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Some identities of Laguerre polynomials arising from differential equations

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Abstract

In this paper, we derive a family of ordinary differential equations from the generating function of the Laguerre polynomials. Then these differential equations are used in order to obtain some properties and new identities for those polynomials.

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Keywords: Laguerre polynomials; differential equations

1 Introduction

The Laguerre polynomials, $L_n(x)$ $(n \geq 0)$, are defined by the generating function

$$\frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n \quad \text{(see [1, 2])}. \tag{1}$$

Indeed, the Laguerre polynomial $L_n(x)$ is a solution of the second order linear differential equation

$$xy'' + (1-x)y + ny = 0 \quad \text{(see [2–5])}. \tag{2}$$

From (1), we can get the following equation:

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n(x)t^n &= \frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m t^m}{m!}(1-t)^{-m-1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m x^m t^m}{m!} \sum_{l=0}^{\infty} \binom{m+l}{l} t^l \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{(-1)^m \binom{n}{m} x^m}{m!} \right) t^n.
\end{aligned} \tag{3}$$

Thus by (3), we get immediately the following equation:

$$L_n(x) = \sum_{m=0}^{n} \frac{(-1)^m \binom{n}{m} x^m}{m!} \quad (n \geq 0) \; \bigl(\text{see [2, 6–8]}\bigr). \tag{4}$$

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Alternatively, the Laguerre polynomials are also defined by the recurrence relation as follows:

$$\begin{aligned}
&L_0(x) = 1, \qquad L_1(x) = 1 - x, \\
&(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \quad (n \geq 1).
\end{aligned} \tag{5}$$

The Rodrigues' formula for the Laguerre polynomials is given by

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}\left(e^{-x}x^n\right) \quad (n \geq 0). \tag{6}$$

The first few of $L_n(x)$ $(n \geq 0)$ are

$$\begin{aligned}
&L_0(x) = 1, \\
&L_1(x) = 1 - x, \\
&L_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right), \\
&L_3(x) = \frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right), \\
&L_4(x) = \frac{1}{24}\left(x^4 - 16x^3 + 72x^2 - 96x + 24\right).
\end{aligned}$$

The Laguerre polynomials arise from quantum mechanics in the radial part of the solution of the Schrödinger equation for a one-electron action. They also describe the static Wigner functions of oscillator system in the quantum mechanics of phase space. They further enter in the quantum mechanics of the Morse potential and of the 3D isotropic harmonic oscillator (see [4, 5, 9]). A contour integral that is commonly taken as the definition of the Laguerre polynomial is given by

$$L_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{-xt}{1-t}}}{1-t} t^{-n-1}\, dt \quad \bigl(\text{see [4, 5, 10, 11]}\bigr), \tag{7}$$

where the contour encloses the origin but not the point $z = 1$.

FDEs (fractional differential equations) have wide applications in such diverse areas as fluid mechanics, plasma physics, dynamical processes and finance, *etc.* Most FDEs do not have exact solutions and hence numerical approximation techniques must be used. Spectral methods are widely used to numerically solve various types of integral and differential equations due to their high accuracy and employ orthogonal systems as basis functions. It is remarkable that a new family of generalized Laguerre polynomials are introduced in applying spectral methods for numerical treatments of FDEs in unbounded domains. They can also be used in solving some differential equations (see [12–17]).

Also, it should be mentioned that the modified generalized Laguerre operational matrix of fractional integration is applied in order to solve linear multi-order FDEs which are important in mathematical physics (see [12–17]).

Many authors have studied the Laguerre polynomials in mathematical physics, combinatorics and special functions (see [1–30]). For the applications of special functions and polynomials, one may referred to the papers (see [18, 19, 28]).

In [22], Kim studied nonlinear differential equations arising from Frobenius-Euler polynomials and gave some interesting identities. In this paper, we derive a family of ordinary differential equations from the generating function of the Laguerre polynomials. Then these differential equations are used in order to obtain some properties and new identities for those polynomials.

