_4 are the same as defined in the proof of Theorem 3.1.

Let $\overline{G}_i(|\mathbf{x}(t)|) = 2c_{qi}(\kappa_i - \bar{\kappa}_i)|\mathbf{x}(t)|^{n_1-1} - 5c_{qi}(q-1)(\lambda_i^2 + \bar{\lambda}_i^2)|\mathbf{x}(t)|^{2n_2-2} + (c_{qi}(2\hat{\kappa}_i - 2\underline{\kappa}_i - 5(q-1)(\hat{\lambda}_i^2 + \underline{\lambda}_i^2) - \frac{(5q-3)(q-2)}{q}) - \frac{2}{q}\sum_{j=1}^N \gamma_{ij}c_{qj})$. From (35), we see that $\bar{\mathscr{B}}(q)c_q = \vec{1}_N$, which can be rewritten in a component-wise form: $c_{qi}(2\hat{\kappa}_i - 2\underline{\kappa}_i - \frac{(5q-3)(q-2)}{q}) - \frac{2}{q}\sum_{j=1}^N \gamma_{ij}c_{qj} = 1$. So $\overline{G}_i(|\mathbf{x}(t)|) = 2c_{qi}(\kappa_i - \bar{\kappa}_i)|\mathbf{x}(t)|^{n_1-1} - 5c_{qi}(q-1)(\lambda_i^2 + \bar{\lambda}_i^2)|\mathbf{x}(t)|^{2n_2-2} + (1 - 5c_{qi}(q-1)(\hat{\lambda}_i^2 +$

$\underline{\lambda}_i^2)$). There exists $|\boldsymbol{x}(t)|^* = (\frac{5c_{qi}(q-1)(\lambda_i^2+\bar{\lambda}_i^2)(2n_2-2)}{2c_{qi}(\kappa_i-\bar{\kappa}_i)(n_1-1)})^{\frac{1}{n_1+1-2n_2}} > 0$ such that $\overline{G}_i'(|\boldsymbol{x}(t)|^*) = 0$ and $\overline{G}_i(|\boldsymbol{x}(t)|^*) = 1 - (\frac{5c_{qi}(q-1)(\lambda_i^2+\bar{\lambda}_i^2)}{n_1-1})^{\frac{n_1-1}{n_1-2n_2+1}}(\frac{n_2-1}{c_{qi}(\kappa_i-\bar{\kappa}_i)})^{\frac{2n_2-2}{n_1-2n_2+1}}(n_1-2n_2+1) - 5c_{qi}(q-1)(\hat{\lambda}_i^2 + \underline{\lambda}_i^2)$. From condition (36), we have $\overline{G}_i(|\boldsymbol{x}(t)|^*) > 0$ and $\overline{G}_i(0) > 0$ for all $i \in S$. We therefore have, for any $t \geq 0$, $\overline{G}_i(|\boldsymbol{x}(t)|) \geq \overline{G}_i(0) \wedge \overline{G}_i(|\boldsymbol{x}(t)|^*) := \bar{b}_i > 0$. Then (38) can be rewritten as

$$\begin{aligned}
&\mathscr{L}V(\boldsymbol{x}(t),t,i) \\
&\quad\leq -\frac{q}{2}\bar{b}_i|\boldsymbol{x}(t)|^q + qc_{qi}\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}J_1 + 2c_{qi}\underline{\kappa}_iJ_2 + \frac{q}{2}c_{qi}5(q-1)\bar{\lambda}_i^2\frac{2n_2}{q+2n_2-2}J_3 \\
&\qquad + 5c_{qi}(q-1)\underline{\lambda}_i^2J_4 + 2c_{qi}\xi_{i1}^{\frac{q}{2}}(t) + 5c_{qi}(q-1)\xi_{i2}^q(t).
\end{aligned}$$

Following the proof of the remainder of Theorem 3.1, we get the required assertion (37). □

From Theorem 4.3, we have another new robustness result of the moment asymptotical stability directly.

