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Hopf bifurcation of a delayed SIQR epidemic model with constant input and nonlinear incidence rate

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Abstract

An SIQR epidemic model with nonlinear incidence rate and two delays is studied under the assumption that a susceptible of the host population has a constant input. Local stability and existence of Hopf bifurcation are analyzed by regarding combination of the time delay due to the latent period of disease and the time delay due to the period that the infective and quarantined individuals need to be cured as the bifurcation parameter. Furthermore, the properties of the Hopf bifurcation are determined by using the normal form method and center manifold theory. Some numerical simulations are also carried out in order to verify our theoretical findings.

Keywords: delays; Hopf bifurcation; SIQR model; periodic solutions

1 Introduction

For the last two decades, various epidemic models have been proposed and investigated in order to understand disease transmissions and behaviours of epidemics. As is well known, the bilinear incidence rate βSI is frequently used in many epidemic models [1–5]. However, the bilinear incidence rate is based on the law of mass action, which is more appropriate for communicable diseases, but not for sexually transmitted diseases [6]. It has been suggested by several authors that the disease transmission process may have a nonlinear incidence rate and the epidemic models with a nonlinear incidence rate have been studied by many researchers [7–13]. In [11], Song and Pang proposed the following SIQR (susceptible-infective-quarantined-recovered) epidemic model with constant input and nonlinear incidence rate:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)I^2(t) - dS(t), \\ \frac{dI(t)}{dt} = \beta S(t)I^2(t) - (d + a + \gamma + \sigma)I(t), \\ \frac{dQ(t)}{dt} = \sigma I(t) - (d + b + p)Q(t), \\ \frac{dR(t)}{dt} = \gamma I(t) + pQ(t) - dR(t), \end{cases} \tag{1}$$

where $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ denote the numbers of the susceptible, infective, quarantined, and recovered individuals at time t, respectively. A is the recruitment rate of the susceptible individual; d is the natural death rate of the susceptible, infective, quarantined and recovered individuals; a and b are the death rates of the infective and quarantined individuals due to the disease, respectively; β, γ, σ, and p are the states transition rates. Song and Pang studied stability of system (1).

Obviously, they neglected the time delay due to the latent period of the disease and the time delay due to the period that the infective and quarantined individuals need to be cured in system (1). As stated in [14], it is necessary to incorporate time delays of one type or another into a dynamical system in order to reflect dynamics of the system depending on its history. It is well known that time delays have a strong impact on dynamics of a dynamical system and effect of time delays on the dynamics of a dynamical system can be found in [1, 9, 14–20]. Therefore, we consider the following system with delays:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t-\tau_1)I^2(t-\tau_1) - dS(t), \\ \frac{dI(t)}{dt} = \beta S(t-\tau_1)I^2(t-\tau_1) - (d+a+\sigma)I(t) - \gamma I(t-\tau_2), \\ \frac{dQ(t)}{dt} = \sigma I(t) - (d+b)Q(t) - pQ(t-\tau_2), \\ \frac{dR(t)}{dt} = \gamma I(t-\tau_2) + pQ(t-\tau_2) - dR(t), \end{cases} \tag{2}$$

where τ_1 is the latent period delay. τ_2 is the time delay due to the period that the infective and quarantined individuals need to be cured.

The structure of this paper is as follows. Section 2 is devoted to the local stability and existence of the Hopf bifurcation. Explicit formulae for determining the properties of the Hopf bifurcation are derived in Section 3. Numerical simulations are presented to verify the obtained theoretical findings in Section 4, and this work is summarised in Section 5.

2 Existence of Hopf bifurcation

According to a direct computation, we know that if $R_0 = \frac{\beta A^2}{4d(d+a+\gamma+\sigma)^2} > 1$, then system (2) has the positive equilibriums $E_*(S_*, I_*, Q_*, R_*)$ where $S_* = \frac{d+a+\gamma+\sigma}{\beta I_*}$, $Q_* = \frac{\sigma I_*}{d+b+p}$, $R_* = \frac{\gamma I_* + pQ_*}{d}$, and I_* is the positive root of the following equation:

$$(d+a+\gamma+\sigma)\beta I^2 - A\beta I + d(d+a+\gamma+\sigma) = 0. \tag{3}$$

Let $u_1(t) = S(t) - S_*$, $u_2(t) = I(t) - I_*$, $u_3(t) = Q(t) - Q_*$, $u_4(t) = R(t) - R_*$. Then system (2) becomes

$$\begin{cases} \frac{dS(t)}{dt} = \alpha_1 S(t) + \beta_1 S(t-\tau_1) + \beta_2 I(t-\tau_1) + f_1, \\ \frac{dI(t)}{dt} = \alpha_2 I(t) + \beta_3 S(t-\tau_1) + \beta_4 I(t-\tau_1) + \beta_5 I(t-\tau_2) + f_2, \\ \frac{dQ(t)}{dt} = \alpha_3 I(t) + \alpha_4 Q(t) + \beta_6 Q(t-\tau_2), \\ \frac{dR(t)}{dt} = \alpha_5 R(t) + \beta_7 I(t-\tau_2) + \beta_8 Q(t-\tau_2), \end{cases} \tag{4}$$

where

$$\alpha_1 = -d, \qquad \alpha_2 = -(d+a+\sigma), \qquad \alpha_3 = \sigma,$$

$$\alpha_4 = -(d+b), \qquad \alpha_5 = -d, \qquad \beta_1 = -\beta I_*^2, \qquad \beta_2 = -2\beta S_* I_*,$$

$$\beta_3 = \beta I_*^2, \qquad \beta_4 = 2\beta S_* I_*,$$

$$\beta_5 = -\gamma, \qquad \beta_6 = -p, \qquad \beta_7 = \gamma, \qquad \beta_8 = p,$$

and

$$f_1 = \beta_{13}S(t-\tau_1)I(t-\tau_1) + \beta_{14}I^2(t-\tau_1) + \beta_{15}S(t-\tau_1)I^2(t-\tau_1),$$
$$f_2 = \beta_{23}S(t-\tau_1)I(t-\tau_1) + \beta_{24}I^2(t-\tau_1) + \beta_{25}S(t-\tau_1)I^2(t-\tau_1),$$

with

$$\beta_{13} = -2\beta S_* I_*, \qquad \beta_{14} = -\beta S_*, \qquad \beta_{15} = -\beta,$$
$$\beta_{23} = 2\beta S_* I_*, \qquad \beta_{24} = \beta S_*, \qquad \beta_{25} = \beta.$$

The linear system of system (4) is

$$\begin{cases} \frac{dS(t)}{dt} = \alpha_1 S(t) + \beta_1 S(t-\tau_1) + \beta_2 I(t-\tau_1), \\ \frac{dI(t)}{dt} = \alpha_2 I(t) + \beta_3 S(t-\tau_1) + \beta_4 I(t-\tau_1) + \beta_5 I(t-\tau_2), \\ \frac{dQ(t)}{dt} = \alpha_3 I(t) + \alpha_4 Q(t) + \beta_6 Q(t-\tau_2), \\ \frac{dR(t)}{dt} = \alpha_5 R(t) + \beta_7 I(t-\tau_2) + \beta_8 Q(t-\tau_2). \end{cases} \tag{5}$$

Thus, we can get the characteristic equation

$$\begin{aligned} &\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + \left(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0\right)e^{-\lambda\tau_1} \\ &\quad + \left(p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0\right)e^{-\lambda\tau_2} + \left(q_2\lambda^2 + q_1\lambda + q_0\right)e^{-\lambda(\tau_1+\tau_2)} \\ &\quad + \left(r_2\lambda^2 + r_1\lambda + r_0\right)e^{-2\lambda\tau_2} + (s_1\lambda + s_0)e^{-\lambda(\tau_1+2\tau_2)} = 0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &m_0 = \alpha_1\alpha_2\alpha_4\alpha_5, \qquad m_1 = -\big(\alpha_1\alpha_2(\alpha_4+\alpha_5) + \alpha_4\alpha_5(\alpha_1+\alpha_2)\big), \\ &m_2 = \alpha_1\alpha_2 + \alpha_4\alpha_5 + (\alpha_1+\alpha_2)(\alpha_4+\alpha_5), \\ &m_3 = -(\alpha_1+\alpha_2+\alpha_4+\alpha_5), \qquad n_0 = \alpha_1\alpha_4\alpha_5\beta_4 + \alpha_2\alpha_4\alpha_5\beta_1, \\ &n_1 = -\big(\beta_1(\alpha_2\alpha_4 + \alpha_2\alpha_5 + \alpha_4\alpha_5) + \beta_4(\alpha_1\alpha_4 + \alpha_1\alpha_5 + \alpha_4\alpha_5)\big), \\ &n_2 = \beta_1(\alpha_2+\alpha_4+\alpha_5) + \beta_4(\alpha_1+\alpha_4+\alpha_5), \qquad n_3 = -(\beta_1+\beta_4), \\ &p_0 = \alpha_1\alpha_4\alpha_5\beta_5 + \alpha_1\alpha_2\alpha_5\beta_6, \\ &p_1 = -\big(\beta_5(\alpha_1\alpha_4 + \alpha_1\alpha_5 + \alpha_4\alpha_5) + \beta_6(\alpha_1\alpha_2 + \alpha_1\alpha_5 + \alpha_2\alpha_5)\big), \\ &p_2 = \beta_5(\alpha_1+\alpha_4+\alpha_5) + \beta_6(\alpha_1+\alpha_2+\alpha_5), \qquad p_3 = -(\beta_5+\beta_6), \\ &q_0 = \alpha_1\alpha_5\beta_4\beta_6 + \alpha_2\alpha_5\beta_1\beta_6 + \alpha_4\alpha_5\beta_1\beta_5, \\ &q_1 = -\big(\beta_1\beta_5(\alpha_4+\alpha_5) + \beta_1\beta_6(\alpha_2+\alpha_5) + \beta_4\beta_6(\alpha_1+\alpha_5)\big), \\ &q_2 = \beta_1\beta_5 + \beta_1\beta_6 + \beta_5\beta_6, \qquad r_0 = \alpha_1\alpha_5\beta_5\beta_6, \\ &r_1 = -\beta_5\beta_6(\alpha_1+\alpha_5), \qquad r_2 = \beta_5\beta_6, \qquad s_0 = \alpha_5\beta_1\beta_5\beta_6, \qquad s_1 = -\beta_1\beta_5\beta_6. \end{aligned}$$

