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Extremal solutions for singular fractional p-Laplacian differential equations with nonlinear boundary conditions

Youzheng Ding[1], Jian Yang[2] and Xiaoyan Zhang[3*]

*Correspondence:
zxysd@sdu.edu.cn
[3]School of Mathematics, Shandong University, Jinan, Shandong 250100, China
Full list of author information is available at the end of the article

Abstract

In this paper, we establish the existence and uniqueness of extremal solutions for nonlinear boundary value problems of a singular fractional p-Laplacian differential equation involving Riemann-Liouville derivatives. Our results are obtained by constructing monotone iterative sequences of upper and lower solutions and applying the comparison result. At last, we present an example to illustrate the results. The compactness of sequences is proved in the Appendix.

MSC: 26A33; 34B15; 34A08

Keywords: fractional differential equation; p-Laplacian operator; nonlinear boundary condition; upper and lower solutions; extremal solution

1 Introduction

Fractional differential equations arise in the mathematical modeling of process in physics, chemistry, aerodynamics, polymer rheology, fluid flow phenomena, wave propagation and signal theory, electrical circuits, control theory, viscoelastic materials, and so on. The fractional calculus and its various applications in many fields of science and engineering have gained much attention and developed rapidly. Consequently, fractional differential equations have been of great interest. For details, see [1–8] and the references therein.

The numerical simulation plays an essential role in the analysis of fractional differential equations, and new numerical techniques are being developed; see, for example, [9, 10]. Recently, many research papers have appeared concerning the existence of solutions for the initial and boundary value problems of fractional differential equations; see [11–17]. The monotone iterative technique, combined with the method of upper and lower solutions, is a powerful tool of obtaining the existence of solutions for fractional boundary value problems; see [18–23].

By means of the monotone iterative method, in [24], the following PBVP of fractional differential equation was considered:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} u(t)|_{t=T}, \end{cases}$$

where $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order $0 < \alpha \leq 1$. The properties of the well-known Mittag-Leffler function and the existence and uniqueness of solution for this problem were given in [24]. However, fewer papers considered p-Laplacian boundary value problems of fractional order via the upper and lower method and the monotone iterative method; see, for instance, [25–27].

In [28], the authors have discussed the following PBVP of fractional p-Laplacian equation:

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u(t))) = f(t, u(t), D^{\alpha}_{0^+}u(t)), & t \in [0,T], \\ u(t)|_{t=0} = u(t)|_{t=T}, \qquad D^{\alpha}_{0^+}u(t)|_{t=0} = D^{\alpha}_{0^+}u(t)|_{t=T}, \end{cases}$$

where $0 < \alpha, \beta \leq 1$, $D^{\alpha}_{0^+}$ is the Caputo fractional derivative, and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. By establishing the continuation theorem, which is an extension of the coincidence degree theory for linear differential operators with PBCs, the existence result of solution of the PBVP was stated under the nonlinear growth restriction of f. To the best of our knowledge, the fractional p-Laplacian differential equation with periodic boundary conditions has rarely been considered up to now.

In this paper, we investigate the existence of extremal solutions and uniqueness of solution for singular fractional p-Laplacian differential equation with general nonlinear boundary conditions

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u(t))) = f(t, u(t), D^{\alpha}_{0^+}u(t)), & t \in (0,T], \\ t^{\frac{1-\beta}{p-1}} D^{\alpha}_{0^+}u(t)|_{t=0} = t^{\frac{1-\beta}{p-1}} D^{\alpha}_{0^+}u(t)|_{t=T}, \\ g(\tilde{u}(0), \tilde{u}(T)) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order α, $\phi_p(t) = |t|^{p-2}t$ $(p > 1)$ is the p-Laplacian operator, and $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. Here $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\tilde{u}(0) = t^{1-\alpha}u(t)|_{t=0}$, and $\tilde{u}(T) = t^{1-\alpha}u(t)|_{t=T}$.

In the problem (1.1), the boundary condition $g(\tilde{u}(0), \tilde{u}(T)) = 0$ is a kind of general condition. When $g(x,y) = x \pm y$ or others, this can cover periodic, antiperiodic, or other nonlinear boundary conditions. Moreover, if $D^{\alpha}_{0^+}u(t)|_{t=0} = D^{\alpha}_{0^+}u(t)|_{t=T}$, then $t^{\frac{1-\beta}{p-1}} D^{\alpha}_{0^+}u(t)|_{t=0} = t^{\frac{1-\beta}{p-1}} D^{\alpha}_{0^+}u(t)|_{t=T}$. From this we can see that the boundary conditions in (1.1) are weaker than those in [28]. Thus, our conclusions can be more extensive. Here we not only obtain the existence of extremal solutions, but also the iterative sequences that converge to the extremal solutions.

In the previous related results on boundary value problems for p-Laplacian differential equations by means of the monotone iterative method, the monotone-type conditions for nonlinear terms f with respect to the functions u or their derivatives are usually required. However, in this paper, we only consider the functions $f + M\phi_p(D^{\alpha}_{0^+}u(t))$, not f, to satisfy the monotone-type conditions (see (H_2)).

The rest of our paper is organized as follows. In Section 2, we provide some preliminaries, the existence results for linear fractional problems with periodic boundary conditions and the comparison result. In Section 3, the existence of extremal solutions and unique solution for (1.1) are established by constructing two well-defined monotone iterative sequences of upper-lower solutions. Finally, an example is given in this section as an application of the theoretical results. Some lengthy proofs of the compactness conclusions used in Theorem 3.1 are settled in the Appendix.

2 Preliminaries and existence results for linear fractional *p*-Laplacian problems

Let $J = [0,T]$ be a compact interval on the real axis $\mathbb{R}$. It is well known that $C[0,T]$ is a Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|u\|_C = \max_{t\in[0,T]} |u(t)|$. Denote

$$C_{1-\alpha}[0,T] = \{u \in C(0,T] : t^{1-\alpha}u \in C[0,T]\}, \quad \alpha \in (0,1].$$

Then $C_{1-\alpha}[0,T]$ is also a Banach space with the norm $\|u\|_{C_{1-\alpha}} = \|t^{1-\alpha}u\|_C$ (see Lemma 2.2). It is clear that $C[0,T] := C_0[0,T] \subset C_{1-\alpha}[0,T] \subset C_{1-\beta}[0,T]$ with $\|u\|_{C_{1-\beta}} \le \|u\|_{C_{1-\alpha}} \le \|u\|_C$ for $1 \ge \alpha \ge \beta > 0$ and $C_{1-\alpha}[0,T] \subset L[0,T]$ ($L[0,T]$ is the space of Lebesgue-integrable real functions on $[0,T]$). Denote

$$C_r^\alpha[0,T] = \{u(t) \in C_{1-\alpha}[0,T] : (D_{0^+}^\alpha u)(t) \in C_r[0,T] \text{ and } t^r D_{0^+}^\alpha u(t)|_{t=0} = t^r D_{0^+}^\alpha u(t)|_{t=T}\},$$

where $r = \frac{1-\beta}{p-1}$, $p > 1$, $0 < \alpha, \beta \le 1$, and $p + \beta > 2$.

For convenience, we first present some useful definitions and fundamental facts of fractional calculus theory, some of which can be found in [1, 2].

Definition 2.1 ([1]) The Riemann-Liouville fractional integral $I_{0^+}^\alpha$ and fractional derivative $D_{0^+}^\alpha$ are defined by

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds$$

and

$$D_{0^+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds = \left(\frac{d}{dt}\right)^n (I_{0^+}^{n-\alpha} f)(t),$$

where $n - 1 < \alpha \le n$, $n \in \mathbb{N}$, provided that the integrals exist.