Theorem 4.4 *If Assumption* 4.1 *holds,* $\kappa_i > \overline{\kappa}_i$ *for all* $i \in S$, *and there exists a series of nonnegative numbers* $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i$ *such that condition* (36) *holds, then for any* $p \in (0,q]$, *any initial data* $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$ *and* $i_0 \in S$, *there is a unique global solution of the stochastically perturbed system* (34) *on* $t \geq -\tau$ *and the solution is pth moment asymptotically stable.*

Proof Since Assumption 4.1 holds, $\kappa_i > \overline{\kappa}_i$ for all $i \in S$, and there exists a series of nonnegative numbers $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i$ such that condition (36) holds, so all conditions of Theorem 4.3 are satisfied for system (34). Hence, by Theorem 4.3, all conclusions of Theorem 4.4 hold. □

Remark 5 Beside of the robustness result of exponential stability, Hu *et al.* [1] obtain the robustness result of H_∞-stability ($\int_0^\infty E|\boldsymbol{x}(s)|^q\,\mathrm{d}s < \infty$). Compared with the robustness result of H_∞-stability in [1], our Theorems 4.1 and 4.4 further get some new robustness results of the moment asymptotical stability by Lemma 3.3.

Then in the case of $\xi_{i1}(t) = \kappa_{i0}e^{-\alpha_{i1}t}$, $\xi_{i2}(t) = \lambda_{i0}e^{-\alpha_{i2}t}$ in Assumption 2.2, similar to Theorem 4.3, we propose another new easily verifiable condition for the exponential stability of the solution of system (34).

Theorem 4.5 *If Assumptions* 2.1, 2.2, 4.1 *hold,* $\xi_{i1}(t) = \kappa_{i0}e^{-\alpha_{i1}t}$, $\xi_{i2}(t) = \lambda_{i0}e^{-\alpha_{i2}t}$ *in Assumption* 2.2, $\kappa_i > \overline{\kappa}_i$ *for all* $i \in S$ *and the condition* (36) *holds, then for any* $r \in (0,q]$ *and any initial data* $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$, $i_0 \in S$, *there is a unique global solution* $\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)$ *of system* (34) *on* $t \geq -\tau$, *and the solution is almost surely exponentially stable and rth moment exponentially stable, i.e.,*

$$\limsup_{t\to\infty}\frac{\log(|\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)|)}{t} \leq -\frac{\bar{\varepsilon}_q}{q} \quad \textit{a.s.}, \qquad \limsup_{t\to\infty}\frac{\log(E|\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)|^r)}{t} \leq -\frac{r}{q}\bar{\varepsilon}_q, \tag{39}$$

where $\kappa_{i0} \geq 0, \lambda_{i0} \geq 0, \alpha_{i1} > 0, \alpha_{i2} > 0$ *for all* $i \in S$, $\bar{\varepsilon}_q$ *is a positive constant which depends on* q *but not on* ζ.

Proof We give an outline focusing on the innovative parts of the proof, and the details of the proof are similar to those of Theorem 3.2.

Since $\bar{\mathscr{B}}(q)$ is a nonsingular M-matrix from Assumption 4.1, using the same technique as applied in the proof of Theorem 3.1, we claim that there exists at least a sufficiently small positive constant $\varepsilon > 0$ such that $(\frac{5\Theta_1'}{n_1-1})^{\frac{n_1-1}{n_1-2n_2+1}}(\frac{n_2-1}{\Theta_2'})^{\frac{2n_2-2}{n_1-2n_2+1}}(n_1-2n_2+1)+5\Theta_3' < 1$ and $\bar{\mathscr{B}}'(q,\varepsilon) = \operatorname{diag}(\bar{\Delta}_1', \bar{\Delta}_2', \ldots, \bar{\Delta}_N') - \frac{(5q-3)(q-2)+2\varepsilon}{q} I_N - \frac{2}{q}\Gamma$ is still a nonsingular M-matrix, where $\Theta_1' = \max_{i\in S} c_{qi}'(q-1)(\lambda_i^2 + \bar{\lambda}_i^2 e^{\varepsilon\tau})$, $\Theta_2' = \min_{i\in S} c_{qi}'(\kappa_i - \bar{\kappa}_i e^{\varepsilon\tau})$, $\Theta_3' = \max_{i\in S} c_{qi}'(q-1)(\hat{\lambda}_i^2 + \underline{\lambda}_i^2 e^{\varepsilon\tau})$ and $\bar{\Delta}_i' = 2(\hat{\kappa}_i - \underline{\kappa}_i e^{\varepsilon\tau}), i \in S$. Using Lemma 3.1, we have $c_q' = (c_{q1}', c_{q2}', \ldots, c_{qN}')^T = \bar{\mathscr{B}}'(q,\varepsilon)^{-1}\vec{1}_N \gg 0$. So we can define $\bar{\varepsilon}_q = \sup\{\varepsilon > 0 : \varepsilon \in \bar{\mathscr{B}}\}$, where $\bar{\mathscr{B}} = \{\varepsilon : \bar{\mathscr{B}}'(q,\varepsilon)$ is a nonsingular M-matrix, $(\frac{5\Theta_1'}{n_1-1})^{\frac{n_1-1}{n_1-2n_2+1}}(\frac{n_2-1}{\Theta_2'})^{\frac{2n_2-2}{n_1-2n_2+1}}(n_1-2n_2+1)+5\Theta_3' < 1, 0 < \varepsilon < \min_{i\in S}\{\frac{q\alpha_{i1}}{2}, q\alpha_{i2}\}\}$.