Case 1 $\tau_1 = \tau_2 = 0$, equation (6) reduces to

$$\lambda^4 + m_{13}\lambda^3 + m_{12}\lambda^2 + m_{11}\lambda + m_{10} = 0, \tag{7}$$

where

$$m_{10} = m_0 + n_0 + p_0 + q_0 + r_0 + s_0, \qquad m_{11} = m_1 + n_1 + p_1 + q_1 + r_1 + s_1,$$
$$m_{12} = m_2 + n_2 + p_2 + q_2 + r_2, \qquad m_{13} = m_3 + n_3 + p_3.$$

Obviously, if the condition (H_1): (8)-(11) holds, all the roots of equation (7) must have negative real parts. Thus, $E_*(S_*, I_*, Q_*, R_*)$ is locally asymptotically stable in the absence of delay. We have

$$Det_1 = m_{13} > 0, \tag{8}$$

$$Det_2 = \begin{vmatrix} m_{13} & 1 \\ m_{11} & m_{12} \end{vmatrix} > 0, \tag{9}$$

$$Det_3 = \begin{vmatrix} m_{13} & 1 & 0 \\ m_{11} & m_{12} & m_{13} \\ 0 & m_{10} & m_{11} \end{vmatrix} > 0, \tag{10}$$

$$Det_4 = \begin{vmatrix} m_{13} & 1 & 0 & 0 \\ m_{11} & m_{12} & m_{13} & 1 \\ 0 & m_{10} & m_{11} & m_{12} \\ 0 & 0 & 0 & m_{10} \end{vmatrix} > 0. \tag{11}$$

Case 2 $\tau_1 > 0$, $\tau_2 = 0$. For $\tau_2 = 0$, equation (6) becomes

$$\lambda^4 + m_{23}\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20} + \left(n_{23}\lambda^3 + n_{22}\lambda^2 + n_{21}\lambda + n_{20}\right)e^{-\lambda\tau_1} = 0, \tag{12}$$

where

$$m_{20} = m_0 + p_0 + r_0, \qquad m_{21} = m_1 + p_1 + r_1, \qquad m_{22} = m_2 + p_2 + r_2,$$
$$m_{23} = m_3 + p_3, \qquad n_{20} = n_0 + q_0 + s_0, \qquad n_{21} = n_1 + q_1 + s_1,$$
$$n_{22} = n_2 + q_2, \qquad n_{23} = n_3.$$

Let $\lambda = i\omega_1$ $(\omega_1 > 0)$ be the root of equation (12), then

$$\begin{cases} (n_{21}\omega_1 - n_{23}\omega_1^3)\sin\tau_1\omega_1 + (n_{20} - n_{22}\omega_1^2)\cos\tau_1\omega_1 = m_{22}\omega_1^2 - \omega_1^4 - m_{20}, \\ (n_{21}\omega_1 - n_{23}\omega_1^3)\cos\tau_1\omega_1 - (n_{20} - n_{22}\omega_1^2)\sin\tau_1\omega_1 = m_{23}\omega_1^3 - m_{21}\omega_1. \end{cases} \tag{13}$$

Then we can get

$$\omega_1^8 + c_{23}\omega_1^6 + c_{22}\omega_1^4 + c_{21}\omega_1^2 + c_{20} = 0, \tag{14}$$

with

$$c_{20} = m_{20}^2 - n_{20}^2, \qquad c_{21} = Am_{21}^2 - n_{21}^2 - 2m_{20}m_{22} + 2n_{20}n_{22},$$
$$c_{22} = m_{22}^2 - n_{22}^2 + 2m_{20} - 2m_{21}m_{23} + 2n_{21}n_{23}, \qquad c_{23} = m_{23}^2 - n_{23}^2 - 2m_{22}.$$

Let $\omega_1^2 = v_1$, then equation (14) becomes

$$v_1^4 + c_{23}v_1^3 + c_{22}v_1^2 + c_{21}v_1 + c_{20} = 0. \tag{15}$$

Define

$$f_1(v_1) = v_1^4 + c_{23}v_1^3 + c_{22}v_1^2 + c_{21}v_1 + c_{20},$$
$$p_{20} = \frac{1}{2}c_{22} - \frac{3}{16}c_{23}^2, \qquad q_{20} = \frac{1}{32}c_{23}^3 - \frac{1}{8}c_{22}c_{23} + c_{21},$$
$$\alpha_{20} = \left(\frac{q_{20}}{2}\right)^2 + \left(\frac{p_{20}}{3}\right)^3, \qquad \beta_{20} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$
$$y_{21} = \sqrt[3]{-\frac{q_{20}}{2} + \sqrt{\alpha_{20}}} + \sqrt[3]{-\frac{q_{20}}{2} - \sqrt{\alpha_{20}}},$$
$$y_{22} = \beta_{20}\sqrt[3]{-\frac{q_{20}}{2} + \sqrt{\alpha_{20}}} + \beta_{20}^2\sqrt[3]{-\frac{q_{20}}{2} - \sqrt{\alpha_{20}}},$$
$$y_{23} = \beta_{20}^2\sqrt[3]{-\frac{q_{20}}{2} + \sqrt{\alpha_{20}}} + \beta_{20}\sqrt[3]{-\frac{q_{20}}{2} - \sqrt{\alpha_{20}}},$$
$$v_{1i} = y_{2i} - \frac{3c_{23}}{4}, \quad i = 1,2,3.$$

The discussion about the distribution of the roots of equation (15) is similar to that in [21]. Thus, we have the following lemma.

Lemma 1 *For equation* (15), *we have*:

(i) *if* $c_{20} < 0$, *equation* (15) *has at least one positive root*;
(ii) *if* $c_{20} \geq 0$ *and* $\alpha_{20} \geq 0$, *equation* (15) *has positive roots if and only if* $v_{11} > 0$ *and* $f_1(v_{11}) < 0$;
(iii) *if* $c_{20} \geq 0$ *and* $\alpha_{20} < 0$, *equation* (15) *has positive roots if and only if there exists at least one* $v_{1*} \in \{v_{11}, v_{12}, v_{13}\}$ *such that* $v_{1*} > 0$ *and* $f_1(v_{1*}) \leq 0$:

In the following, we assume that (H_{21}): *the coefficients in* $f_1(v_1)$ *satisfy one of the following conditions in* (a)-(c):

(a) $c_{20} < 0$;
(b) $c_{20} \geq 0$, $\alpha_{20} \geq 0$, $v_{11} > 0$, *and* $f_1(v_{11}) < 0$;
(c) $c_{20} \geq 0$, $\alpha_{20} < 0$, *and there exists at least one* $v_{1*} \in \{v_{11}, v_{12}, v_{13}\}$ *such that* $v_{1*} > 0$ *and* $f_1(v_{1*}) \leq 0$.

If (H_{21}) *holds, we can conclude that equation* (14) *has at least a positive root* ω_{10} *such that equation* (12) *has a pair of purely imaginary root* $\pm i\omega_{10}$. *For* ω_{10},

$$\begin{aligned}\tau_{10} = {} & \frac{1}{\omega_{10}} \arccos\bigl(\bigl(\bigl(n_{21}\omega_{10} - n_{23}\omega_{10}^3\bigr) \times \bigl(m_{23}\omega_{10}^3 - m_{21}\omega_{10}\bigr) \\ & + \bigl(n_{20} - n_{22}\omega_{22}^2\bigr) \times \bigl(m_{22}\omega_{10}^2 - \omega_{10}^4 - m_{20}\bigr)\bigr) \\ & / \bigl(\bigl(n_{20} - n_{22}\omega_{10}^2\bigr)^2 + \bigl(n_{21}\omega_{10} - n_{23}\omega_{10}^3\bigr)^2\bigr)\bigr).\end{aligned} \tag{16}$$

Next, we will verify the transversality condition. Differentiating the two sides of equation (12) *with respect to* τ_1, *we get*

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = -\frac{4\lambda^3 + 3m_{23}\lambda^2 + 2m_{22}\lambda + m_{21}}{\lambda(\lambda^4 + m_{23}\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20})} + \frac{3n_{23}\lambda^2 + 2n_{22}\lambda + n_{21}}{\lambda(n_{23}\lambda^3 + n_{22}\lambda^2 + n_{21}\lambda + n_{20})} - \frac{\tau_1}{\lambda}.$$

Further, we have

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}} = \frac{f_1'(v_{1*})}{(n_{21}\omega_{10} - n_{23}\omega_{10}^3)^2 + (n_{20} - n_{22}\omega_{10}^2)^2},$$

where $f_1(v_1) = v_1^4 + c_{23}v_1^3 + c_{22}v_1^2 + c_{21}v_1 + c_{20}$ *and* $v_{1*} = \omega_{10}^2$. *Thus, if the condition* (H_{22}) $f_1'(v_{1*}) \neq 0$ *holds, then* $\operatorname{Re}[\frac{d\lambda}{d\tau_1}]^{-1}_{\tau_1=\tau_{10}} \neq 0$. *According to the Hopf bifurcation theorem in* [22], *we have the following for system* (2).