Lemma 2.1 ([1]) *Assume that* $f \in C(0,T] \cap L(0,T]$ *with a fractional derivative of order* α ($0 < \alpha \le 1$) *that belongs to* $C(0,T] \cap L(0,T]$. *Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha f(t) = f(t) - ct^{\alpha-1} \quad \text{for some } c \in \mathbb{R}.$$

Lemma 2.2 $(C_{1-\alpha}[0,T], \|\cdot\|_{C_{1-\alpha}})$ *and* $(C_r^\alpha[0,T], \|\cdot\|_{C_r^\alpha})$ *are Banach spaces, where*

$$\|u\|_{C_{1-\alpha}} = \|t^{1-\alpha}u\|_C, \qquad \|u\|_{C_r^\alpha} = \|u\|_{C_{1-\alpha}} + \|D_{0^+}^\alpha u\|_{C_r}.$$

Proof Let $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence in the space $(C_{1-\alpha}[0,T], \|\cdot\|_{C_{1-\alpha}})$. Then there exist $v_n \in C[0,T]$ such that $v_n(t) = t^{1-\alpha}u_n(t)$, $t \in [0,T]$, and thus $u_n(t) = t^{\alpha-1}v_n(t)$, $t \in (0,T]$. For any $\varepsilon > 0$, there exists $N > 0$ such that

$$\|u_n - u_m\|_{C_{1-\alpha}} = \|v_n - v_m\|_C < \varepsilon, \quad n, m \ge N,$$

which implies that there exists $v(t) \in C[0,T]$ such that $v_n(t) \to v(t)$, $t \in [0,T]$, and so $u_n(t) = t^{\alpha-1}v_n(t) \to t^{\alpha-1}v(t)$, $t \in (0,T]$. Let $u(t) = t^{\alpha-1}v(t)$, $t \in (0,T]$. Then $\{t^{1-\alpha}u_n(t)\}_{n=1}^{\infty}$ converges uniformly to $t^{1-\alpha}u(t)$, and we can easily find that $u \in C_{1-\alpha}[0,T]$.

Next, we shall prove that $C_r^\alpha[0,T]$ is a Banach space. It is clear that $\|\cdot\|_{C_r^\alpha}$ is a norm. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(C_r^\alpha[0,T], \|\cdot\|_{C_r^\alpha})$. Evidently, $\{u_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in the space $(C_{1-\alpha}[0,T], \|\cdot\|_{C_{1-\alpha}})$; thus, $\lim_{n\to\infty} t^{1-\alpha}u_n(t) = t^{1-\alpha}u(t)$, and $u \in C_{1-\alpha}[0,T]$. Moreover, $\{t^r(D_{0^+}^\alpha u_n)(t)\}_{n=1}^{\infty}$ converges uniformly to some $w(t) \in C[0,T]$. We need to verify that $w(t) = t^r(D_{0^+}^\alpha u)(t)$, $t \in [0,T]$.

For $\varepsilon = 1$, there exists $N > 0$ such that $|t^r(D_{0^+}^\alpha u_n)(t) - w(t)| < 1$ for any $t \in [0,T]$ and $n > N$. Denoting

$$M^* = \max\Big\{1 + \sup_{t\in[0,T]}\big|w(t)\big|, \sup_{t\in[0,T]}\big|t^r\big(D_{0^+}^\alpha u_i\big)(t)\big|, i = 1,2,\ldots,N\Big\},$$

we have

$$\left|t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-r}s^r D_{0^+}^\alpha u_n(s)\,ds\right| \le M^* t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-r}\,ds \le M^* B(\alpha, 1-r)T^{1-r},$$

where $B(\cdot,\cdot)$ is the Beta function. By Lemma 2.1 we get

$$\begin{aligned} t^{1-\alpha}u_n(t) &= t^{1-\alpha}I_{0^+}^\alpha D_{0^+}^\alpha u_n(t) + c = t^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}D_{0^+}^\alpha u_n(s)\,ds + c \\ &= t^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-r}s^r D_{0^+}^\alpha u_n(s)\,ds + c, \quad t \in [0,T]. \end{aligned} \tag{2.1}$$

Letting $n \to \infty$, by the Lebesgue dominated convergence theorem from (2.1) we derive that

$$t^{1-\alpha}u(t) = t^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-r}w(s)\,ds + c = t^{1-\alpha}I_{0^+}^\alpha\big[t^{-r}w(t)\big] + c, \quad t \in [0,T],$$

that is, $u(t) = I_{0^+}^\alpha[t^{-r}w(t)] + ct^{\alpha-1}$, $t \in (0,T]$, and so $w(t) = t^r D_{0^+}^\alpha u(t)$, $t \in (0,T]$. Obviously, $t^r D_{0^+}^\alpha u(t)|_{t=0} = t^r D_{0^+}^\alpha u(t)|_{t=T}$; hence, $\|u_n - u\|_{C_r^\alpha} \to 0$, and $u \in C_r^\alpha$. The proof of the lemma is complete. □

Lemma 2.3 ([24], Lemma 1.1) *Assume that* $0 < \beta \le 1$, $M > 0$ *is a constant*, $u(t) \in C_{1-\beta}[0,T]$, *and* $h(t) \in C_{1-\beta}[0,T]$. *Then the linear fractional periodic boundary value problem*

$$\begin{cases} D_{0^+}^\beta u(t) + Mu(t) = h(t), & t \in (0,T], \\ t^{1-\beta}u(t)|_{t=0} = t^{1-\beta}u(t)|_{t=T}, \end{cases}$$

has the following integral representation of the solution:

$$\begin{aligned} u(t) = {} & \frac{\Gamma(\beta)T^{1-\beta}t^{\beta-1}E_{\beta,\beta}(-Mt^\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(-MT^\beta)}\int_0^T (T-s)^{\beta-1}E_{\beta,\beta}\big(-M(T-s)^\beta\big)h(s)\,ds \\ & + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(-M(t-s)^\beta\big)h(s)\,ds, \end{aligned}$$

where $E_{\beta,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\beta+\beta)}$ *is the Mittag-Leffler function*; *see* [1, 15].

Remark 2.1 Note that $E_{\beta,\beta}(x) > 0$ for all $x \in \mathbb{R}$ and $E_{\beta,\beta}(x) < \frac{1}{\Gamma(\beta)}$ for $x < 0$ (see [24], Lemma 2.2), so we know that $1 - \Gamma(\beta)E_{\beta,\beta}(-MT^\beta) > 0$.

Lemma 2.4 *Assume that* $0 < \alpha, \beta \leq 1$, $M > 0$ *is a constant*, $k \in \mathbb{R}$, $u(t) \in C_r^\alpha[0,T]$, *and* $\eta(t) \in C_{1-\beta}[0,T]$. *Then the linear fractional periodic boundary value problem*

$$\begin{cases} D_{0^+}^\beta(\phi_p(D_{0^+}^\alpha u(t))) + M\phi_p(D_{0^+}^\alpha u(t)) = \eta(t), \quad t \in (0,T], \\ t^r D_{0^+}^\alpha u(t)|_{t=0} = t^r D_{0^+}^\alpha u(t)|_{t=T}, \qquad \tilde{u}(0) = k, \end{cases} \tag{2.2}$$

has a unique solution of the following integral form:

$$\begin{aligned} u(t) = kt^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \biggl[\frac{\Gamma(\beta)T^{1-\beta}s^{\beta-1}E_{\beta,\beta}(-Ms^\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(-MT^\beta)} \\ \times \int_0^T (T-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(T-s)^\beta\bigr)\eta(s)\,ds \\ + \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta(\tau)\,d\tau \biggr]. \end{aligned} \tag{2.3}$$

Proof Let $v(t) = \phi_p(D_{0^+}^\alpha u(t))$. Then $\phi_p(t^r D_{0^+}^\alpha u(t)) = t^{1-\beta}v(t)$ for $0 < t \leq T$. Thus, problem (2.2) is changed to the following fractional periodic boundary problem:

$$\begin{cases} D_{0^+}^\beta v(t) + Mv(t) = \eta(t), \quad t \in (0,T], \\ t^{1-\beta}v(t)|_{t=0} = t^{1-\beta}v(t)|_{t=T}. \end{cases}$$

By Lemma 2.3 we get

$$\begin{aligned} v(t) = \frac{\Gamma(\beta)T^{1-\beta}t^{\beta-1}E_{\beta,\beta}(-Mt^\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(-MT^\beta)} \int_0^T (T-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(T-s)^\beta\bigr)\eta(s)\,ds \\ + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(t-s)^\beta\bigr)\eta(s)\,ds. \end{aligned} \tag{2.4}$$