Applying the generalized Itô formula to $V(\mathbf{x},t,i) = c_{qi}' e^{\varepsilon t}|\mathbf{x}|^q$, $\varepsilon \in \bar{\mathscr{B}}$, we find

$$
\begin{aligned}
&\mathscr{L}V(\mathbf{x}(t),t,i) \\
&\quad\le c_{qi}' e^{\varepsilon t}\Bigl[\varepsilon|\mathbf{x}(t)|^q + \frac{q}{2}|\mathbf{x}(t)|^{q-2}\bigl(2\mathbf{x}^T(t)\mathbf{f}(\mathbf{x}_t,t,r(t)) + (q-1)|\mathbf{g}(\mathbf{x}_t,t,r(t))|^2\bigr)\Bigr] \\
&\qquad + \sum_{j=1}^N \gamma_{ij} c_{qj}' |\mathbf{x}(t)|^q e^{\varepsilon t} \\
&\quad\le \frac{q}{2} c_{qi}' e^{\varepsilon t}\biggl[\frac{2}{q}\varepsilon|\mathbf{x}(t)|^q + |\mathbf{x}(t)|^{q-2}\biggl[-2\kappa_i|\mathbf{x}(t)|^{n_1+1} + 2\bar{\kappa}_i\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{n_1+1}\,\mathrm{d}\eta_1(\theta) \\
&\qquad - 2\hat{\kappa}_i|\mathbf{x}(t)|^2 + 2\underline{\kappa}_i\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2\,\mathrm{d}\eta_2(\theta) + 2\kappa_{i0}e^{-\alpha_{i1}t} + 5(q-1)\biggl(\lambda_i^2|\mathbf{x}(t)|^{2n_2} \\
&\qquad + \bar{\lambda}_i^2\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^{2n_2}\,\mathrm{d}\eta_3(\theta) + \hat{\lambda}_i^2|\mathbf{x}(t)|^2 + \underline{\lambda}_i^2\int_{-\tau}^0 |\mathbf{x}(t+\theta)|^2\,\mathrm{d}\eta_4(\theta) + \lambda_{i0}e^{-2\alpha_{i2}t}\biggr)\biggr]\biggr] \\
&\qquad + \sum_{j=1}^N \gamma_{ij} c_{qj}' |\mathbf{x}(t)|^q e^{\varepsilon t} + 2c_{qi}'\kappa_{i0}^{\frac{p}{2}} e^{(\varepsilon - \frac{q}{2}\alpha_{i1})t} + 5c_{qi}'(q-1)\lambda_{i0}^p e^{(\varepsilon - p\alpha_{i2})t} \\
&\quad\le \frac{q}{2} e^{\varepsilon t}\biggl[-2c_{qi}'\bigl(\kappa_i - \bar{\kappa}_i e^{\varepsilon\tau}\bigr)|\mathbf{x}(t)|^{q+n_1-1} + 5c_{qi}'(q-1)\bigl(\lambda_i^2 + \bar{\lambda}_i^2 e^{\varepsilon\tau}\bigr)|\mathbf{x}(t)|^{q+2n_2-2} \\
&\qquad - \biggl(c_{qi}'\biggl(2\hat{\kappa}_i - 2\underline{\kappa}_i e^{\varepsilon\tau} - 5(q-1)\bigl(\hat{\lambda}_i^2 + \underline{\lambda}_i^2 e^{\varepsilon\tau}\bigr) - \frac{(5q-3)(q-2)+2\varepsilon}{q}\biggr) \\
&\qquad - \frac{2}{q}\sum_{j=1}^N \gamma_{ij} c_{qj}'\biggr)\cdot|\mathbf{x}(t)|^q\biggr] \\
&\qquad + 2c_{qi}'\underline{\kappa}_i e^{\varepsilon t}\bar{J}_2 + qc_{qi}'\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}e^{\varepsilon t}\bar{J}_1 + \frac{q}{2}c_{qi}' 5(q-1)\bar{\lambda}_i^2\frac{2n_2}{q+2n_2-2}e^{\varepsilon t}\bar{J}_3 \\
&\qquad + 5c_{qi}'(q-1)\underline{\lambda}_i^2 e^{\varepsilon t}\bar{J}_4 + 2c_{qi}'\kappa_{i0}^{\frac{p}{2}} e^{(\varepsilon - \frac{q}{2}\alpha_{i1})t} + 5c_{qi}'(q-1)\lambda_{i0}^p e^{(\varepsilon - p\alpha_{i2})t},
\end{aligned}
\tag{40}
$$