Theorem 1 *If the conditions* (H_{21})-(H_{22}) *hold, then*:
- (i) *the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *of system* (2) *is asymptotically stable for* $\tau_1 \in [0, \tau_{10})$;
- (ii) *system* (2) *undergoes a Hopf bifurcation at the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *when* $\tau_1 = \tau_{10}$.

Case 3 $\tau_1 = 0$, $\tau_2 > 0$. For $\tau_1 = 0$, equation (6) becomes

$$\begin{aligned} &\lambda^4 + m_{33}\lambda^3 + m_{32}\lambda^2 + m_{31}\lambda + m_{30} \\ &\quad + (p_{33}\lambda^3 + p_{32}\lambda^2 + p_{31}\lambda + p_{30})e^{-\lambda\tau_2} \\ &\quad + (r_{32}\lambda^2 + r_{31}\lambda + r_{30})e^{-2\lambda\tau_2} = 0, \end{aligned} \tag{17}$$

where

$$\begin{aligned} &m_{30} = m_0 + n_0, \qquad m_{31} = m_1 + n_1, \qquad m_{32} = m_2 + n_2, \qquad m_{33} = m_3 + n_3, \\ &p_{30} = p_0 + q_0, \qquad p_{31} = p_1 + q_1, \qquad p_{32} = p_2 + q_2, \qquad p_{33} = q_3, \\ &r_{30} = r_0 + s_0, \qquad r_{31} = r_1 + s_1, \qquad r_{32} = r_2. \end{aligned}$$

Multiplying by $e^{\lambda\tau_2}$, equation (17) becomes

$$\begin{aligned} &p_{33}\lambda^3 + p_{32}\lambda^2 + p_{31}\lambda + p_{30} \\ &\quad + (\lambda^4 + m_{33}\lambda^3 + m_{32}\lambda^2 + m_{31}\lambda + m_{30})e^{\lambda\tau_2} \\ &\quad + (r_{32}\lambda^2 + r_{31}\lambda + r_{30})e^{-\lambda\tau_2} = 0. \end{aligned} \tag{18}$$

Let $\lambda = i\omega_2$ ($\omega_2 > 0$) be the root of equation (18), then

$$\begin{cases} M_{31}(\omega_2)\cos\tau_2\omega_2 - M_{32}(\omega_2^2)\sin\tau_2\omega_2 = M_{33}(\omega_2), \\ M_{33}(\omega_2)\sin\tau_2\omega_2 + M_{35}(\omega_2^2)\cos\tau_2\omega_2 = M_{36}(\omega_2), \end{cases} \tag{19}$$

where

$$M_{31}(\omega_2) = \omega_2^4 - (m_{32} + r_{32})\omega_2^2 + m_{30} + r_{30},$$
$$M_{32}(\omega_2) = (m_{31} - r_{31})\omega_2 - m_{33}\omega_2^3, \qquad M_{33}(\omega_2) = p_{32}\omega_2^2 - p_{30},$$
$$M_{34}(\omega_2) = \omega_2^4 - (m_{32} - r_{32})\omega_2^2 + m_{30} - r_{30},$$
$$M_{35}(\omega_2) = (m_{31} + r_{31})\omega_2 - m_{33}\omega_2^3, \qquad M_{36}(\omega_2) = p_{33}\omega_2^2 - p_{31}\omega_2.$$

Then we have

$$\cos\tau_2\omega_2 = \frac{M_{32}(\omega_2)\times M_{36}(\omega_2) + M_{33}(\omega_2)\times M_{34}(\omega_2)}{M_{31}(\omega_2)\times M_{34}(\omega_2) + M_{32}(\omega_2)\times M_{35}(\omega_2)},$$
$$\sin\tau_2\omega_2 = \frac{M_{31}(\omega_2)\times M_{36}(\omega_2) - M_{33}(\omega_2)\times M_{35}(\omega_2)}{M_{31}(\omega_2)\times M_{34}(\omega_2) + M_{32}(\omega_2)\times M_{35}(\omega_2)}.$$

Therefore, we can obtain the following equation with respect to ω_2:

$$\cos^2\tau_2\omega_2 + \sin^2\tau_2\omega_2 = 1. \tag{20}$$

Next, we make the following assumption. (H_{31}): equation (20) has at least one positive root. If the condition (H_{31}) holds, then equation (20) has one positive root ω_{20} such that equation (18) has a pair of purely imaginary roots $\pm i\omega_{20}$. For ω_{20}, we have

$$\tau_{20} = \frac{1}{\omega_{20}}\arccos\frac{M_{32}(\omega_{20})\times M_{36}(\omega_{20}) + M_{33}(\omega_{20})\times M_{34}(\omega_{20})}{M_{31}(\omega_{20})\times M_{34}(\omega_{20}) + M_{32}(\omega_{20})\times M_{35}(\omega_{20})}. \tag{21}$$

In the following, we can obtain

$$\begin{aligned}\left[\frac{d\lambda}{d\tau_2}\right]^{-1} &= \bigl(3p_{33}\lambda^2 + 2p_{32}\lambda + p_{31} + \bigl(4\lambda^3 + 3m_{33}\lambda^2 + 2m_{32}\lambda + m_{31}\bigr)e^{\lambda\tau_2} \\ &\quad + (2r_{32}\lambda + r_{31})e^{-\lambda\tau_2}\bigr) \\ &\quad \big/\bigl(\lambda\bigl(r_{32}\lambda^2 + r_{31}\lambda + r_{30}\bigr)e^{-\lambda\tau_2} - \lambda\bigl(\lambda^4 + m_{33}\lambda^3 + m_{32}\lambda^2 + m_{31}\lambda + m_{30}\bigr)e^{\lambda\tau_2}\bigr) \\ &\quad - \frac{\tau_2}{\lambda},\end{aligned}$$

which leads to

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{20}} = \frac{P_{3R}Q_{3R} + P_{3I}Q_{3I}}{Q_{3R}^2 + Q_{3I}^2},$$

where

$$P_{3R} = \bigl(m_{31} + r_{31} - 3m_{33}\omega_{20}^2\bigr)\cos\tau_{20}\omega_{20} - 2\bigl((m_{32} - r_{32})\omega_{20} - 2\omega_{20}^3\bigr)\sin\tau_{20}\omega_{20},$$
$$P_{3I} = \bigl(m_{31} - r_{31} - 3m_{33}\omega_{20}^2\bigr)\sin\tau_{20}\omega_{20} - 2\bigl((m_{32} + r_{32})\omega_{20} - 2\omega_{20}^3\bigr)\cos\tau_{20}\omega_{20},$$
$$\begin{aligned}Q_{3R} &= \bigl[(r_{30} + m_{30})\omega_{20} - (r_{32} + m_{32})\omega_{20}^3 - \omega_{20}^5\bigr]\sin\tau_{20}\omega_{20} \\ &\quad - \bigl((r_{31} - m_{31})\omega_{20}^2 + m_{33}\omega_{20}^4\bigr)\cos\tau_{20}\omega_{20},\end{aligned}$$

$$Q_{3I} = \left[(r_{30} - m_{30})\omega_{20} - (r_{32} - m_{32})\omega_{20}^3 - \omega_{20}^5\right]\cos\tau_{20}\omega_{20} + \left((r_{31} + m_{31})\omega_{20}^2 - m_{33}\omega_{20}^4\right)\sin\tau_{20}\omega_{20}.$$

Obviously, if the condition (H_{32}) $P_{3R}Q_{3R} + P_{3I}Q_{3I}$ holds, then $\mathrm{Re}[\frac{d\lambda}{d\tau_2}]^{-1}_{\tau_2=\tau_{20}} \neq 0$. That is, if the condition (H_{32}) holds, then the transversality condition is satisfied. Thus, we have the following results according to the Hopf bifurcation theorem in [22].

Theorem 2 *If the conditions* (H_{31})-(H_{32}) *hold, then*:
(i) *the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *of system* (2) *is asymptotically stable for* $\tau_2 \in [0, \tau_{20})$;
(ii) *system* (2) *undergoes a Hopf bifurcation at the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *when* $\tau_2 = \tau_{20}$.