Hence, $v(t) \in C_{1-\beta}[0,T]$, and

$$\begin{aligned} D_{0^+}^\alpha u(t) = \phi_q \biggl[\frac{\Gamma(\beta)T^{1-\beta}t^{\beta-1}E_{\beta,\beta}(-Mt^\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(-MT^\beta)} \int_0^T (T-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(T-s)^\beta\bigr)\eta(s)\,ds \\ + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(t-s)^\beta\bigr)\eta(s)\,ds \biggr]. \end{aligned} \tag{2.5}$$

Since $v(t) \in C(0,T] \cap L(0,T]$, we have $D_{0^+}^\alpha u(t) \in C(0,T] \cap L(0,T]$. By Lemma 2.1 we arrive at

$$\begin{aligned} u(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \biggl[\frac{\Gamma(\beta)T^{1-\beta}s^{\beta-1}E_{\beta,\beta}(-Ms^\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(-MT^\beta)} \\ \times \int_0^T (T-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(T-s)^\beta\bigr)\eta(s)\,ds \\ + \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta(\tau)\,d\tau \biggr]\,ds. \end{aligned}$$

In view of $\tilde{u}(0) = k$, we find $c = k$ and

$$\begin{aligned} u(t) = kt^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \Bigg[\frac{\Gamma(\beta) T^{1-\beta} s^{\beta-1} E_{\beta,\beta}(-Ms^\beta)}{1-\Gamma(\beta) E_{\beta,\beta}(-MT^\beta)} \\ \times \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}\big(-M(T-s)^\beta\big)\eta(s)\,ds \\ + \int_0^s (s-\tau)^{\beta-1} E_{\beta,\beta}\big(-M(s-\tau)^\beta\big)\eta(\tau)\,d\tau \Bigg]\,ds. \end{aligned} \tag{2.6}$$

Conversely, it is obvious that $u(t) \in C_{1-\alpha}[0,T]$ and $\tilde{u}(0) = k$. Note that $D^\alpha_{0^+} t^{\alpha-1} = 0$ and $D^\alpha_{0^+} I^\alpha u = u$ for all $u \in C(0,T] \cap L(0,T]$. Differentiating (2.6) with order α, we get (2.5). Since $\eta(t) \in C_{1-\beta}[0,T]$, we have $\phi_p(D^\alpha_{0^+} u(t)) \in C_{1-\beta}[0,T]$ and $D^\alpha_{0^+} u(t) \in C_r[0,T]$. By (2.4) we see that

$$\begin{aligned} t^{1-\beta} v(t) = \frac{\Gamma(\beta) T^{1-\beta} E_{\beta,\beta}(-Mt^\beta)}{1-\Gamma(\beta) E_{\beta,\beta}(-MT^\beta)} \\ \times \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}\big(-M(T-s)^\beta\big)\eta(s)\,ds \\ + t^{1-\beta} \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(-M(t-s)^\beta\big)\eta(s)\,ds \end{aligned}$$

and

$$\begin{aligned} t^{1-\beta} v(t)|_{t=0} &= t^{1-\beta} v(t)|_{t=T} \\ &= \frac{T^{1-\beta}}{1-\Gamma(\beta) E_{\beta,\beta}(-MT^\beta)} \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}\big(-M(T-s)^\beta\big)\eta(s)\,ds. \end{aligned}$$

Thus, $t^r D^\alpha_{0^+} u(t)|_{t=0} = t^r D^\alpha_{0^+} u(t)|_{t=T}$. Differentiating (2.4) with order β, by Lemma 2.3 we obtain

$$D^\beta_{0^+}\big(\phi_p\big(D^\alpha_{0^+} u(t)\big)\big) + M\phi_p\big(D^\alpha_{0^+} u(t)\big) = \eta(t).$$

This completes the proof. □

Lemma 2.5 (Comparison result) *If $u(t) \in C^\alpha_r[0,T]$ and satisfies*

$$\begin{cases} D^\beta_{0^+}(\phi_p(D^\alpha_{0^+} u(t))) + M\phi_p(D^\alpha_{0^+} u(t)) \ge 0, & t \in (0,T], \\ t^r D^\alpha_{0^+} u(t)|_{t=0} = t^r D^\alpha_{0^+} u(t)|_{t=T}, \\ \tilde{u}(0) \ge 0, \end{cases}$$

where $M > 0$ is a constant, then $D^\alpha_{0^+} u(t) \ge 0$ and $u(t) \ge 0$ for $t \in (0,T]$.

Proof Let $w(t) = \phi_p(D^\alpha_{0^+} u(t))$. Then $w(t) \in C_{1-\beta}[0,T]$ and satisfies

$$\begin{cases} D^\beta_{0^+} w(t) + Mw(t) \ge 0, & t \in (0,T], \\ t^{1-\beta} w(t)|_{t=0} = t^{1-\beta} w(t)|_{t=T}, \end{cases}$$

and hence $w(t) \geq 0$ for $t \in (0, T]$ by Lemma 2.3 and Remark 2.1. Since $\phi_p(x)$ is nondecreasing, $u(t) \in C_r^\alpha[0, T]$ satisfies

$$\begin{cases} D_{0^+}^\alpha u(t) \geq 0, & t \in (0, T], \\ \tilde{u}(0) \geq 0, \end{cases}$$

and so we get $u(t) \geq 0$, $t \in (0, T]$, by (2.5) and (2.6). This lemma is complete. □

Remark 2.2 In fact, from the above proof, we can see that Lemma 2.5 unifies and includes two separate comparison results, which are applied to the next Theorem 3.1 directly.

3 Main results

We first introduce the definition of a pair of lower and upper solutions for using the monotone iterative method.

Definition 3.1 A function $u(t) \in C_r^\alpha[0, T]$ is called a lower solution of problem (1.1) if it satisfies

$$\begin{cases} D_{0^+}^\beta(\phi_p(D_{0^+}^\alpha u(t))) \leq f(t, u(t), D_{0^+}^\alpha u(t)), & t \in (0, T], \\ t^r D_{0^+}^\alpha u(t)|_{t=0} = t^r D_{0^+}^\alpha u(t)|_{t=T}, & g(\tilde{u}(0), \tilde{u}(T)) \geq 0. \end{cases} \tag{3.1}$$

A function $v(t) \in C_r^\alpha[0, T]$ is called an upper solution of problem (1.1) if it satisfies

$$\begin{cases} D_{0^+}^\beta(\phi_p(D_{0^+}^\alpha v(t))) \geq f(t, v(t), D_{0^+}^\alpha v(t)), & t \in (0, T], \\ t^r D_{0^+}^\alpha v(t)|_{t=0} = t^r D_{0^+}^\alpha v(t)|_{t=T}, & g(\tilde{v}(0), \tilde{v}(T)) \leq 0. \end{cases} \tag{3.2}$$

For our main results, we need the following assumptions.

(H_1) Assume that $u_0, v_0 \in C_r^\alpha[0, T]$ are lower and upper solutions of problem (1.1), respectively, and $u_0(t) \leq v_0(t)$, $t \in (0, T]$.

(H_2) There exists a constant $M > 0$ such that

$$f(t, u(t), D_{0^+}^\alpha u(t)) - f(t, v(t), D_{0^+}^\alpha v(t)) \leq M\bigl[\phi_p(D_{0^+}^\alpha v(t)) - \phi_p(D_{0^+}^\alpha u(t))\bigr]$$

for $u_0(t) \leq u(t) \leq v(t) \leq v_0(t)$, $D_{0^+}^\alpha u_0(t) \leq D_{0^+}^\alpha u(t) \leq D_{0^+}^\alpha v(t) \leq D_{0^+}^\alpha v_0(t)$, $t \in (0, T]$.

(H_3) There exist constants $\lambda > 0$ and $\mu \geq 0$ such that

$$g(x_1, y_1) - g(x_2, y_2) \leq \lambda(x_2 - x_1) - \mu(y_2 - y_1)$$

for $\tilde{u}_0(0) \leq x_1 \leq x_2 \leq \tilde{v}_0(0)$ and $\tilde{u}_0(T) \leq y_1 \leq y_2 \leq \tilde{v}_0(T)$.