where $\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{J}_4$ are the same as defined in the proof of Theorem 3.2.

Let $\overline{H}_i(|\mathbf{x}(t)|) = 2c_{qi}'(\kappa_i - \bar{\kappa}_i e^{\varepsilon\tau})|\mathbf{x}(t)|^{n_1-1} - 5c_{qi}'(q-1)(\lambda_i^2 + \bar{\lambda}_i^2 e^{\varepsilon\tau})|\mathbf{x}(t)|^{2n_2-2} + (c_{qi}'(2\hat{\kappa}_i - 2\underline{\kappa}_i e^{\varepsilon\tau} - 5(q-1)(\hat{\lambda}_i^2 + \underline{\lambda}_i^2 e^{\varepsilon\tau}) - \frac{(5q-3)(q-2)+2\varepsilon}{q}) - \frac{2}{q}\sum_{j=1}^N \gamma_{ij}c_{qj}')$. From the definition of c_q', we have $\bar{\mathscr{B}}'(q,\varepsilon)c_q' = \vec{1}_N$, which can be rewritten in a component-wise form: $c_{qi}'(2\hat{\kappa}_i - 2\underline{\kappa}_i e^{\varepsilon\tau} -$

$\frac{(5q-3)(q-2)+2\varepsilon}{q}) - \frac{2}{q}\sum_{j=1}^{N}\gamma_{ij}c'_{qj} = 1$. So $\overline{H}_i(|\boldsymbol{x}(t)|) = 2c'_{qi}(\kappa_i - \bar{\kappa}_i e^{\varepsilon\tau})|\boldsymbol{x}(t)|^{n_1-1} - 5c'_{qi}(q-1)(\lambda_i^2 + \bar{\lambda}_i^2 e^{\varepsilon\tau})|\boldsymbol{x}(t)|^{2n_2-2} + (1 - 5c'_{qi}(q-1)(\hat{\lambda}_i^2 + \underline{\lambda}_i^2 e^{\varepsilon\tau}))$. There exists

$$\bigl|\boldsymbol{x}(t)\bigr|^{**} = \left(\frac{5c'_{qi}(q-1)(\lambda_i^2 + \bar{\lambda}_i^2 e^{\varepsilon\tau})(2n_2-2)}{2c'_{qi}(\kappa_i - \bar{\kappa}_i e^{\varepsilon\tau})(n_1-1)}\right)^{\frac{1}{n_1+1-2n_2}} > 0$$

such that $\overline{H}'_i(|\boldsymbol{x}(t)|^{**}) = 0$ and

$$\begin{aligned}\overline{H}_i\bigl(\bigl|\boldsymbol{x}(t)\bigr|^{**}\bigr) = 1 &- \left(\frac{5c'_{qi}(q-1)(\lambda_i^2 + \overline{\lambda}_i^2 e^{\varepsilon\tau})}{n_1-1}\right)^{\frac{n_1-1}{n_1-2n_2+1}} \\ &\times \left(\frac{n_2-1}{c'_{qi}(\kappa_i - \overline{\kappa}_i e^{\varepsilon\tau})}\right)^{\frac{2n_2-2}{n_1-2n_2+1}}(n_1 - 2n_2 + 1) - 5c'_{qi}(q-1)\bigl(\hat{\lambda}_i^2 + \underline{\lambda}_i^2 e^{\varepsilon\tau}\bigr) > 0.\end{aligned}$$