Case 4 $\tau_1 = \tau_2 = \tau > 0$. For $\tau_1 = \tau_2 = \tau$, equation (6) becomes

$$\lambda^4 + m_{43}\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40} + \left(n_{43}\lambda^3 + n_{42}\lambda^2 + n_{41}\lambda + n_{40}\right)e^{-\lambda\tau} + \left(q_{42}\lambda^2 + q_{41}\lambda + q_{40}\right)e^{-2\lambda\tau} + (s_{41}\lambda + s_{40})e^{-3\lambda\tau} = 0, \tag{22}$$

where

$$m_{40} = m_0, \quad m_{41} = m_1, \quad m_{42} = m_2, \quad m_{43} = m_3,$$
$$n_{40} = n_0 + p_0, \quad n_{41} = n_1 + p_1, \quad n_{42} = n_2 + p_2, \quad n_{43} = n_3 + p_3,$$
$$q_{40} = q_0 + r_0, \quad q_{41} = q_1 + r_1, \quad q_{42} = q_2 + r_2, \quad s_{40} = s_0, \quad s_{41} = s_1.$$

Multiplying by $e^{\lambda\tau_2}$, equation (22) becomes

$$n_{43}\lambda^3 + n_{42}\lambda^2 + n_{41}\lambda + n_{40} + \left(\lambda^4 + m_{43}\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40}\right)e^{\lambda\tau} + \left(q_{42}\lambda^2 + q_{41}\lambda + q_{40}\right)e^{-\lambda\tau} + (s_{41}\lambda + s_{40})e^{-2\lambda\tau} = 0. \tag{23}$$

Let $\lambda = i\omega$ ($\omega > 0$) be the root of equation (23), then

$$\begin{cases} g_{41}\cos\tau\omega - g_{42}\sin\tau\omega + g_{43} = h_{41}\sin 2\tau\omega - h_{42}\cos 2\tau\omega, \\ g_{44}\sin\tau\omega + g_{45}\cos\tau\omega + g_{46} = h_{41}\sin 2\tau\omega + h_{42}\cos 2\tau\omega, \end{cases} \tag{24}$$

where

$$g_{41} = \omega^4 - (m_{42} + q_{42})\omega^2 + m_{40} + q_{40},$$
$$g_{42} = m_{43}\omega^3 - (m_{41} - q_{41})\omega,$$
$$g_{43} = n_{40} - n_{42}\omega^2,$$
$$g_{44} = \omega^4 - (m_{42} - q_{42})\omega^2 + m_{40} - q_{40},$$

$$g_{45} = (m_{41} + q_{41})\omega - m_{43}\omega^3,$$
$$g_{46} = n_{41}\omega - n_{43}\omega^3,$$
$$h_{41} = -s_{41}\omega, \qquad h_{42} = s_{40}.$$

Then we can get

$$(g_{41}\cos\tau\omega - g_{42}\sin\tau\omega + g_{43})^2 + (g_{44}\sin\tau\omega + g_{45}\cos\tau\omega + g_{46})^2 = h_{41}^2 + h_{42}^2. \tag{25}$$

As is well known, $\sin\tau\omega = \pm\sqrt{1-\cos^2\tau\omega}$. Therefore, we consider the following two cases.

Case I $\sin\tau\omega = \sqrt{1-\cos^2\tau\omega}$, then equation (25) can be transformed to the following form:

$$\begin{aligned}&\bigl(g_{41}\cos\tau\omega - g_{42}\sqrt{1-\cos^2\tau\omega} + g_{43}\bigr)^2 + \bigl(g_{44}\sqrt{1-\cos^2\tau\omega} + g_{45}\cos\tau\omega + g_{46}\bigr)^2\\ &\quad = h_{41}^2 + h_{42}^2,\end{aligned} \tag{26}$$

from which we can obtain

$$c_{44}\cos^4\tau\omega + c_{43}\cos^3\tau\omega + c_{42}\cos^2\tau\omega + c_{41}\cos\tau\omega + c_{40} = 0, \tag{27}$$

where

$$\begin{aligned}
c_{40} &= \bigl(g_{42}^2 + g_{43}^2 + g_{44}^2 + g_{46}^2 - h_{41}^2 - h_{42}^2\bigr)^2 - 4(g_{44}g_{46} - g_{42}g_{43})^2,\\
c_{41} &= 4\bigl(g_{42}^2 + g_{43}^2 + g_{44}^2 + g_{46}^2 - h_{41}^2 - h_{42}^2\bigr)^2(g_{41}g_{43} + g_{45}g_{46})\\
&\quad - 8(g_{44}g_{45} - g_{41}g_{42})(g_{44}g_{46} - g_{42}g_{43}),\\
c_{42} &= 4(g_{41}g_{43} + g_{45}g_{46})^2 - 4(g_{44}g_{45} - g_{41}g_{42})^2 + 4(g_{44}g_{46} - g_{42}g_{43})^2\\
&\quad + 2\bigl(g_{41}^2 + g_{45}^2 - g_{42}^2 - g_{44}^2\bigr)\bigl(g_{42}^2 + g_{43}^2 + g_{44}^2 + g_{46}^2 - h_{41}^2 - h_{42}^2\bigr),\\
c_{43} &= (g_{41}g_{43} + g_{45}g_{46})\bigl(g_{41}^2 + g_{45}^2 - g_{42}^2 - g_{44}^2\bigr) + 8(g_{44}g_{45} - g_{41}g_{42})(g_{44}g_{46} - g_{42}g_{43}),\\
c_{44} &= \bigl(g_{41}^2 + g_{45}^2 - g_{42}^2 - g_{44}^2\bigr)^2 + 4(g_{44}g_{45} - g_{41}g_{42})^2.
\end{aligned}$$

Let $\cos\tau\omega = v_4$ and denote

$$f_4(v_4) = v_4^4 + \frac{c_{43}}{c_{44}}v_4^3 + \frac{c_{42}}{c_{44}}v_4^2 + \frac{c_{41}}{c_{44}}v_4 + \frac{c_{40}}{c_{44}}.$$

Thus,

$$f_4'(v_4) = 4v_4^3 + \frac{3c_{43}}{c_{44}}v_4^2 + \frac{2c_{42}}{c_{44}}v_4 + \frac{c_{41}}{c_{44}}.$$

Set

$$4v_4^3 + \frac{3c_{43}}{c_{44}}v_4^2 + \frac{2c_{42}}{c_{44}}v_4 + \frac{c_{41}}{c_{44}} = 0. \tag{28}$$

Let $y_4 = v_4 + \frac{c_{43}}{4c_{44}}$. Then equation (28) becomes

$$y_4^3 + \gamma_{41} y_4 + \gamma_{40} = 0,$$

where

$$\gamma_{41} = \frac{c_{41}}{2c_{44}} - \frac{3c_{43}^2}{16c_{44}^2}, \qquad \gamma_{40} = \frac{c_{43}^3}{32c_{44}^3} - \frac{c_{42}c_{43}}{8c_{44}^2} + \frac{c_{41}}{c_{44}}.$$

Define

$$\beta_{41} = \left(\frac{\gamma_{40}}{2}\right)^2 + \left(\frac{\gamma_{41}}{3}\right)^3, \qquad \beta_{42} = \frac{-1+\sqrt{3}i}{2},$$

$$y_{41} = \sqrt[3]{-\frac{\gamma_{40}}{2} + \sqrt{\beta_{41}}} + \sqrt[3]{-\frac{\gamma_{40}}{2} - \sqrt{\beta_{41}}},$$

$$y_{42} = \sqrt[3]{-\frac{\gamma_{40}}{2} + \sqrt{\beta_{41}}}\beta_{42} + \sqrt[3]{-\frac{\gamma_{40}}{2} - \sqrt{\beta_{41}}}\beta_{42}^2,$$

$$y_{43} = \sqrt[3]{-\frac{\gamma_{40}}{2} + \sqrt{\beta_{41}}}\beta_{42}^2 + \sqrt[3]{-\frac{\gamma_{40}}{2} - \sqrt{\beta_{41}}}\beta_{42}.$$

Thus, we can obtain the expression of $\cos\tau\omega$ and we denote $f_{41}(\omega) = \cos\tau\omega$. Then we can obtain the expression of $\sin\tau\omega$ from equation (26) and we denote $f_2(\omega) = \sin\tau\omega$. Thus, a function with respect to ω in the following form can be obtained:

$$f_{41}^2(\omega) + f_{42}^2(\omega) = 1. \tag{29}$$

In order to obtain the main results in this section, we make the following assumption. (H_{41}): equation (29) has at least one positive root ω_{01}. For ω_{01}, the corresponding critical value of the delay is

$$\tau_{01} = \frac{1}{\omega_{01}} \arccos f_{41}(\omega_{01}). \tag{30}$$

Case II $\sin\tau\omega = -\sqrt{1-\cos^2\tau\omega}$, then equation (25) becomes

$$\begin{aligned}&\left(g_{41}\cos\tau\omega + g_{42}\sqrt{1-\cos^2\tau\omega} + g_{43}\right)^2 + \left(g_{45}\cos\tau\omega - g_{44}\sqrt{1-\cos^2\tau\omega} + g_{46}\right)^2\\ &\quad = h_{41}^2 + h_{42}^2.\end{aligned} \tag{31}$$

Similar to Case I, we can obtain the expression of $\cos\tau\omega$, which is denoted as $f_{41}'(\omega)$, and the expression of $\sin\tau\omega$, which is denoted as $f_{42}'(\omega)$. Then we get

$$\left(f_{41}'\right)^2(\omega) + \left(f_{42}'\right)^2(\omega) = 1. \tag{32}$$

We assumed that equation (32) has at least one positive root denoted as ω_{02}. Then we get the corresponding critical value of the delay

$$\tau_{02} = \frac{1}{\omega_{02}} \arccos f_{41}'(\omega_{01}). \tag{33}$$

Let

$$\tau_0 = \min\{\tau_{01}, \tau_{02}\}. \tag{34}$$

Thus, we know that equation (22) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_0$. Differentiating the two sides of equation (22) regarding τ, we can get