Theorem 3.1 *Suppose that* $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, *and* ($H_1$), ($H_2$), *and* ($H_3$) *hold. Then there exist sequences* $\{u_n(t)\}, \{v_n(t)\} \subset C_r^\alpha[0, T]$ *such that* $\lim_{n\to\infty} u_n = x$, $\lim_{n\to\infty} v_n = y$ *on* $(0, T]$ *and* x, y *are minimal and maximal solutions on the interval* $[u_0, v_0]$ *of problem* (1.1), *respectively, where*

$$[u_0, v_0] = \bigl\{u \in C_r^\alpha[0, T] : u_0(t) \leq u(t) \leq v_0(t), t \in (0, T], \tilde{u}_0(0) \leq \tilde{u}(0) \leq \tilde{v}_0(0)\bigr\},$$

that is, for any solution $u \in [u_0, v_0]$,

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le x \le u \le y \le \cdots \le v_n \le \cdots \le v_1 \le v_0.$$

Moreover, we have

$$\begin{aligned} D_{0^+}^{\alpha} u_0 &\le D_{0^+}^{\alpha} u_1 \le \cdots \le D_{0^+}^{\alpha} u_n \le \cdots \le D_{0^+}^{\alpha} x \\ &\le D_{0^+}^{\alpha} u \le D_{0^+}^{\alpha} y \le \cdots \le D_{0^+}^{\alpha} v_n \le \cdots \le D_{0^+}^{\alpha} v_1 \le D_{0^+}^{\alpha} v_0. \end{aligned}$$

Proof Let $F(u(t)) := f(t, u(t), D_{0^+}^{\alpha} u(t))$. For $n = 1, 2, \ldots$, we define

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha} u_n(t))) + M\phi_p(D_{0^+}^{\alpha} u_n(t)) \\ \quad = F(u_{n-1}(t)) + M\phi_p(D_{0^+}^{\alpha} u_{n-1}(t)), \quad t \in (0, T], \\ t^r D_{0^+}^{\alpha} u_n(t)|_{t=0} = t^r D_{0^+}^{\alpha} u_n(t)|_{t=T}, \\ \tilde{u}_n(0) = \tilde{u}_{n-1}(0) + \frac{1}{\lambda} g(\tilde{u}_{n-1}(0), \tilde{u}_{n-1}(T)), \end{cases} \tag{3.3}$$

and

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha} v_n(t))) + M\phi_p(D_{0^+}^{\alpha} v_n(t)) \\ \quad = F(v_{n-1}(t)) + M\phi_p(D_{0^+}^{\alpha} v_{n-1}(t)), \quad t \in (0, T], \\ t^r D_{0^+}^{\alpha} v_n(t)|_{t=0} = t^r D_{0^+}^{\alpha} v_n(t)|_{t=T}, \\ \tilde{v}_n(0) = \tilde{v}_{n-1}(0) + \frac{1}{\lambda} g(\tilde{v}_{n-1}(0), \tilde{v}_{n-1}(T)). \end{cases} \tag{3.4}$$

Since $u_0, v_0 \in C_r^{\alpha}[0, T]$, we know that $D_{0^+}^{\alpha} u_0(t), D_{0^+}^{\alpha} v_0(t) \in C_r[0, T]$, and so $F(u_0(t)) + \phi_p(D_{0^+}^{\alpha} u_0(t)), F(v_0(t)) + \phi_p(D_{0^+}^{\alpha} v_0(t)) \in C_{1-\beta}[0, T]$. In view of Lemma 2.4, the functions u_1 and v_1 are well defined in the space $C_r^{\alpha}[0, T]$. By induction, we can infer that u_n and v_n are well defined in the space $C_r^{\alpha}[0, T]$.

First, we prove that $u_0(t) \le u_1(t) \le v_1(t) \le v_0(t)$, $t \in (0, T]$, and $D_{0^+}^{\alpha} u_0(t) \le D_{0^+}^{\alpha} u_1(t) \le D_{0^+}^{\alpha} v_1(t) \le D_{0^+}^{\alpha} v_0(t)$, $t \in (0, T]$. Let $\delta(t) := \phi_p(D_{0^+}^{\alpha} u_1(t)) - \phi_p(D_{0^+}^{\alpha} u_0(t))$. The definition of u_1 and the assumption that u_0 is a lower solution imply that

$$D_{0^+}^{\beta} \delta(t) + M\delta(t) = F\bigl(u_0(t)\bigr) - D_{0^+}^{\beta}\bigl(\phi_p\bigl(D_{0^+}^{\alpha} u_0(t)\bigr)\bigr) \ge 0$$

and $t^{1-\beta}\delta(t)|_{t=0} = t^{1-\beta}\delta(t)|_{t=T}$, $\tilde{u}_1(0) - \tilde{u}_0(0) = \frac{1}{\lambda} g(\tilde{u}_0(0), \tilde{u}_0(T)) \ge 0$. Thus, we have $D_{0^+}^{\alpha} u_0(t) \le D_{0^+}^{\alpha} u_1(t)$ and $u_1(t) \ge u_0(t)$, $t \in (0, T]$ by Lemma 2.5.

Using a similar method, we can show that $v_1(t) \le v_0(t)$ and $D_{0^+}^{\alpha} v_1(t) \le D_{0^+}^{\alpha} v_0(t)$ for all $t \in (0, T]$. Now, we put $\xi(t) = \phi_p(D_{0^+}^{\alpha} v_1(t)) - \phi_p(D_{0^+}^{\alpha} u_1(t))$. From (3.3), (3.4), and (H_2) we get

$$D_{0^+}^{\beta} \xi(t) + M\xi(t) = F\bigl(v_0(t)\bigr) - F\bigl(u_0(t)\bigr) + M\bigl[\phi_p\bigl(D_{0^+}^{\alpha} v_0(t)\bigr) - \phi_p\bigl(D_{0^+}^{\alpha} u_0(t)\bigr)\bigr] \ge 0 \tag{3.5}$$

and

$$t^{1-\beta}\xi(t)|_{t=0} = t^{1-\beta}\xi(t)|_{t=T}. \tag{3.6}$$

We find, by (H_3) and (H_1), that

$$\begin{aligned}\tilde{v}_1(0)-\tilde{u}_1(0) &= \tilde{v}_0(0)+\frac{1}{\lambda}g\big(\tilde{v}_0(0),\tilde{v}_0(T)\big)-\left[\tilde{u}_0(0)+\frac{1}{\lambda}g\big(\tilde{u}_0(0),\tilde{u}_0(T)\big)\right]\\ &= \frac{1}{\lambda}\big[\lambda\big(\tilde{v}_0(0)-\tilde{u}_0(0)\big)+g\big(\tilde{v}_0(0),\tilde{v}_0(T)\big)-g\big(\tilde{u}_0(0),\tilde{u}_0(T)\big)\big]\\ &\geq \frac{\mu}{\lambda}\big(\tilde{v}_0(T)-\tilde{u}_0(T)\big)\geq 0. \end{aligned} \tag{3.7}$$

It follows from (3.5)-(3.7) and Lemma 2.5 that $D^{\alpha}_{0^+}v_1(t)\geq D^{\alpha}_{0^+}u_1(t)$ and $v_1(t)\geq u_1(t)$, $t\in(0,T]$.