We therefore have, for any $t \ge 0$, $\overline{H}_i(|\boldsymbol{x}(t)|) \ge \overline{H}_i(0) \wedge \overline{H}_i(|\boldsymbol{x}(t)|^{**}) := \bar{b}'_i > 0$. Then (40) can be rewritten as

$$\begin{aligned}&\mathscr{L}V\bigl(\boldsymbol{x}(t),t,i\bigr) \\ &\quad\le -\frac{q}{2}\bar{b}'_i e^{\varepsilon t}\bigl|\boldsymbol{x}(t)\bigr|^q + 2c'_{qi}\underline{\kappa}_i e^{\varepsilon t}\bar{J}_2 + qc'_{qi}\bar{\kappa}_i\frac{n_1+1}{p+n_1-1}e^{\varepsilon t}\bar{J}_1 + 5c'_{qi}(q-1)\underline{\lambda}_i^2 e^{\varepsilon t}\bar{J}_4 \\ &\qquad + \frac{q}{2}c'_{qi}5(q-1)\bar{\lambda}_i^2\frac{2n_2}{q+2n_2-2}e^{\varepsilon t}\bar{J}_3 + 2c'_{qi}\kappa_{i0}^{\frac{p}{2}}e^{(\varepsilon-\frac{q}{2}\alpha_{i1})t} + 5c'_{qi}(q-1)\lambda_{i0}^{p}e^{(\varepsilon-p\alpha_{i2})t}.\end{aligned}$$

Following the proof of the remainder of Theorem 3.2, we get the assertions (39). □

Obviously, Theorem 4.4 is a simple application of Theorem 4.3. Similarly, we get another new robustness result of exponential stability directly from Theorem 4.5.

Theorem 4.6 *If Assumption* 4.1 *holds*, $\xi_{i1}(t) = \kappa_{i0}e^{-\alpha_{i1}t}$ *in condition* (11), $\kappa_i > \overline{\kappa}_i$ *for all* $i \in S$, *and there exist a series of nonnegative numbers* $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i, \lambda_{i0}$ *and positive numbers* α_{i2} *such that* $\xi_{i2}(t) = \lambda_{i0}e^{-\alpha_{i2}t}$ *in condition* (12) *and condition* (36) *holds, then for any* $r \in (0,q]$, *any initial data* $\boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$ *and* $i_0 \in S$, *there is a unique global solution* $\boldsymbol{x}(t,\boldsymbol{\zeta},i_0)$ *of the stochastically perturbed system* (34) *on* $t \ge -\tau$ *and the solution is almost surely exponentially stable and rth moment exponentially stable, where* $\kappa_{i0} \ge 0$, $\alpha_{i1} > 0$ *for all* $i \in S$.

Proof Since Assumption 4.1 holds, $\xi_{i1}(t) = \kappa_{i0}e^{-\alpha_{i1}t}$ in condition (11), $\kappa_i > \overline{\kappa}_i$ for all $i \in S$, and there exist a series of nonnegative numbers $\lambda_i, \bar{\lambda}_i, \hat{\lambda}_i, \underline{\lambda}_i, \lambda_{i0}$ and positive numbers α_{i2} such that $\xi_{i2}(t) = \lambda_{i0}e^{-\alpha_{i2}t}$ in condition (12) and condition (36) holds, all conditions of Theorem 4.5 are satisfied for system (34). Hence, by Theorem 4.5, all conclusions of Theorem 4.6 hold. □

Remark 6 Compared with [1], our Theorems 4.1, 4.2, 4.4, and 4.6 about robust stability are different from their conclusions in [1].