2 Laguerre polynomials arising from linear differential equations

Let

$$F = F(t,x) = \frac{1}{1-t}e^{\frac{-xt}{1-t}}. \tag{8}$$

From (8), we note that

$$F^{(1)} = \frac{dF(t,x)}{dt} = \bigl((1-t)^{-1} - x(1-t)^{-2}\bigr)F. \tag{9}$$

Thus, by (3), we get

$$F^{(2)} = \frac{dF^{(1)}}{dt} = \bigl(2(1-t)^{-2} - 4x(1-t)^{-3} + x^2(1-t)^{-4}\bigr)F \tag{10}$$

and

$$F^{(3)} = \frac{dF^{(2)}}{dt} = \bigl(6(1-t)^{-3} - 18x(1-t)^{-4} + 9x^2(1-t)^{-5} - x^3(1-t)^{-6}\bigr)F. \tag{11}$$

So we are led to put

$$F^{(N)} = \left(\sum_{i=N}^{2N} a_{i-N}(N,x)(1-t)^{-i}\right)F, \tag{12}$$

where $N = 0,1,2,\ldots$.

From (12), we can get equation (13):

$$\begin{aligned}
F^{(N+1)} &= \left(\sum_{i=N}^{2N} a_{i-N}(N,x)i(1-t)^{-i-1}\right)F + \left(\sum_{i=N}^{2N} a_{i-N}(N,x)(1-t)^{-i}\right)F^{(1)} \\
&= \left(\sum_{i=N}^{2N} a_{i-N}(N,x)i(1-t)^{-i-1}\right)F \\
&\quad + \left(\sum_{i=N}^{2N} a_{i-N}(N,x)(1-t)^{-i}\right)\bigl((1-t)^{-1} - x(1-t)^{-2}\bigr)F \\
&= \left(\sum_{i=N}^{2N} (i+1)a_{i-N}(N,x)(1-t)^{-i-1} - x\sum_{i=N}^{2N} a_{i-N}(N,x)(1-t)^{-i-2}\right)F \\
&= \left(\sum_{i=N+1}^{2N+1} ia_{i-N-1}(N,x)(1-t)^{-i} - x\sum_{i=N+2}^{2N+2} a_{i-N-2}(N,x)(1-t)^{-i}\right)F.
\end{aligned} \tag{13}$$

Replacing N by $N+1$ in (12), we get

$$F^{(N+1)} = \left(\sum_{i=N+1}^{2N+2} a_{i-N-1}(N+1,x)(1-t)^{-i}\right)F. \tag{14}$$

Comparing the coefficients on both sides of (13) and (14), we have

$$a_0(N+1,x) = (N+1)a_0(N,x), \tag{15}$$

$$a_{N+1}(N+1,x) = -xa_N(N,x), \tag{16}$$

and

$$a_{i-N-1}(N+1,x) = ia_{i-N-1}(N,x) - xa_{i-N-2}(N,x) \quad (N+2 \le i \le 2N+1). \tag{17}$$

We note that

$$F = F^{(0)} = a_0(0,x)F. \tag{18}$$

Thus, by (18), we get

$$a_0(0,x) = 1. \tag{19}$$

From (9) and (12), we note that

$$\big((1-t)^{-1} - x(1-t)^{-2}\big)F = F^{(1)} = \big(a_0(1,x)(1-t)^{-1} + a_1(1,x)(1-t)^{-2}\big)F. \tag{20}$$

Thus, by comparing the coefficients on both sides of (20), we get

$$a_0(1,x) = 1, \qquad a_1(1,x) = -x. \tag{21}$$

From (15), (16), we get

$$\begin{aligned} a_0(N+1,x) &= (N+1)a_N(N,x) = (N+1)Na_{N-1}(N-1,x)\cdots \\ &= (N+1)N(N-1)\cdots 2a_0(1,x) = (N+1)! \end{aligned} \tag{22}$$

and

$$\begin{aligned} a_{N+1}(N+1,x) &= (-x)a_N(N,x) = (-x)^2 a_{N-1}(N-1,x)\cdots \\ &= (-x)^N a_1(1,x) = (-x)^{N+1}. \end{aligned} \tag{23}$$