Next, we show that u_1 and v_1 are lower and upper solutions of problem (1.1), respectively. From (3.3) and assumptions (H_2) and (H_3) we have

$$\begin{aligned}D^{\beta}_{0^+}\big(\phi_p\big(D^{\alpha}_{0^+}u_1(t)\big)\big) &= F\big(u_0(t)\big)-F\big(u_1(t)\big)+F\big(u_1(t)\big)\\ &\quad -M\big[\phi_p\big(D^{\alpha}_{0^+}u_1(t)\big)-\phi_p\big(D^{\alpha}_{0^+}u_0(t)\big)\big]\\ &\leq F\big(u_1(t)\big)\end{aligned}$$

and

$$\begin{aligned}0 &= g\big(\tilde{u}_0(0),\tilde{u}_0(T)\big)-g\big(\tilde{u}_1(0),\tilde{u}_1(T)\big)+g\big(\tilde{u}_1(0),\tilde{u}_1(T)\big)-\lambda\big[\tilde{u}_1(0)-\tilde{u}_0(0)\big]\\ &\leq g\big(\tilde{u}_1(0),\tilde{u}_1(T)\big)-\mu\big(\tilde{u}_1(T)-\tilde{u}_0(T)\big).\end{aligned}$$

Since $\tilde{u}_1(T)\geq\tilde{u}_0(T)$, the last inequality implies $g(\tilde{u}_1(0),\tilde{u}_1(T))\geq 0$. This proves that u_1 is a lower solution of problem (1.1). In the same way, we can show that v_1 is an upper solution of (1.1).

Using mathematical induction, we have

$$\begin{aligned}&u_0(t)\leq u_1(t)\leq\cdots\leq u_n(t)\leq u_{n+1}(t)\leq v_{n+1}(t)\leq v_n(t)\leq\cdots\leq v_1(t)\leq v_0(t),\\ &D^{\alpha}_{0^+}u_0\leq D^{\alpha}_{0^+}u_1\leq\cdots\leq D^{\alpha}_{0^+}u_n\leq D^{\alpha}_{0^+}u_{n+1}\\ &\qquad\leq D^{\alpha}_{0^+}v_{n+1}\leq D^{\alpha}_{0^+}v_n\leq\cdots\leq D^{\alpha}_{0^+}v_1\leq D^{\alpha}_{0^+}v_0\end{aligned} \tag{3.8}$$

for $t\in(0,T]$ and $n=1,2,3,\ldots$.

The sequences $\{t^{1-\alpha}u_n\}$ and $\{t^rD^{\alpha}_{0^+}u_n\}$ are uniformly bounded and equicontinuous (see Lemma A.1 in the Appendix). Similarly, we can prove that the sequences $\{t^{1-\alpha}v_n\}$ and $\{t^rD^{\alpha}_{0^+}v_n\}$ are uniformly bounded and equicontinuous. The Arzelà-Ascoli theorem guarantees that $\{t^{1-\alpha}u_n\}$ and $\{t^{1-\alpha}v_n\}$ converge to $t^{1-\alpha}x(t)$ and $t^{1-\alpha}y(t)$ uniformly on $[0,T]$, respectively, and $\{t^rD^{\alpha}_{0^+}u_n\}$ and $\{t^rD^{\alpha}_{0^+}v_n\}$ converge to $\{t^rD^{\alpha}_{0^+}x(t)\}$ and $\{t^rD^{\alpha}_{0^+}y(t)\}$ uniformly on $[0,T]$, respectively. Therefore, $\|u_n-x\|_{C^{\alpha}_r}\to 0$, $\|v_n-y\|_{C^{\alpha}_r}\to 0$ ($n\to\infty$).

By the integral representation (2.3) for the linear fractional problem, the solution $u_n(t)$ of problem (3.3) can be expressed as

$$\begin{aligned}u_n(t) &= t^{\alpha-1}\left[\tilde{u}_{n-1}(0)+\frac{1}{\lambda}g\big(\tilde{u}_{n-1}(0),\tilde{u}_{n-1}(T)\big)\right]\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\Big[M_0s^{\beta-1}E_{\beta,\beta}\big(-Ms^{\beta}\big)\end{aligned}$$

$$\times \int_0^T (T-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(T-s)^\beta\bigr)\eta_{n-1}(s)\,ds$$
$$+ \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta_{n-1}(\tau)\,d\tau\Biggr], \quad t\in(0,T],$$

where $\eta_{n-1}(s) = F(u_{n-1}(s)) + M\phi_p(D^\alpha_{0^+}u_{n-1}(s))$ and

$$M_0 := \frac{\Gamma(\beta)T^{1-\beta}}{1-\Gamma(\beta)E_{\beta,\beta}(-MT^\beta)}. \tag{3.9}$$

By the assumption on f, applying the dominated convergence theorem, we get that $x(t)$ satisfies the following integral equation:

$$x(t) = t^{\alpha-1}\tilde{x}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\Biggl[M_0 s^{\beta-1}E_{\beta,\beta}\bigl(-Ms^\beta\bigr)$$
$$\times \int_0^T (T-s)^{\beta-1}E_{\beta,\beta}\bigl(-M(T-s)^\beta\bigr)\eta(s)\,ds$$
$$+ \int_0^s (s-\tau)^{\beta-1}E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta(\tau)\,d\tau\Biggr], \quad t\in(0,T],$$

where $\eta(s) = F(x(s)) + M\phi_p(D^\alpha_{0^+}x(s))$. By Lemma 2.4 we have that $x(t)$ is a solution of problem (1.1). Meanwhile, $y(t)$ is also a solution of problem (1.1) and satisfies $u_0 \le x \le y \le v_0$ on $(0,T]$.

To prove that $x(t)$ and $y(t)$ are extremal solutions of (1.1), let $u \in [u_0, v_0]$ be any solution of problem (1.1). We suppose that $u_n \le u \le v_n$, $t\in(0,T]$, for some n. Let $\zeta(t) = \phi_p(D^\alpha_{0^+}u(t)) - \phi_p(D^\alpha_{0^+}u_{n+1}(t))$, $\eta(t) = \phi_p(D^\alpha_{0^+}v_{n+1}(t)) - \phi_p(D^\alpha_{0^+}u(t))$. Thus, by condition (H_2) we have

$$D^\beta_{0^+}\zeta(t) + M\zeta(t) = F\bigl(u(t)\bigr) - F\bigl(u_n(t)\bigr) + M\bigl[\phi_p\bigl(D^\alpha_{0^+}u\bigr) - \phi_p\bigl(D^\alpha_{0^+}u_n\bigr)\bigr] \ge 0$$

and

$$D^\beta_{0^+}\eta(t) + M\eta(t) = F\bigl(v_n(t)\bigr) - F\bigl(u(t)\bigr) + M\bigl[\phi_p\bigl(D^\alpha_{0^+}v_n\bigr) - \phi_p\bigl(D^\alpha_{0^+}v\bigr)\bigr] \ge 0.$$

Moreover, from condition (H_3) we find

$$\begin{aligned}\tilde{u}(0) - \tilde{u}_{n+1}(0) &= \frac{1}{\lambda}\bigl[\lambda\tilde{u}(0) + g\bigl(\tilde{u}(0),\tilde{u}(T)\bigr) - \bigl(\lambda\tilde{u}_n(0) + g\bigl(\tilde{u}_n(0),\tilde{u}_n(T)\bigr)\bigr)\bigr]\\ &\ge \frac{\mu}{\lambda}\bigl(\tilde{u}(T) - \tilde{u}_n(T)\bigr) \ge 0\end{aligned}$$

and

$$\begin{aligned}\tilde{v}_{n+1}(0) - \tilde{u}(0) &= \frac{1}{\lambda}\bigl[\lambda\tilde{v}_n(0) + g\bigl(\tilde{u}(0),\tilde{u}(T)\bigr) - \bigl(\lambda\tilde{u}(0) + g\bigl(\tilde{u}_n(0),\tilde{u}_{n+1}(T)\bigr)\bigr)\bigr]\\ &\ge \frac{\mu}{\lambda}\bigl(\tilde{v}_{n+1}(T) - \tilde{u}(T)\bigr) \ge 0.\end{aligned}$$

These inequalities and Lemma 2.5 imply that $D^\alpha_{0^+}u_{n+1}(t) \le D^\alpha_{0^+}u(t) \le D^\alpha_{0^+}v_{n+1}(t)$ and $u_{n+1}(t) \le u(t) \le v_{n+1}(t)$, $t\in(0,T]$, so by induction $x(t) \le u(t) \le y(t)$ and $D^\alpha_{0^+}x \le D^\alpha_{0^+}u \le D^\alpha_{0^+}y$ on $(0,T]$ by taking the limits as $n\to\infty$. This finishes the proof. □