The conclusions in [1] illustrate that a stable determined nonlinear delay differential equation with Markovian switching with drift coefficient $\boldsymbol{f}$ can tolerate a stochastic perturbation $\mathbf{g}\dot{B}(t)$ requiring $\boldsymbol{x}^T\boldsymbol{f}$ and $|\mathbf{g}|^2$ to be bounded by polynomials with the same orders.

Theorems 4.1, 4.2, 4.4, and 4.6 in this paper illustrate that a stable determined nonlinear FDEwMS with drift coefficient $\boldsymbol{f}$ can tolerate a stochastic perturbation $\mathbf{g}\dot{B}(t)$ requiring $\boldsymbol{x}^T\boldsymbol{f}$ and $|\mathbf{g}|^2$ to be controlled by polynomials with different orders.

5 Example

In this section, we shall discuss some examples to illustrate our theorems.

Example 5.1 Let us consider a one-dimensional SFDEwMS as follows:

$$\begin{aligned} \mathrm{d}\boldsymbol{x}(t) &= \bigl(a\bigl(r(t)\bigr)\boldsymbol{x}^5(t) + b\bigl(r(t)\bigr)\boldsymbol{x}(t) + 2D_1^3(\boldsymbol{x}_t) + 2D_2(\boldsymbol{x}_t) + \xi(t)\bigr)\,\mathrm{d}t \\ &\quad + \frac{1}{2}\bigl(\boldsymbol{x}^2(t) + \boldsymbol{x}(t) + D_3^2(\boldsymbol{x}_t) + D_4(\boldsymbol{x}_t) + \xi(t)\bigr)\,\mathrm{d}B(t), \end{aligned} \tag{41}$$

with initial data $\{\boldsymbol{x}(\theta) : -\tau \le \theta \le 0\} = \boldsymbol{\zeta} \in C^b_{\mathscr{F}_0}([-\tau,0];R^n), i_0 \in S$, where $B(t)$ is a scalar Brownian motion and $r(t)$ is a right-continuous Markovian chain taking values in $S = \{1,2\}$ with generator $\Gamma = \bigl(\begin{smallmatrix} -1 & 1 \\ \frac{15}{2} & -\frac{15}{2} \end{smallmatrix}\bigr)$, $a(1) = -4, a(2) = -5, b(1) = -10, b(2) = -3$, D_j are bounded linear operators from $C([-\tau,0];R)$ to R satisfying $|D_j(\boldsymbol{x}_t)| \le \int_{-\tau}^0 |\boldsymbol{x}(t+\theta)|\,\mathrm{d}\eta_j(\theta)$, where $\eta_j(.)$ are probability measures on $[-\tau,0]$, $j = 1,2,3,4$.

If the function $\xi(t)$ is defined by

$$\xi(t) = \begin{cases} 1 - 4^n|t-n|, & \text{when } t \in [n - \frac{1}{4^n}, n + \frac{1}{4^n}], n = 1,2,3,\ldots, \\ 0, & \text{otherwise}, \end{cases}$$

then it is easy to show that $\xi(t)$ is bounded, and $\int_0^{+\infty} \xi(t)\,\mathrm{d}t = 1$. And we compute

$$\begin{aligned} &\boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{x}_t,t,1) \\ &\quad \le -4\bigl|\boldsymbol{x}(t)\bigr|^6 + \int_{-\tau}^0 \bigl|\boldsymbol{x}(t+\theta)\bigr|^6\,\mathrm{d}\eta_1(\theta) - \frac{15}{2}\bigl|\boldsymbol{x}(t)\bigr|^2 + \int_{-\tau}^0 \bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,\mathrm{d}\eta_2(\theta) + \frac{1}{2}\xi^2(t), \\ &\boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{x}_t,t,2) \\ &\quad \le -5\bigl|\boldsymbol{x}(t)\bigr|^6 + \int_{-\tau}^0 \bigl|\boldsymbol{x}(t+\theta)\bigr|^6\,\mathrm{d}\eta_1(\theta) - \frac{1}{2}\bigl|\boldsymbol{x}(t)\bigr|^2 + \int_{-\tau}^0 \bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,\mathrm{d}\eta_2(\theta) + \frac{1}{2}\xi^2(t), \\ &\bigl|\mathbf{g}(\boldsymbol{x}_t,t,1)\bigr| = \bigl|\mathbf{g}(\boldsymbol{x}_t,t,2)\bigr| \\ &\qquad \le \frac{1}{2}\biggl(\bigl|\boldsymbol{x}(\boldsymbol{t})\bigr|^2 + \int_{-\tau}^0 \bigl|\boldsymbol{x}(t+\theta)\bigr|^2\,\mathrm{d}\eta_3(\theta) + \bigl|\boldsymbol{x}(t)\bigr| + \int_{-\tau}^0 \bigl|\boldsymbol{x}(t+\theta)\bigr|\,\mathrm{d}\eta_4(\theta) + \xi(t)\biggr). \end{aligned}$$