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{P_{41}(\lambda)}{Q_{41}(\lambda)} - \frac{\tau}{\lambda},$$

where

$$\begin{aligned}
P_{41}(\lambda) &= 4\lambda^3 + 3m_{43}\lambda^2 + 2m_{42}\lambda + m_{41} + \left(3n_{43}\lambda^3 + 2n_{42}\lambda + n_{41}\right)e^{-\lambda\tau} \\
&\quad + (2q_{42}\lambda + q_{41})e^{-2\lambda\tau} + s_{41}e^{-3\lambda\tau}, \\
Q_{41}(\lambda) &= \lambda\left(n_{43}\lambda^3 + n_{42}\lambda^2 + n_{41}\lambda + n_{40}\right)e^{-\lambda\tau} + 2\lambda\left(q_{42}\lambda^2 + q_{41}\lambda + q_{40}\right)e^{-2\lambda\tau} \\
&\quad + 3\lambda(s_{41}\lambda + s_{40})e^{-3\lambda\tau}.
\end{aligned}$$

Further we get

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{P_{4R}Q_{4R} + P_{4I}Q_{4I}}{Q_{4R}^2 + Q_{4I}^2},$$

where

$$\begin{aligned}
P_{4R} &= 2n_{42}\omega_0 \sin\tau_0\omega_0 + \left(n_{41} - 3n_{43}\omega_0^2\right)\cos\tau_0\omega_0 \\
&\quad + 2q_{42}\omega_0 \sin 2\tau_0\omega_0 + q_{41}\cos 2\tau_0\omega_0 \\
&\quad + s_{41}\cos 3\tau_0\omega_0 + m_{41} - 3m_{43}\omega_0^2, \\
P_{4I} &= 2n_{42}\omega_0 \cos\tau_0\omega_0 - \left(n_{41} - 3n_{43}\omega_0^2\right)\sin\tau_0\omega_0 \\
&\quad + 2q_{42}\omega_0 \cos 2\tau_0\omega_0 - q_{41}\sin 2\tau_0\omega_0 \\
&\quad - s_{41}\sin 3\tau_0\omega_0 + 2m_{42}\omega_0 - 4\omega_0^3, \\
Q_{4R} &= \left(n_{40}\omega_0 - n_{42}\omega_0^3\right)\sin\tau_0\omega_0 + \left(n_{43}\omega_0^4 - n_{41}\omega_0^2\right)\cos\tau_0\omega_0 \\
&\quad + 2\left(q_{40}\omega_0 - q_{42}\omega_0^3\right)\sin 2\tau_0\omega_0 - 2q_{41}\omega_0^2\cos 2\tau_0\omega_0 \\
&\quad + 3s_{40}\omega_0\sin 3\tau_0\omega_0 - 3s_{41}\omega_0^2\cos 3\tau_0\omega_0, \\
Q_{4I} &= \left(n_{40}\omega_0 - n_{42}\omega_0^3\right)\cos\tau_0\omega_0 - \left(n_{43}\omega_0^4 - n_{41}\omega_0^2\right)\sin\tau_0\omega_0 \\
&\quad + 2\left(q_{40}\omega_0 - q_{42}\omega_0^3\right)\cos 2\tau_0\omega_0 + 2q_{41}\omega_0^2\sin 2\tau_0\omega_0 \\
&\quad + 3s_{40}\omega_0\cos 3\tau_0\omega_0 + 3s_{41}\omega_0^2\sin 3\tau_0\omega_0.
\end{aligned}$$

Clearly, if the condition (H_{42}) $P_{4R}Q_{4R} + P_{4I}Q_{4I} \neq 0$ holds, then $\operatorname{Re}[\frac{d\lambda}{d\tau}]^{-1}_{\tau=\tau_0} \neq 0$. According to the discussion above and the Hopf bifurcation theorem in [22], we have the following results.

Theorem 3 *If the conditions* (H_{41})-(H_{42}) *hold, then*:

(i) *the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *of system* (2) *is asymptotically stable for* $\tau \in [0, \tau_0)$;

(ii) *system* (2) *undergoes a Hopf bifurcation at the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *when* $\tau = \tau_0$.

Case 5 $\tau_1 > 0$, $\tau_2 > 0$, and $\tau_2 \in (0, \tau_{20})$.

Let $\lambda = i\omega_{1*}$ ($\omega_{1*} > 0$) be the root of equation (3), then

$$\begin{cases} M_{51}(\omega_{1*}) \sin \tau_1\omega_{1*} + M_{52}(\omega_{1*}) \cos \tau_1\omega_{1*} = M_{53}(\omega_{1*}), \\ M_{51}(\omega_{1*}) \cos \tau_1\omega_{1*} - M_{52}(\omega_{1*}) \sin \tau_1\omega_{1*} = M_{54}(\omega_{1*}), \end{cases} \tag{35}$$

where

$$\begin{aligned} M_{51}(\omega_{1*}) &= n_1\omega_{1*} - n_3\omega_{1*}^3 + q_1\omega_{1*} \cos \tau_2\omega_{1*} - \left(q_0 - q_2\omega_{1*}^2\right) \sin \tau_2\omega_{1*} \\ &\quad + s_1\omega_{1*} \cos 2\tau_2\omega_{1*} - s_0 \sin 2\tau_2\omega_{1*}, \\ M_{52}(\omega_{1*}) &= n_0 - n_2\omega_{1*}^2 + q_1\omega_{1*} \sin \tau_2\omega_{1*} + \left(q_0 - q_2\omega_{1*}^2\right) \cos \tau_2\omega_{1*} \\ &\quad + s_1\omega_{1*} \sin 2\tau_2\omega_{1*} + s_0 \cos 2\tau_2\omega_{1*}, \\ M_{53}(\omega_{1*}) &= m_2\omega_{1*}^2 - \omega_{1*}^4 - m_0 - \left(p_0 - p_2\omega_{1*}^2\right) \cos \tau_2\omega_{1*} - \left(p_1\omega_{1*} - p_3\omega_{1*}^3\right) \sin \tau_2\omega_{1*} \\ &\quad - \left(r_0 - r_2\omega_{1*}^2\right) \cos 2\tau_2\omega_{1*} - r_1\omega_{1*} \sin 2\tau_2\omega_{1*}, \\ M_{53}(\omega_{1*}) &= m_3\omega_{1*}^3 - m_1\omega_{1*} + \left(p_0 - p_2\omega_{1*}^2\right) \sin \tau_2\omega_{1*} - \left(p_1\omega_{1*} - p_3\omega_{1*}^3\right) \cos \tau_2\omega_{1*} \\ &\quad + \left(r_0 - r_2\omega_{1*}^2\right) \sin 2\tau_2\omega_{1*} - r_1\omega_{1*} \cos 2\tau_2\omega_{1*}. \end{aligned}$$

Thus, we can obtain

$$\begin{aligned} &f_{50}(\omega_{1*}) + 2f_{51}(\omega_{1*}) \cos \tau_2\omega_{1*} + 2f_{52}(\omega_{1*}) \sin \tau_2\omega_{1*} \\ &\quad + 2f_{53}(\omega_{1*}) \cos 2\tau_2\omega_{1*} + 2f_{54}(\omega_{1*}) \sin 2\tau_2\omega_{1*} = 0, \end{aligned} \tag{36}$$

where

$$\begin{aligned} f_{50}(\omega_{1*}) &= \omega_{1*}^8 + \left(p_3^2 + m_3^2 - n_3^2 - 2m_2\right)\omega_{1*}^6 \\ &\quad + \left(m_2^2 + p_2^2 - n_2^2 - q_2^2 - 2m_1m_3 - 2p_1p_3 + 2n_1n_3 + 2m_0\right)\omega_{1*}^4 \\ &\quad + \left(p_1^2 + r_1^2 + m_1^2 - n_1^2 - q_1^2 - s_1^2 - 2m_0m_2 - 2p_0p_2 \right. \\ &\quad \left. + 2n_0n_2 - 2r_0r_2 + 2q_0q_2\right)\omega_{1*}^2 + p_0^2 + m_0^2 + r_0^2 - n_0^2 - q_0^2 - s_0^2, \\ f_{51}(\omega_{1*}) &= (m_3p_3 - p_2)\omega_{1*}^6 + (m_2p_2 + p_2r_2 - p_3r_1 - m_1p_3 - m_3p_1 \\ &\quad - n_2q_2 + n_3q_1 + p_0)\omega_{1*}^4 + (m_1p_1 - m_0p_2 - m_2p_0 - p_0r_2 \\ &\quad - p_2r_0 + p_1r_1 - n_1q_1 + n_0q_2 + n_2q_0 - q_1s_1 + q_2s_0)\omega_{1*}^2 \\ &\quad + m_0p_0 + p_0r_0 - n_0q_0 - q_0s_0, \\ f_{52}(\omega_{1*}) &= -p_3\omega_{1*}^7 + (m_2p_3 - m_3p_2 - p_3r_2 + n_3q_2 + p_1)\omega_{1*}^5 \\ &\quad + (m_1p_2 - m_2p_1 - m_0p_3 + m_3p_0 - p_2r_1 - p_1r_2 - p_3r_0 - n_1q_2 \end{aligned}$$

$$
\begin{aligned}
&\quad + n_2q_1 - n_3q_0 + s_1q_2)\omega_{1*}^3 \\
&\quad + (m_0p_1 - m_1p_0 + p_0r_1 - p_1r_0 - n_0q_1 + n_1q_0 + q_0s_1 - q_1s_0)\omega_{1*}, \\
f_{53}(\omega_{1*}) &= -r_2\omega_{1*}^6 + (m_2r_2 - m_3r_1 + n_3s_1 + r_0)\omega_{1*}^4 \\
&\quad + (m_1r_1 - m_0r_2 - m_2r_0 - n_1s_1 + n_2s_0)\omega_{1*}^2 + m_0r_0 - n_0s_0, \\
f_{54}(\omega_{1*}) &= (r_1 - m_3r_2)\omega_{1*}^5 + (m_1r_2 - m_2r_1 + m_3r_0 - n_3s_0 + n_2s_1)\omega_{1*}^3 \\
&\quad + (m_0r_1 - m_1r_0 - n_0s_1 + n_1s_0)\omega_{1*}.
\end{aligned}
$$