Remark 3.1 In Definition 3.1, we also can use $g(\tilde{u}(0),\tilde{u}(T)) \le 0$ instead of $g(\tilde{u}(0),\tilde{u}(T)) \ge 0$ to define the lower solution of problem (1.1) and use $g(\tilde{v}(0),\tilde{v}(T)) \ge 0$ instead of $g(\tilde{v}(0),\tilde{v}(T)) \le 0$ to define the upper solution of problem (1.1), with the remaining conditions unchanged. However, the conclusions of Theorem 3.1 hold under assumptions (H_1), (H_2), and

(H_3') there exist constants $\lambda' > 0$, $\mu' \ge 0$ such that

$$g(x_1,y_1) - g(x_2,y_2) \ge -\lambda'(x_2 - x_1) + \mu'(y_2 - y_1)$$

for $\tilde{u}_0(0) \le x_1 \le x_2 \le \tilde{v}_0(0)$ and $\tilde{u}_0(T) \le y_1 \le y_2 \le \tilde{v}_0(T)$. Meanwhile, in the proof, we need to transform the definitions of $\tilde{u}_n(0)$ and $\tilde{v}_n(0)$ in (3.3) and (3.4) into the forms

$$\tilde{u}_n(0) = \tilde{u}_{n-1}(0) - \frac{1}{\lambda'}g\big(\tilde{u}_{n-1}(0),\tilde{u}_{n-1}(T)\big), \qquad \tilde{v}_n(0) = \tilde{v}_{n-1}(0) - \frac{1}{\lambda'}g\big(\tilde{v}_{n-1}(0),\tilde{v}_{n-1}(T)\big)$$

and make the corresponding modification in view of (H_3').

Theorem 3.2 *The assumptions of Theorem* 3.1 *hold, and there exists a constant* $N > 0$ *such that*

$$N\big[\phi_p\big(D^\alpha_{0^+}v(t)\big) - \phi_p\big(D^\alpha_{0^+}u(t)\big)\big] \le f\big(t,u(t),D^\alpha_{0^+}u(t)\big) - f\big(t,v(t),D^\alpha_{0^+}v(t)\big) \tag{3.10}$$

for $u_0(t) \le u(t) \le v(t) \le v_0(t)$, $D^\alpha_{0^+}u_0(t) \le D^\alpha_{0^+}u(t) \le D^\alpha_{0^+}v(t) \le D^\alpha_{0^+}v_0(t)$, $t \in (0,T]$, *and* $\tilde{u}_0(0) = \tilde{v}_0(0)$. *Then problem* (1.1) *has a unique solution in the order interval* $[u_0,v_0]$.

Proof By Theorem 3.1 we see that $x(t)$ and $y(t)$ are extremal solutions and $x(t) \le y(t)$, $t \in (0,T]$. In order to prove that $x(t) \ge y(t)$, $t \in (0,T]$, we let $w(t) = \phi_p(D^\alpha_{0^+}x(t)) - \phi_p(D^\alpha_{0^+}y(t))$, $t \in (0,T]$. From (3.10) we arrive at

$$\begin{cases} D^\beta_{0^+}w(t) = F(x(t)) - F(y(t)) \ge N[\phi_p(D^\alpha_{0^+}y(t)) - \phi_p(D^\alpha_{0^+}x(t))] = -Nw(t), \\ t^{1-\beta}w(t)|_{t=0} = t^{1-\beta}w(t)|_{t=T}. \end{cases}$$

Then $w(t) \ge 0$, $t \in (0,T]$, that is, $D^\alpha_{0^+}x(t) \ge D^\alpha_{0^+}y(t)$, $t \in (0,T]$. Also, by (3.8), since $\tilde{u}_0(0) = \tilde{v}_0(0)$, we have $\tilde{x}(0) = \tilde{y}(0)$. Therefore, Lemma 2.5 implies $x(t) \ge y(t)$, $t \in (0,T]$. Thus, we obtain $x = y$. The proof is complete. □

Example 3.1 Consider the following fractional periodic boundary value problem:

$$\begin{cases} D^\beta_{0^+}(\phi_p(D^\alpha_{0^+}u(t))) = t^{1/2}(1-t) - 2[D^\alpha_{0^+}u(t)]^2 + u(t), \quad t \in (0,1], \\ t^{1/6}D^\alpha_{0^+}u(t)|_{t=0} = t^{1/6}D^\alpha_{0^+}u(t)|_{t=1}, \\ \tilde{u}(0)(\frac{\Gamma(5/6)}{2\Gamma(4/3)} - \tilde{u}(1)) = 0, \end{cases} \tag{3.11}$$

where $\alpha = 1/2$, $\beta = 2/3$, $p = 3$, $T = 1$, and $f(t,u,D^\alpha_{0^+}u) = t^{1/2}(1-t) - 2[D^\alpha_{0^+}u(t)]^2 + u(t)$, $g(x,y) = x(\frac{\Gamma(5/6)}{2\Gamma(4/3)} - y)$. Set

$$u_0(t) \equiv 0, \qquad v_0(t) = \frac{\Gamma(5/6)}{\Gamma(4/3)}t^{1/3}, \quad t \in [0,1].$$

It is easy to verify that $D_{0^+}^{1/2}u_0(t) \equiv 0$ and $D_{0^+}^{1/2}v_0(t) = t^{-1/6}$ for $t \in (0,1]$ and

$$\begin{aligned}
&t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=0} = 0 = t^{1/6}D_{0^+}^{1/2}u_0(t)|_{t=1}, \qquad t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=0} = 1 = t^{1/6}D_{0^+}^{1/2}v_0(t)|_{t=1},\\
&D_{0^+}^{2/3}\big(\phi_3\big(D_{0^+}^{1/2}u_0(t)\big)\big) \equiv 0 \le f\big(t, u_0, D_{0^+}^{1/2}u_0\big) = t^{1/2}(1-t),\\
&D_{0^+}^{2/3}\big(\phi_3\big(D_{0^+}^{1/2}v_0(t)\big)\big) = D_{0^+}^{2/3}\big(t^{-1/3}\big) = 0 \ge f\big(t, v_0, D_{0^+}^{1/2}v_0\big)\\
&\qquad = t^{1/2}(1-t) - 2t^{-1/3} + \frac{\Gamma(5/6)}{\Gamma(4/3)}t^{1/3},\\
&g\big(\tilde{u}_0(0), \tilde{u}_0(1)\big) = 0, \qquad g\big(\tilde{v}_0(0), \tilde{v}_0(1)\big) = 0.
\end{aligned}$$

These show that u_0 and v_0 are the lower and upper solutions of (3.11), respectively, and $u_0(t) \le v_0(t)$ on $[0,1]$.

For $u_0 \le u \le v \le v_0$, we have $\phi_3(D_{0^+}^{1/2}v) - \phi_3(D_{0^+}^{1/2}u) = (D_{0^+}^{1/2}v)^2 - (D_{0^+}^{1/2}u)^2$ and

$$f\big(t, u, D_{0^+}^{1/2}u\big) + 2\phi_3\big(D_{0^+}^{1/2}u\big) - \big[f\big(t, v, D_{0^+}^{1/2}v\big) + 2\phi_3\big(D_{0^+}^{1/2}v\big)\big] = u - v \le 0.$$

Thus, $f(t, u, D_{0^+}^{1/2}u) - f(t, v, D_{0^+}^{1/2}v) \le M[\phi_3(D_{0^+}^{1/2}v) - \phi_3(D_{0^+}^{1/2}u)]$, where $M = 2$.