We observe that the matrix $[a_i(j,x)]_{0\le i,j\le N}$ is given by

$$\begin{bmatrix} 1 & 1! & 2!\cdots & N! \\ 0 & (-x) & \cdots & \\ 0 & 0 & (-x)^2 & \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & (-x)^N \end{bmatrix}.$$

From (17), we can get the following equations:

$$\begin{aligned}
a_1(N+1,x) &= -xa_0(N,x) + (N+2)a_1(N,x) \\
&= -x\{a_0(N,x) + (N+2)a_0(N-1,x)\} + (N+2)(N+1)a_1(N-1,x) \\
&= \cdots \\
&= -x\sum_{i=0}^{N-1}(N+2)_i a_0(N-i,x) + (N+2)(N+1)\cdots 3a_1(1,x) \\
&= -x\sum_{i=0}^{N-1}(N+2)_i a_0(N-i,x) + (N+2)(N+1)\cdots 3(-x) \\
&= -x\sum_{i=0}^{N}(N+2)_i a_0(N-i,x), \qquad (24)
\end{aligned}$$

$$\begin{aligned}
a_2(N+1,x) &= -xa_1(N,x) + (N+3)a_2(N,x) \\
&= -x\{a_1(N,x) + (N+3)a_1(N-1,x)\} + (N+3)(N+2)a_2(N-1,x) \\
&= \cdots \\
&= -x\sum_{i=0}^{N-2}(N+3)_i a_1(N-i,x) + (N+3)(N+2)\cdots 5a_2(2,x) \\
&= -x\sum_{i=0}^{N-2}(N+3)_i a_1(N-i,x) + (N+3)(N+2)\cdots 5(-x)^2 \\
&= -x\sum_{i=0}^{N-1}(N+3)_i a_1(N-i,x), \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
a_3(N+1,x) &= -xa_2(N,x) + (N+4)a_3(N,x) \\
&= -x\{a_2(N,x) + (N+4)a_2(N-1,x)\} + (N+4)(N+3)a_3(N-1,x) \\
&= \cdots \\
&= -x\sum_{i=0}^{N-3}(N+4)_i a_2(N-i,x) + (N+4)(N+3)\cdots 7a_3(3,x) \\
&= -x\sum_{i=0}^{N-3}(N+4)_i a_2(N-i,x) + (N+4)(N+3)\cdots 7(-x)^3 \\
&= -x\sum_{i=0}^{N-2}(N+4)_i a_2(N-i,x), \qquad (26)
\end{aligned}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \geq 1)$, and $(x)_0 = 1$.

Continuing this process, we have

$$a_j(N+1,x) = -x\sum_{i=0}^{N-j+1}(N+j+1)_i a_{j-1}(N-i,x), \qquad (27)$$

where $j = 1, 2, \ldots, N$. Now we give explicit expressions for $a_j(N+1,x)$, $j = 1, 2, \ldots, N$. From (22) and (24), we note that

$$\begin{aligned} a_1(N+1,x) &= -x\sum_{i_1=0}^{N}(N+2)_{i_1}a_0(N-i_1,x) \\ &= -x\sum_{i_1=0}^{N}(N+2)_{i_1}(N-i_1)!. \end{aligned} \tag{28}$$

By (25) and (28), we get

$$\begin{aligned} a_2(N+1,x) &= -x\sum_{i_2=0}^{N-1}(N+3)_{i_2}a_1(N-i_2,x) \\ &= (-x)^{-2}\sum_{i_2=0}^{N-1}\sum_{i_1=0}^{N-i_2-1}(N+3)_{i_2}(N-i_2+1)_{i_1}(N-i_2-i_1-1)!. \end{aligned} \tag{29}$$

From (26) and (29), we get

$$\begin{aligned} a_3(N+1,x) &= -x\sum_{i_3=0}^{N-2}(N+4)_{i_3}a_2(N-i_3,x) \\ &= (-x)^{-3}\sum_{i_3=0}^{N-2}\sum_{i_2=0}^{N-i_3-2}\sum_{i_1=0}^{N-i_3-i_2-2}(N+4)_{i_3}(N-i_3+2)_{i_2}(N-i_3-i_2)_{i_1} \\ &\quad \times (N-i_3-i_2-i_1-2)!. \end{aligned} \tag{30}$$