Suppose that (H_{51}): equation (36) has at least one positive root. Then there exists $\omega_{10}^* > 0$ such that equation (6) has a pair of purely imaginary roots $\pm i\omega_{10}^*$. For ω_{10}^*,

$$
\tau_{10}^* = \frac{1}{\omega_{10}^*} \arccos \frac{M_{51}(\omega_{10}^*) \times M_{54}(\omega_{10}^*) + M_{52}(\omega_{10}^*) \times M_{53}(\omega_{10}^*)}{M_{51}^2(\omega_{10}^*) + M_{52}^2(\omega_{10}^*)}. \tag{37}
$$

In addition, we can get

$$
\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{P_{51}(\lambda)}{Q_{51}(\lambda)} - \frac{\tau_1}{\lambda},
$$

with

$$
\begin{aligned}
P_{51}(\lambda) &= 4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1 + \left(3n_3\lambda^2 + 2n_2\lambda + n_1\right)e^{-\lambda\tau_1} \\
&\quad - \left(\tau_2p_3\lambda^3 - (3p_3 - \tau_2p_2)\lambda^2 - (2p_2 - \tau_2p_1)\lambda - p_1 + \tau_2p_0\right)e^{-\lambda\tau_2} \\
&\quad - \left(\tau_2q_2\lambda^2 - (2q_2 - \tau_2q_1)\lambda - q_1 + \tau_2q_0\right)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad - \left(2\tau_2r_2\lambda^2 - (2r_2 - 2\tau_2r_1)\lambda - r_1 + 2\tau_2r_0\right)e^{-2\lambda\tau_2} \\
&\quad - (2\tau_2s_2\lambda + 2\tau_2s_0)e^{-\lambda(\tau_1+2\tau_2)}, \\
Q_{51}(\lambda) &= \left(n_3\lambda^4 + n_2\lambda^3 + n_1\lambda^2 + n_0\lambda\right)e^{-\lambda\tau_1} + \left(q_2\lambda^3 + q_1\lambda^2 + q_0\lambda\right)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + \left(s_1\lambda^2 + s_0\lambda\right)e^{-\lambda(\tau_1+2\tau_2)}.
\end{aligned}
$$

Define

$$
\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}^*} = \frac{P_{5R}Q_{5R} + P_{5I}Q_{5I}}{Q_{5R}^2 + Q_{5I}^2}.
$$

Clearly, if the condition (H_{52}) $P_{5R}Q_{5R} + P_{5I}Q_{5I} \neq 0$ holds, then $\operatorname{Re}[\frac{d\lambda}{d\tau_1}]^{-1}_{\tau_1=\tau_{10}^*} \neq 0$. Therefore, we can have the following results according to the Hopf bifurcation theorem in [22].

Theorem 4 *If the conditions* (H_{51})-(H_{52}) *hold, then*:

(i) *the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *of system* (2) *is asymptotically stable for* $\tau_2 \in [0, \tau_{20}^*)$;

(ii) *system* (2) *undergoes a Hopf bifurcation at the positive equilibrium* $E_*(S_*, I_*, Q_*, R_*)$ *when* $\tau_2 = \tau_{20}^*$.

3 Properties of the Hopf bifurcation

In this section, we investigate the direction, stability, and period of the Hopf bifurcation for $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$. We assume that $\tau_{20}^* < \tau_{10}^*$, where $\tau_{20}^* \in (0, \tau_{20})$ in this section. Let $\tau_1 = \tau_{10}^* + \mu$, $\mu \in R$, $u_1 = S(\tau_1 t)$, $u_2 = I(\tau_1 t)$, $u_3 = Q(\tau_1 t)$, $u_4 = R(\tau_1 t)$. Then system (2) becomes

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{38}$$

where

$$L_\mu \phi = \left(\tau_{10}^* + \mu\right)\left(A_{\max}\phi(0) + B_{2\max}\phi\left(-\frac{\tau_{20}^*}{\tau_{10}^*}\right) + B_{1\max}\phi(-1)\right),$$

and

$$F(\mu, u_t) = \left(\tau_{10}^* + \mu\right)(F_1, F_2, 0, 0)^T,$$

with

$$\phi(\theta) = \left(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta)\right)^T \in C\left([-1, 0], R^4\right),$$

$$A_{\max} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_3 & \alpha_4 & 0 \\ 0 & 0 & 0 & \alpha_5 \end{pmatrix}, \qquad B_{1\max} = \begin{pmatrix} \beta_1 & \beta_2 & 0 & 0 \\ \beta_3 & \beta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{2\max} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta_5 & 0 & 0 \\ 0 & 0 & \beta_6 & 0 \\ 0 & \beta_7 & \beta_8 & 0 \end{pmatrix},$$

and

$$F_1 = \beta_{13}\phi_1(-1)\phi_2(-1) + \beta_{14}\phi_2^2(-1) + \beta_{15}\phi_1(-1)\phi_2^2(-1),$$
$$F_2 = \beta_{23}\phi_1(-1)\phi_2(-1) + \beta_{24}\phi_2^2(-1) + \beta_{25}\phi_1(-1)\phi_2^2(-1).$$

Therefore, according to the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ for $\theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C\left([-1, 0], R^4\right).$$

In fact, we choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_{10}^* + \mu)(A_{\max} + B_{1\max} + B_{2\max}), & \theta = 0, \\ (\tau_{10}^* + \mu)(B_{1\max} + B_{2\max}), & \theta \in [-\frac{\tau_{20}^*}{\tau_{10}^*}, 0), \\ (\tau_{10}^* + \mu)B_{1\max}, & \theta \in (-1, -\frac{\tau_{20}^*}{\tau_{10}^*}), \\ 0, & \theta = -1. \end{cases}$$

For $\phi \in C([-1,0],R^4)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (38) can be transformed into the following form:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t.$$

For $\varphi \in C([-1,0],(R^4)^*)$, the adjoint operator A^* of A is defined by

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases}$$

associated with a bilinear form

$$\langle \varphi(s),\phi(\theta)\rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \tag{39}$$

where $\eta(\theta) = \eta(\theta,0)$.

Next, we calculate the eigenvector $\rho(\theta)$ of $A(0)$ belonging to the eigenvalue $+i\tau_{10}^*\omega_{10}^*$ and the eigenvector $\rho^*(s)$ belonging to the eigenvalue $-i\tau_{10}^*\omega_{10}^*$. According to a direction computation, we obtain $\rho(\theta) = (1,\rho_2,\rho_3,\rho_4)^T e^{i\tau_{10}^*\omega_{10}^*\theta}$, $\rho^*(\theta) = V(1,\rho_2^*,\rho_3^*,\rho_4^*)e^{i\tau_{10}^*\omega_{10}^* s}$, where

$$\begin{aligned} \rho_2 &= \frac{i\omega_{10}^* - \alpha_1 - \beta_1 e^{-i\tau_{10}^*\omega_{10}^*}}{\beta_2 e^{-i\tau_{10}^*\omega_{10}^*}}, \qquad \rho_3 = \frac{\alpha_3(i\omega_{10}^* - \alpha_1 - \beta_1 e^{-i\tau_{10}^*\omega_{10}^*})}{\beta_2(i\omega_{10}^* - \alpha_4 - \beta_6)e^{-i\tau_{10}^*\omega_{10}^*}}, \\ \rho_4 &= \frac{\beta_7\rho_2 + \beta_8\rho_3}{i\omega_{10}^* - \alpha_5}, \qquad \rho_2^* = -\frac{i\omega_{10}^* + \alpha_1 + \beta_1 e^{i\tau_{10}^*\omega_{10}^*}}{\beta_3 e^{i\tau_{10}^*\omega_{10}^*}}, \\ \rho_3^* &= \frac{(\beta_2\beta_8 e^{i\tau_{10}^*\omega_{10}^*} + i\omega_{10}^* + \alpha_2 + \beta_4 e^{i\tau_{10}^*\omega_{10}^*} + \beta_5 e^{i\tau_{20}^*\omega_{10}^*})\rho_2^*}{\beta_4(i\omega_{10}^* + \alpha_4 + \beta_8 e^{i\tau_{20}^*\omega_{10}^*}) - \alpha_3\beta_8}, \\ \rho_4^* &= -\frac{(i\omega_{10}^* + \alpha_4 + \beta_6 e^{i\tau_{20}^*\omega_{10}^*})\rho_3^*}{\beta_8 e^{i\tau_{20}^*\omega_{10}^*}}. \end{aligned}$$