And the parameters used in Assumption 2.2 are $\kappa_1 = 4, \bar{\kappa}_1 = 1, \hat{\kappa}_1 = \frac{15}{2}, \underline{\kappa}_1 = 1, \kappa_2 = 5, \bar{\kappa}_2 = 1, \hat{\kappa}_2 = \frac{1}{2}, \underline{\kappa}_2 = 1, \gamma_1 = \gamma_2 = \bar{\gamma}_1 = \bar{\gamma}_2 = \hat{\gamma}_1 = \hat{\gamma}_2 = \underline{\gamma}_1 = \underline{\gamma}_2 = \frac{1}{2}, n_1 = 5, n_2 = 2$. So, we obtain $L = (n_1 - 2n_2 + 1)(2n_2 - 2)^{\frac{2n_2-2}{n_1-2n_2+1}}(n_1-1)^{\frac{1-n_1}{n_1-2n_2+1}} = (5-4+1)(4-2)^{\frac{2}{2}}(5-1)^{\frac{-4}{2}} = \frac{1}{4}$. Moreover, the matrix defined in Theorem 3.1 becomes $\mathscr{B}(p) = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_N) - \frac{2}{p}\Gamma = \bigl(\begin{smallmatrix} 13 - \frac{3}{2} - \frac{6(p-1)}{7-p} + \frac{2}{p} & -\frac{2}{p} \\ -\frac{15}{p} & -1-2-\frac{8(p-1)}{9-p} + \frac{15}{p} \end{smallmatrix}\bigr)$. By Lemma 3.1, $\mathscr{B}(p)$ is a nonsingular M-matrix if and only if all the leading principal minors of $\mathscr{B}(p)$ are positive, that is

$$13 - \frac{3}{2} - \frac{6(p-1)}{7-p} + \frac{2}{p} > 0, \qquad \biggl(13 - \frac{3}{2} - \frac{6(p-1)}{7-p} + \frac{2}{p}\biggr)\biggl(-1-2-\frac{8(p-1)}{9-p} + \frac{15}{p}\biggr) - \frac{30}{p^2} > 0.$$

Simple computations show that $\mathscr{B}(p)$ is a nonsingular M-matrix when $p \in (0, 2.694)$. So by Theorem 3.1, we claim that for any $p \in (0, 2.694)$, any initial data $\boldsymbol{\zeta}$ and $i_0 \in S$, the global solution of system (41) is pth moment asymptotically stable.

Example 5.2 Let us recall the generalized logistic differential system with Markovian switching (8) with a nonlinear stochastic perturbation in Section 1,

$$\begin{aligned} \mathrm{d}\boldsymbol{x}(t) &= \boldsymbol{x}(t)\left[\overline{c\big(r(t)\big)} - a\big(r(t)\big)\big|\boldsymbol{x}(t)\big|^2 + b\big(r(t)\big)\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^2\,d\eta(\theta)\right]\mathrm{d}t \\ &\quad + \boldsymbol{x}(t)\big|\rho\big(r(t)\big)\big|\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^{\frac{1}{3}}\,d\eta(\theta)\,\mathrm{d}B(t), \end{aligned}$$

with initial data $\boldsymbol{x}(0) \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$, where the switching is governed by a Markovian chain $r(t)$ on the state space $S = \{1, 2\}$ with generator $\Gamma = \left(\begin{smallmatrix} -1 & 1 \\ 5 & -5 \end{smallmatrix}\right)$, $\overline{c(1)} = -2, \overline{c(2)} = 1, a(1) = a(2) = 2, b(1) = b(2) = \frac{1}{2}$. Then we have