In addition, $\frac{\partial g(x,y)}{\partial x} = \frac{\Gamma(5/6)}{2\Gamma(4/3)} - y \ge -\frac{\Gamma(5/6)}{2\Gamma(4/3)}$, $\frac{\partial g(x,y)}{\partial y} = -x$ for $\tilde{u}_0(0) \le x \le \tilde{v}_0(0)$, $y \in [\tilde{u}_0(1), \tilde{v}_0(1)] = [0, \frac{\Gamma(5/6)}{\Gamma(4/3)}]$. Therefore, $g(u_1, v_1) - g(u_2, v_2) \le \frac{\Gamma(5/6)}{2\Gamma(4/3)}(u_2 - u_1)$ for $\tilde{u}_0(0) \le u_1 \le u_2 \le \tilde{v}_0(0)$, $\tilde{u}_0(1) \le v_1 \le v_2 \le \tilde{v}_0(1)$. Hence, conditions (H_1), (H_2), and (H_3) are satisfied. There exist two monotone iterative sequences $\{u_k\}$ and $\{v_k\}$ that converge uniformly to the minimal and maximal solutions of fractional periodic boundary problem (3.11) in $[u_0, v_0]$ by Theorem 3.1.

Appendix

Lemma A.1 *The sequences $\{t^{1-\alpha}u_n\}$ and $\{t^r D^\alpha u_n\}$ are uniformly bounded and equicontinuous in $C[0,T]$, where u_n is defined by* (3.3) *in Theorem* 3.1.

Proof We first show that $\{t^{1-\alpha}u_n\}$ are uniformly bounded in $C[0,T]$. Since $u_0, v_0 \in C_r^\alpha[0,T]$, we have $\phi_p(D_{0^+}^\alpha v_0(t)) \in C_{1-\beta}[0,T]$, that is, $t^{1-\beta}\phi_p(D_{0^+}^\alpha v_0(t)) \in C[0,T]$. Thus, there exists a constant $\gamma_1 > 0$ such that

$$\big\| t^{1-\beta}\phi_p\big(D_{0^+}^\alpha v_0\big)\big\|_C \le \gamma_1, \quad t \in [0,T],$$

which is equivalent to

$$\big|\phi_p\big(D_{0^+}^\alpha v_0(t)\big)\big| \le \gamma_1 t^{\beta-1}, \quad t \in (0,T]. \tag{A.1}$$

Let

$$\eta_{n-1}(t) = F\big(u_{n-1}(t)\big) + M\phi_p\big(D_{0^+}^\alpha u_{n-1}(t)\big), \quad t \in (0,T]. \tag{A.2}$$

By condition (H_2) and (A.1) we get

$$\eta_{n-1}(t) \le F\big(v_0(t)\big) + M\phi_p\big(D_{0^+}^\alpha v_0(t)\big) \le F\big(v_0(t)\big) + M\gamma_1 t^{\beta-1}, \quad t \in (0,T].$$

Hence,

$$\|\eta_{n-1}\|_{C_{1-\beta}} \le \|F(v_0)\|_C T + M\gamma_1 =: \gamma_2, \quad t \in [0,T]. \tag{A.3}$$

Let

$$x_{n-1}(s) = M_0 s^{\beta-1} E_{\beta,\beta}\bigl(-Ms^\beta\bigr) \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(T-s)^\beta\bigr)\eta_{n-1}(s)\,ds$$
$$+ \int_0^s (s-\tau)^{\beta-1} E_{\beta,\beta}\bigl(-M(s-\tau)^\beta\bigr)\eta_{n-1}(\tau)\,d\tau, \tag{A.4}$$

where M_0 is defined in (3.9). Then $x_{n-1} \in C_{1-\beta}[0,T]$. Noting that $E_{\beta,\beta}(x) < \frac{1}{\Gamma(\beta)}$ for $x < 0$, by (A.4) and (A.3) we have

$$\bigl|x_{n-1}(s)\bigr| \le M_0 s^{\beta-1} \frac{1}{\Gamma^2(\beta)} \int_0^T (T-s)^{\beta-1} s^{\beta-1} \|\eta_{n-1}\|_{C_{1-\beta}}\,ds$$
$$+ \frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1}\tau^{\beta-1}\|\eta_{n-1}\|_{C_{1-\beta}}\,d\tau$$
$$\le M_0 s^{\beta-1}\frac{\gamma_2}{\Gamma^2(\beta)} B(\beta,\beta) T^{2\beta-1} + \frac{\gamma_2}{\Gamma(\beta)} B(\beta,\beta) s^{2\beta-1}, \quad s \in (0,T], \tag{A.5}$$

which yields

$$\bigl|s^{1-\beta} x_{n-1}(s)\bigr| \le M_0 \frac{\gamma_2}{\Gamma^2(\beta)} B(\beta,\beta) T^{2\beta-1} + \frac{\gamma_2}{\Gamma(\beta)} B(\beta,\beta) T^\beta, \quad s \in (0,T].$$

Thus, for $s \in (0,T]$, we get

$$s^r \phi_q\bigl(x_{n-1}(s)\bigr) = \phi_q\bigl(s^{1-\beta} x_{n-1}(s)\bigr)$$
$$\le \phi_q\left(M_0 \frac{\gamma_2}{\Gamma^2(\beta)} B(\beta,\beta) T^{2\beta-1} + \frac{\gamma_2}{\Gamma(\beta)} B(\beta,\beta) T^\beta\right) =: C. \tag{A.6}$$

This implies that $\phi_q(x_{n-1}(s)) = D_{0^+}^\alpha u_n(s)$ is bounded in $C_r[0,T]$. From (3.3) and Lemma 2.4 we find

$$t^{1-\alpha} u_n(t) = \tilde{u}_{n-1}(0) + \frac{1}{\lambda} g\bigl(\tilde{u}_{n-1}(0), \tilde{u}_{n-1}(T)\bigr) + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\bigl(x_{n-1}(s)\bigr)\,ds. \tag{A.7}$$

Using (A.7), (A.6), and condition (H_3), we get

$$\bigl|t^{1-\alpha} u_n(t)\bigr| = \left|\tilde{u}_{n-1}(0) + \frac{1}{\lambda} g\bigl(\tilde{u}_{n-1}(0), \tilde{u}_{n-1}(T)\bigr)\right| + \left|\frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_q\bigl(x_{n-1}(s)\bigr)\,ds\right|$$
$$\le \left|\tilde{v}_0(0) + \frac{1}{\lambda} g\bigl(\tilde{v}_0(0), \tilde{v}_0(T)\bigr)\right| + \frac{Ct^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-r}\,ds$$
$$\le \left|\tilde{v}_0(0) + \frac{1}{\lambda} g\bigl(\tilde{v}_0(0), \tilde{v}_0(T)\bigr)\right| + \frac{CB(\alpha,1-r)}{\Gamma(\alpha)} T^{1-r}, \quad t \in [0,T].$$

Hence, $\{t^{1-\alpha} u_n\}$ are uniformly bounded in $C[0,T]$.

Next, we prove that $\{t^{1-\alpha}u_n\}$ are equicontinuous in $C[0,T]$. Suppose $0 < t_1 \le t_2 \le T$. From (A.7) and (A.6) we have

$$\begin{aligned}
&\bigl|t_2^{1-\alpha}u_n(t_2) - t_1^{1-\alpha}u_n(t_1)\bigr| \\
&\quad = \left|\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\phi_q\bigl(x_{n-1}(s)\bigr)\,ds - \frac{t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\phi_q\bigl(x_{n-1}(s)\bigr)\,ds\right| \\
&\quad \le \left|\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\phi_q\bigl(x_{n-1}(s)\bigr)\,ds\right| \\
&\qquad + \left|\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}\bigl[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr]\phi_q\bigl(x_{n-1}(s)\bigr)\,ds\right| \\
&\qquad + \left|\frac{t_2^{1-\alpha} - t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\phi_q\bigl(x_{n-1}(s)\bigr)\,ds\right| \\
&\quad \le \frac{Ct_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}s^{-r}\,ds + \frac{Ct_2^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}\bigl|\bigl[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr]s^{-r}\bigr|\,ds \\
&\qquad + \frac{C(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}s^{-r}\,ds \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

For part I, we have

$$\int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}s^{-r}\,ds \le t_2^{1-\alpha}t_2^{\alpha-r}B(\alpha,1-r) = t_2^{1-r}B(\alpha,1-r) < \infty.$$

By the absolute continuity of the integral, we have that I can be sufficiently small when t_1 is sufficiently close to t_2. For part II, we have

$$\begin{aligned}
\mathrm{II} &= \frac{C}{\Gamma(\alpha)}\left[t_2^{1-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}s^{-r}\,ds - t_2^{1-\alpha}\int_0^{t_1}(t_2-s)^{\alpha-1}s^{-r}\,ds\right] \\
&= \frac{C}{\Gamma(\alpha)}\left[t_2^{1-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}s^{-r}\,ds - t_2^{1-\alpha}\left(\int_0^{t_2}(t_2-s)^{\alpha-1}s^{-r}\,ds - \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}s^{-r}\,ds\right)\right] \\
&= \frac{C}{\Gamma(\alpha)}\left(\left(\frac{t_2}{t_1}\right)^{1-\alpha}\cdot t_1^{\alpha} - t_2^{\alpha}\right)B(\alpha,1-r) + t_2^{1-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}s^{-r}\,ds. \qquad \text{(A.8)}
\end{aligned}$$

It is easy to see that as t_1 approaches t_2, II goes to zero.