From equation (39), we have

$$\begin{aligned} \langle \rho^*(s),\rho(\theta)\rangle = \bar{V}\bigl[& 1 + \rho_2\bar{\rho}_2^* + \rho_3\bar{\rho}_3^* + \rho_4\bar{\rho}_4^* + \tau_{10}^* e^{-i\tau_{10}^*\omega_{10}^*}\bigl(\beta_1 + \beta_3\bar{\rho}_2^* + \rho_2\bigl(\beta\beta_2 + \beta_4\bar{\rho}_3^*\bigr)\bigr) \\ & + \tau_{20}^* e^{-i\tau_{20}^*\omega_{10}^*}\bigl(\rho\rho_2\bigl(\beta_5\bar{\rho}_2^* + \beta_7\bar{\rho}_4^*\bigr) + \rho_4\bigl(\beta_6\bar{\rho}_3^* + \beta_8\bar{\rho}_4^*\bigr)\bigr)\bigr]. \end{aligned}$$

Let $\langle\rho^*(s),\rho(\theta)\rangle = 1$, then

$$\begin{aligned}\bar{V} &= \big[1 + \rho_2\bar{\rho}_2^* + \rho_3\bar{\rho}_3^* + \rho_4\bar{\rho}_4^* + \tau_{10}^* e^{-i\tau_{10}^*\omega_{10}^*}\big(\beta_1 + \beta_3\bar{\rho}_2^* + \rho_2\big(\beta\beta_2 + \beta_4\bar{\rho}_3^*\big)\big) \\ &\quad + \tau_{20}^* e^{-i\tau_{20}^*\omega_{10}^*}\big(\rho\rho_2\big(\beta_5\bar{\rho}_2^* + \beta_7\bar{\rho}_4^*\big) + \rho_4\big(\beta_6\bar{\rho}_3^* + \beta_8\bar{\rho}_4^*\big)\big)\big]^{-1}.\end{aligned}$$

Next, we can get expressions of g_{20}, g_{11}, g_{02}, and g_{21}, which can determine the properties of the Hopf bifurcation by the algorithms introduced in [22], and using a similar computation process as used in [23, 24]:

$$\begin{aligned}
g_{20} &= 2\tau_{10}^*\bar{V}\big[\big(\beta_{13} + \beta_{23}\bar{\rho}_2^*\big)\rho^{(1)}(-1)\rho^{(2)}(-1) + \big(\beta_{14} + \beta_{24}\bar{\rho}_2^*\big)\big(\rho^{(2)}(-1)\big)^2\big], \\
g_{11} &= \tau_{10}^*\bar{V}\big[\big(\beta_{13} + \beta_{23}\bar{\rho}_2^*\big)\big(\rho^{(1)}(-1)\bar{\rho}^{(2)}(-1) + \bar{\rho}^{(1)}(-1)\rho^{(2)}(-1)\big) \\
&\quad + 2\big(\beta_{14} + \beta_{24}\bar{\rho}_2^*\big)\rho^{(2)}(-1)\bar{\rho}^{(2)}(-1)\big], \\
g_{02} &= 2\tau_{10}^*\bar{V}\big[\big(\beta_{13} + \beta_{23}\bar{\rho}_2^*\big)\bar{\rho}^{(1)}(-1)\bar{\rho}^{(2)}(-1) + \big(\beta_{14} + \beta_{24}\bar{\rho}_2^*\big)\big(\bar{\rho}^{(2)}(-1)\big)^2\big], \\
g_{21} &= 2\tau_{10}^*\bar{V}\bigg[\big(\beta_{13} + \beta_{23}\bar{\rho}_2^*\big)\bigg(W_{11}^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(-1)\bar{\rho}^{(2)}(-1) + W_{11}^{(2)}(-1)\rho^{(1)}(-1) \\
&\quad + \frac{1}{2}W_{20}^{(2)}(-1)\bar{\rho}^{(-1)}(-1)\bigg) \\
&\quad + \big(\beta_{14} + \beta_{24}\bar{\rho}_2^*\big)\big(2W_{11}^{(2)}(-1)\rho^{(2)}(-1) + W_{20}^{(2)}(-1)\bar{\rho}^{(2)}(-1)\big) \\
&\quad + \big(\beta_{15} + \beta_{25}\bar{\rho}_2^*\big)\big(\bar{\rho}^{(1)}(-1)\big(\rho^{(2)}(-1)\big)^2 + 2\rho^{(1)}(-1)\rho^{(2)}(-1)\bar{\rho}^{(2)}(-1)\big)\bigg],
\end{aligned}$$

with

$$\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}\rho(0)}{\tau_{10}^*\omega_{10}^*}e^{i\tau_{10}^*\omega_{10}^*\theta} + \frac{i\bar{g}_{02}\bar{\rho}(0)}{3\tau_{10}^*\omega_{10}^*}e^{-i\tau_{10}^*\omega_{10}^*\theta} + E_1 e^{2i\tau_{10}^*\omega_{10}^*\theta}, \\
W_{11}(\theta) &= -\frac{ig_{11}\rho(0)}{\tau_{10}^*\omega_{10}^*}e^{i\tau_{10}^*\omega_{10}^*\theta} + \frac{i\bar{g}_{11}\bar{\rho}(0)}{\tau_{10}^*\omega_{10}^*}e^{-i\tau_{10}^*\omega_{10}^*\theta} + E_2,
\end{aligned}$$

where E_1 and E_2 can be obtained by the following two equations, respectively:

$$\begin{aligned}
E_1 &= 2\begin{pmatrix} \alpha_1' & -\beta_2 e^{-2i\tau_{10}^*\omega_{10}^*} & 0 & 0 \\ -\beta_3 e^{-2i\tau_{10}^*\omega_{10}^*} & \alpha_2' & 0 & 0 \\ 0 & -\alpha_3 & \alpha_4' & 0 \\ 0 & -\beta_7 e^{-2i\tau_{20}^*\omega_{10}^*} & -\beta_8 e^{-2i\tau_{20}^*\omega_{10}^*} & \alpha_5' \end{pmatrix}^{-1}\begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ 0 \\ 0 \end{pmatrix}, \\
E_2 &= -\begin{pmatrix} \alpha_1 + \beta_1 & \beta_2 & a_{13} & 0 \\ \beta_3 & \alpha_2 + \beta_4 + \beta_5 & 0 & 0 \\ 0 & \alpha_3 & \alpha_4 + \beta_6 & 0 \\ 0 & \beta_7 & \beta_8 & \alpha_5 \end{pmatrix}^{-1}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1' &= 2i\omega_{10}^* - \alpha_1 - \beta_1 e^{-2i\tau_{10}^*\omega_{10}^*}, \\
\alpha_2' &= 2i\omega_{10}^* - \alpha_2 - \beta_4 e^{-2i\tau_{10}^*\omega_{10}^*} - \beta_5 e^{-2i\tau_{20}^*\omega_{10}^*},
\end{aligned}$$

$$\alpha_4' = 2i\omega_{10}^* - \alpha_4 - \beta_6 e^{-2i\tau_{20}^*\omega_{10}^*},$$
$$\alpha_5' = 2i\omega_{10}^* - \alpha_5,$$

and

$$\begin{aligned}
E_1^{(1)} &= \beta_{13}\rho^{(1)}(-1)\rho^{(2)}(-1) + \beta_{14}\big(\rho^{(2)}(-1)\big)^2, \\
E_1^{(2)} &= \beta_{23}\rho^{(1)}(-1)\rho^{(2)}(-1) + \beta_{24}\big(\rho^{(2)}(-1)\big)^2, \\
E_2^{(1)} &= \beta_{13}\big(\rho^{(1)}(-1)\bar{\rho}^{(2)}(-1) + \bar{\rho}^{(1)}(-1)\rho^{(2)}(-1)\big) + 2\beta_{14}\rho^{(2)}(-1)\bar{\rho}^{(2)}(-1), \\
E_2^{(2)} &= \beta_{23}\big(\rho^{(1)}(-1)\bar{\rho}^{(2)}(-1) + \bar{\rho}^{(1)}(-1)\rho^{(2)}(-1)\big) + 2\beta_{24}\rho^{(2)}(-1)\bar{\rho}^{(2)}(-1).
\end{aligned}$$

Then we get the following coefficients:

$$\begin{aligned}
C_1(0) &= \frac{i}{2\tau_{10}^*\omega_{10}^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_{10}^*)\}}, \\
\beta_2 &= 2\operatorname{Re}\big\{C_1(0)\big\}, \\
T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_{10}^*)\}}{\tau_{10}^*\omega_{10}^*}.
\end{aligned} \tag{40}$$

According to the analysis of the properties of Hopf bifurcation in [22], we have the following results.

Theorem 5 *For system* (2), *if* $\mu_2 > 0$ ($\mu_2 < 0$), *the Hopf bifurcation is supercritical* (*subcritical*). *If* $\beta_2 < 0$ ($\beta_2 > 0$), *the bifurcating periodic solutions are stable* (*unstable*). *If* $T_2 > 0$ ($T_2 < 0$), *the period of the bifurcating periodic solutions increases* (*decreases*).