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{x}_t, t, 1) &\le -\frac{7}{4}\big|\boldsymbol{x}(t)\big|^4 + \frac{1}{4}\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^4\,\mathrm{d}\eta(\theta) - 2\big|\boldsymbol{x}(t)\big|^2, \\ \boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{x}_t, t, 2) &\le -\frac{7}{4}\big|\boldsymbol{x}(t)\big|^4 + \frac{1}{4}\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^4\,\mathrm{d}\eta(\theta) + \big|\boldsymbol{x}(t)\big|^2, \\ \big|\mathbf{g}(\boldsymbol{x}_t, t, 1)\big| &\le \frac{3}{4}\big|\rho(1)\big|\big|\boldsymbol{x}(t)\big|^{\frac{4}{3}} + \frac{1}{4}\big|\rho(1)\big|\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^{\frac{4}{3}}\,\mathrm{d}\eta(\theta), \\ \big|\mathbf{g}(\boldsymbol{x}_t, t, 2)\big| &\le \frac{3}{4}\big|\rho(2)\big|\big|\boldsymbol{x}(t)\big|^{\frac{4}{3}} + \frac{1}{4}\big|\rho(2)\big|\int_{-\tau}^{0}\big|\boldsymbol{x}(t+\theta)\big|^{\frac{4}{3}}\,\mathrm{d}\eta(\theta). \end{aligned}$$

And the parameters used in Assumption 2.2 are $\kappa_1 = \kappa_2 = \frac{7}{4}, \bar{\kappa}_1 = \bar{\kappa}_2 = \frac{1}{4}, \hat{\kappa}_1 = 2, \hat{\kappa}_2 = -1, \underline{\kappa}_1 = \underline{\kappa}_2 = 0, \lambda_1 = \frac{3}{4}|\rho(1)|, \lambda_2 = \frac{3}{4}|\rho(2)|, \bar{\lambda}_1 = \frac{1}{4}|\rho(1)|, \bar{\lambda}_2 = \frac{1}{4}|\rho(2)|, \hat{\lambda}_1 = \hat{\lambda}_2 = 0, \underline{\lambda}_1 = \underline{\lambda}_2 = 0, n_1 = 3, n_2 = \frac{4}{3}$.

Hence, the problem of robust stability is to choose appropriate parameters $q, \rho(1), \rho(2)$ such that Assumption 4.1 and condition (36) hold. By simple computation, we have $\bar{\Delta}_1 = 2(\hat{\kappa}_1 - \underline{\kappa}_1) = 4, \bar{\Delta}_2 = 2(\hat{\kappa}_2 - \underline{\kappa}_2) = -2$. We choose $q = 2$ such that $\bar{\mathscr{B}}(q) = \mathrm{diag}(\bar{\Delta}_1, \bar{\Delta}_2) - \Gamma = \left(\begin{smallmatrix} 4 & 0 \\ 0 & -2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} -1 & 1 \\ 5 & -5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 5 & -1 \\ -5 & 3 \end{smallmatrix}\right)$ is a nonsingular M-matrix. And we compute $c_q = (c_{q1}, c_{q2})^T = \bar{\mathscr{B}}(q)^{-1}\vec{1} = (\frac{2}{5}, 1)^T$. Choosing $\rho(1) = \rho(2) = \frac{1}{3}$, we see that $\lambda_1 = \lambda_2 = \frac{1}{4}, \bar{\lambda}_1 = \bar{\lambda}_2 = \frac{1}{12}$, $\Theta_1 = \max_{i\in S} c_{qi}(q-1)(\lambda_i^2 + \overline{\lambda}_i^2) = \frac{5}{72}, \Theta_2 = \min_{i\in S} c_{qi}(\kappa_i - \overline{\kappa}_i) = \frac{3}{5}, \Theta_3 = \max_{i\in S} c_{qi}(q-1)(\hat{\lambda}_i^2 + \underline{\lambda}_i^2) = 0$ and condition (36) holds.

By Theorem 4.4, there is a unique global solution of the stochastically perturbed system (8) on $t \ge -\tau$ and the solution is pth moment asymptotically stable, for $p \in (0, 2]$. By Theorem 4.6, the solution is almost surely exponentially stable and rth moment exponentially stable, for $r \in (0, 2]$.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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