For part III, we have

$$\mathrm{III} \le \frac{C(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)t_1^{1-\alpha}}t_1^{1-r}B(\alpha,1-r).$$

Combining the results of I, II, and III, we have that $|t_2^{1-\alpha}u_n(t_2) - t_1^{1-\alpha}u_n(t_1)| \to 0$ as $t_1 \to t_2$.

When $t_1 = 0 \le t_2 \le T$, from (A.7) and (A.6) we have

$$\begin{aligned}
\bigl|t_2^{1-\alpha}u_n(t_2) - \tilde{u}_n(0)\bigr| &= \left|\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\phi_q\bigl(x_{n-1}(s)\bigr)\,ds\right| \\
&\le \frac{C}{\Gamma(\alpha)}\int_0^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}s^{-r}\,ds
\end{aligned}$$

$$= \frac{C}{\Gamma(\alpha)} t_2^{1-r} B(\alpha, 1-r)$$
$$\to 0 \quad \text{as } t_2 \to 0.$$

This shows that $\{t^{1-\alpha} u_n\}$ are equicontinuous in $C[0,T]$.

In the following, we will check that $\{t^r D_{0^+}^{\alpha} u_n\}$ are relatively compact in $C[0,T]$.

First, we prove that $\{t^r D_{0^+}^{\alpha} u_n\}$ are uniformly bounded in $C[0,T]$. By (3.3) and (2.5) we get

$$\begin{aligned} D_{0^+}^{\alpha} u_n(t) = \phi_q \biggl[& M_0 t^{\beta-1} E_{\beta,\beta}\bigl(-Mt^{\beta}\bigr) \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(T-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \\ & + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \biggr], \end{aligned} \tag{A.9}$$

where η_{n-1} is defined in (A.3). By (A.9) and the definition of x_{n-1} in (A.4) and (A.6), we still have

$$\bigl|t^r D_{0^+}^{\alpha} u_n(t)\bigr| = \bigl|t^r \phi_q\bigl(x_{n-1}(t)\bigr)\bigr| = \bigl|\phi_q\bigl(t^{1-\beta} x_{n-1}(t)\bigr)\bigr| \le C, \quad t \in (0,T].$$

Therefore, $\{t^r D_{0^+}^{\alpha} u_n\}$ are uniformly bounded in $C[0,T]$.

Second, we prove that $\{t^r D_{0^+}^{\alpha} u_n\}$ are equicontinuous in $C[0,T]$. Since $\phi_p(t^r D_{0^+}^{\alpha} u(t)) = t^{1-\beta} \phi_p(D_{0^+}^{\alpha} u(t))$, we need to deal with the equicontinuity of $\{t^{1-\beta} \phi_p(D_{0^+}^{\alpha} u_n)\}$ in $C[0,T]$. Choosing $0 < t_1 \le t_2 \le T$, by (A.9) and (A.3) we have

$$\begin{aligned} & \bigl|t_2^{1-\beta} \phi_p\bigl(D_{0^+}^{\alpha} u_n(t_2)\bigr) - t_1^{1-\beta} \phi_p\bigl(D_{0^+}^{\alpha} u_n(t_1)\bigr)\bigr| \\ &\quad = \biggl| M_0 \bigl[E_{\beta,\beta}\bigl(-Mt_2^{\beta}\bigr) - E_{\beta,\beta}\bigl(-Mt_1^{\beta}\bigr)\bigr] \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(T-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \\ &\qquad + t_2^{1-\beta} \int_0^{t_2} (t_2-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_2-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \\ &\qquad - t_1^{1-\beta} \int_0^{t_1} (t_1-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_1-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \biggr| \\ &\quad \le M_0 \bigl|E_{\beta,\beta}\bigl(-Mt_2^{\beta}\bigr) - E_{\beta,\beta}\bigl(-Mt_1^{\beta}\bigr)\bigr| \int_0^T (T-s)^{\beta-1} \frac{1}{\Gamma(\beta)} s^{\beta-1} \|\eta_{n-1}\|_{C_{1-\beta}}\,ds \\ &\qquad + \biggl| t_2^{1-\beta} \int_0^{t_2} (t_2-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_2-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \\ &\qquad - t_1^{1-\beta} \int_0^{t_1} (t_1-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_1-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \biggr| \\ &\quad \le t_2^{1-\beta} \biggl| \int_0^{t_1} \bigl[(t_2-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_2-s)^{\beta}\bigr) - (t_1-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_1-s)^{\beta}\bigr)\bigr] \eta_{n-1}(s)\,ds \biggr| \\ &\qquad + \biggl| \bigl(t_2^{1-\beta} - t_1^{1-\beta}\bigr) \int_0^{t_1} (t_1-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_1-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \biggr| \\ &\qquad + \biggl| t_2^{1-\beta} \int_{t_1}^{t_2} (t_2-s)^{\beta-1} E_{\beta,\beta}\bigl(-M(t_2-s)^{\beta}\bigr) \eta_{n-1}(s)\,ds \biggr| \\ &\qquad + C^* \bigl|E_{\beta,\beta}\bigl(-Mt_2^{\beta}\bigr) - E_{\beta,\beta}\bigl(-Mt_1^{\beta}\bigr)\bigr| \\ &\quad =: \mathrm{I}' + \mathrm{II}' + \mathrm{III}' + \mathrm{IV}', \end{aligned}$$

where

$$C^* = M_0 \frac{\gamma_2}{\Gamma(\beta)} T^{2\beta-1} B(\beta,\beta).$$

It is easy to verify that II′, III′, and IV′ go to zero as $t_1 \to t_2$. In the following, we only consider I′:

$$\begin{aligned}
\mathrm{I}' &\le \gamma_2 t_2^{1-\beta} \left| \int_0^{t_1} \left[(t_2-s)^{\beta-1} E_{\beta,\beta}\left(-M(t_2-s)^\beta\right) - (t_1-s)^{\beta-1} E_{\beta,\beta}\left(-M(t_1-s)^\beta\right) \right] s^{\beta-1}\, ds \right| \\
&\le \gamma_2 t_2^{1-\beta} \int_0^{t_1} (t_1-s)^{\beta-1} \left| E_{\beta,\beta}\left(-M(t_2-s)^\beta\right) - E_{\beta,\beta}\left(-M(t_1-s)^\beta\right) \right| s^{\beta-1}\, ds \\
&\quad + \frac{\gamma_2}{\Gamma(\beta)} t_2^{1-\beta} \int_0^{t_1} \left((t_1-s)^{\beta-1} - (t_2-s)^{\beta-1} \right) s^{\beta-1}\, ds.
\end{aligned}$$

By the continuity of the Mittag-Leffler function and (A.8) we have that I′ goes to zero as $t_1 \to t_2$. It is easy to verify that the equicontinuity of $\{t^r D_{0^+}^\alpha u_n\}$ is true for $t_1 = 0$ by (A.9) and similar estimates. This completes the proof of the lemma. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shandong Jianzhu University, Jinan, Shandong 250101, China. [2]School of Mathematical Science, University of Jinan, Jinan, Shandong 250022, China. [3]School of Mathematics, Shandong University, Jinan, Shandong 250100, China.

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