4 Numerical solutions

In this section, a numerical example of system (2) is provided to illustrate the validity of our obtained theoretical results in Sections 2 and 3. We take $A = 10$, $\beta = 0.002$, $d = 0.01$, $a = 0.01$, $\sigma = 0.3$, $\gamma = 0.2$, $b = 0.25$, $p = 0.6$. Then we can get a specific case of system (2).

By a direct computation, we get $R_0 = 18.4911 > 1$ and the positive equilibrium $E_*(13.1765, 19.7365, 6.8848, 776.6628)$. It can be verified that the condition (H_1) and other conditions for existence of the Hopf bifurcation are satisfied.

For $\tau_1 > 0$, $\tau_2 = 0$. We can obtain $\omega_{10} = 3.6828$, $\tau_{10} = 0.5720$. According to Theorem 1, we know that $E_*(13.1765, 19.7365, 6.8848, 776.6628)$ is asymptotically stable when $\tau_1 \in [0, 0.5720)$, which can be illustrated by the simulation in Figure 1. However, once the value of τ_1 is above the critical value $\tau_{10} = 0.5720$, then $E_*(13.1765, 19.7365, 6.8848, 776.6628)$ will lose stability and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from $E_*(13.1765, 19.7365, 6.8848, 776.6628)$. This phenomenon is described in Figure 2. Similarly, we have $\omega_{20} = 1.9205$, $\tau_{20} = 1.1461$ for $\tau_1 = 0$, $\tau_2 > 0$, and $\omega_0 = 1.6026$, $\tau_0 = 0.4085$ for $\tau_1 = \tau_2 = \tau > 0$, respectively. The corresponding waveforms are omitted here.

Lastly, we have $\omega_{10}^* = 2.9448$, $\tau_{10}^* = 0.3727$ when $\tau_1 > 0$ and $\tau_2 = 0.45 \in (0, \tau_{20})$. The corresponding waveforms can be shown in Figures 3-4. Further, we obtain $C_1(0) = -7.6083 -$

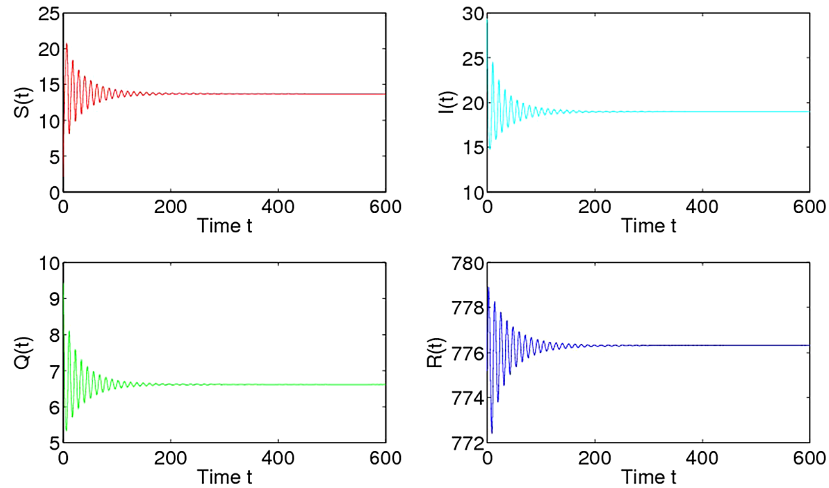

Figure 1 **The trajectory graphs with $\tau_1 = 0.45 < \tau_{10} = 0.5720$ and initial value data '2.15, 29.39, 9.36, 775.105'.**

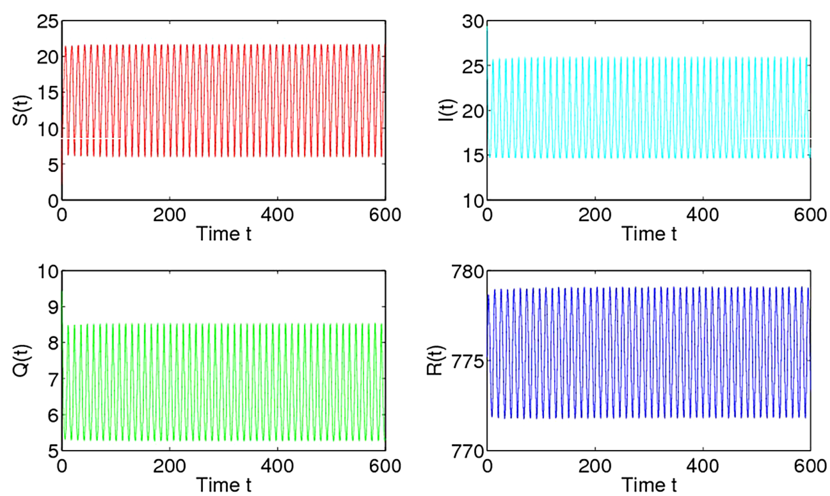

Figure 2 **The trajectory graphs with $\tau_1 = 0.75 < \tau_{10} = 0.5720$ and initial value data '2.15, 29.39, 9.36, 775.105'.**

$0.2944i$ and $\lambda(\tau_{10}^*) = 0.9642 + 1.0027i$. Further, we have $\mu_2 = 7.8908 > 0$, $\beta_2 = -15.2166 < 0$, and $T_2 = -6.9408 > 0$. Therefore, we can conclude that the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable and the period of the bifurcating periodic solutions decreases according to Theorem 5.

5 Conclusions

In the present paper, a delayed SIQR epidemic model with constant input and nonlinear incidence rate is investigated based on the model studied in [11]. Compared with the epidemic model studied in [11], we not only consider the time delay due to the latent period of the disease, but also the time delay due to the period that the infective and quarantined individuals need to be cured. Therefore, the proposed model in the present paper is more general. The main results are given in terms of local stability and Hopf bifurcation.

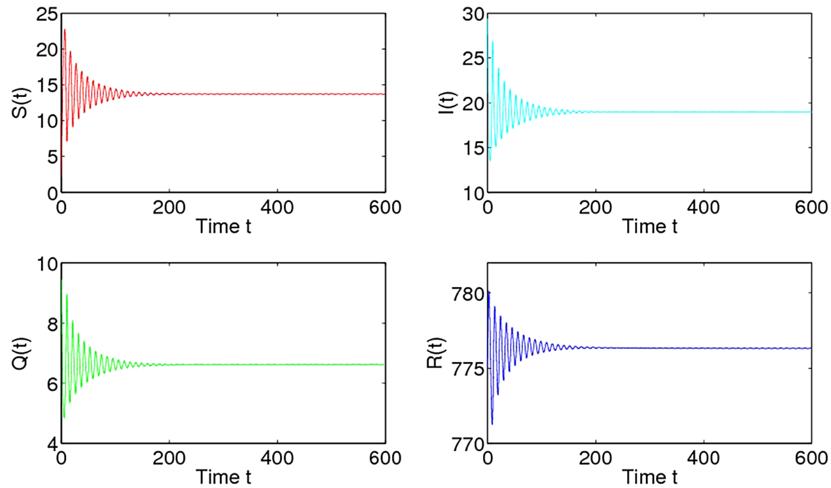

Figure 3 The trajectory graphs with $\tau_1 = 0.27 < \tau_{10}^* = 0.3727$, $\tau_2 = 0.45$ and initial value data '2.15, 29.39, 9.36, 775.105'.

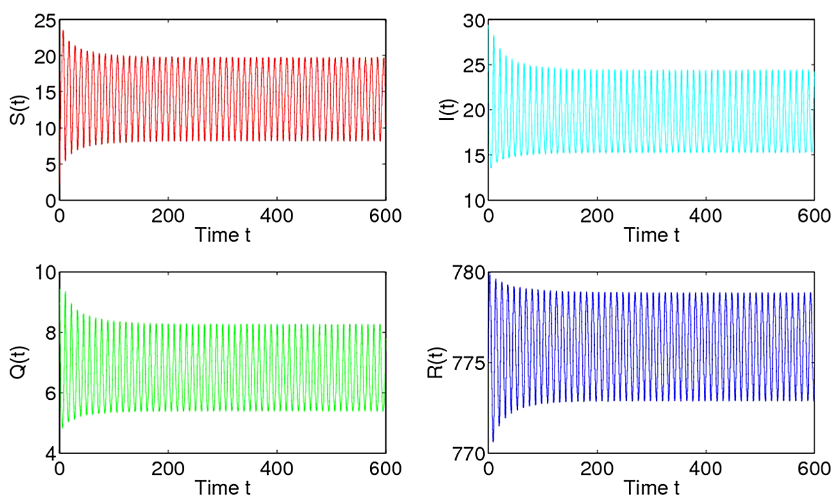

Figure 4 The trajectory graphs with $\tau_1 = 0.47 < \tau_{10}^* = 0.3727$, $\tau_2 = 0.45$ and initial value data '2.15, 29.39, 9.36, 775.105'.

Sufficient conditions for local stability and existence of the Hopf bifurcation are obtained by regarding different combination of the two delays as a bifurcation parameter and analyzing distribution of roots of corresponding characteristic equation. It is found that both the two delays can affect the stability of the model. When the delay is suitable small, the model is asymptotically stable. In this case, the disease can be controlled easily. However, once the value of the delay is above the critical value, the epidemic model will lose its stability and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from the positive equilibrium of the model. In this case, the disease is out of control and this is not helpful to predict the law of propagation of the disease. Therefore, we should take some measures such as introduced in [25] to control the occurrence of the Hopf bifurcation and we leave this for future work.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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