By continuing this process, we get

$$\begin{aligned} a_j(N+1,x) &= (-x)^j\sum_{i_j=0}^{N-j+1}\sum_{i_{j-1}=0}^{N-i_j-j+1}\cdots\sum_{i_1=0}^{N-i_j-\cdots-i_2-j+1}(N+j+1)_{i_j} \\ &\quad \times\left(\prod_{k=2}^{j}N-i_j-\cdots-i_k-\bigl(j-(2k-1)\bigr)_{i_{k-1}}\right) \\ &\quad \times (N-i_j-\cdots-i_1-j+1)!. \end{aligned} \tag{31}$$

Therefore, we obtain the following theorem.

Theorem 1 *The linear differential equation*

$$F^{(N)} = \left(\sum_{i=N}^{2N}a_{i-N}(N,x)(1-t)^{-i}\right)F \quad (N \in \mathbb{N})$$

has a solution $F = F(t,x) = (1-t)^{-1}\exp(-\frac{xt}{1-t})$, *where* $a_0(N,x) = N!$, $a_N(N,x) = (-x)^N$,

$$a_j(N,x) = (-x)^j \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-i_j-j} \cdots \sum_{i_1=0}^{N-i_j-\cdots-i_2-j} (N+j)_{i_j} \times \left(\prod_{k=2}^{j} \bigl(N - i_j - \cdots - i_k - \bigl(j - (2k-2)\bigr)\bigr)_{i_{k-1}} \right) (N - i_j - \cdots - i_1 - j)!.$$

From (1), we note that

$$F = F(t,x) = \frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n. \tag{32}$$

Thus, by (32), we get

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \sum_{n=N}^{\infty} L_n(x)(n)_N t^{n-N} = \sum_{n=0}^{\infty} L_{n+N}(x)(n+N)_N t^n. \tag{33}$$

On the other hand, by Theorem 1, we have

$$\begin{aligned} F^{(N)} &= \left(\sum_{i=N}^{2N} a_{i-N}(N,x)(1-t)^{-i} \right) F \\ &= \sum_{i=N}^{2N} a_{i-N}(N,x) \sum_{l=0}^{\infty} \binom{i+l-1}{l} t^l \sum_{k=0}^{\infty} L_k(x) t^k \\ &= \sum_{i=N}^{2N} a_{i-N}(N,x) \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{i+l-1}{l} L_{n-l}(x) \right) t^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=N}^{2N} a_{i-N}(N,x) \sum_{l=0}^{N} \binom{i+l-1}{l} L_{n-l}(x) \right) t^n. \end{aligned} \tag{34}$$

Therefore, by comparing the coefficients on both sides of (33) and (34), we have the following theorem.

Theorem 2 *For* $n \in \mathbb{N} \cup \{0\}$ *and* $N \in \mathbb{N}$, *we have*

$$L_{n+N}(x) = \frac{1}{(n+N)_N} \sum_{i=N}^{2N} a_{i-N}(N,x) \sum_{l=0}^{N} \binom{i+l-1}{l} L_{n-l}(x),$$

where $a_0(N,x) = N!$, $a_N(N,x) = (-x)^N$,

$$a_j(N,x) = (-x)^j \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-i_j-j} \cdots \sum_{i_1=0}^{N-i_j-\cdots-i_2-j} (N+j)_{i_j} \times \left(\prod_{k=2}^{j} \bigl(N - i_j - \cdots - i_k - \bigl(j - (2k-2)\bigr)\bigr)_{i_{k-1}} \right) (N - i_j - \cdots - i_1 - j)!.$$

3 Conclusion

It has been demonstrated that it is a fascinating idea to use differential equations associated with the generating function (or a slight variant of generating function) of special polynomials or numbers. Immediate applications of them have been in deriving interesting identities for the special polynomials or numbers. Along this line of research, here we derived a family of differential equations from the generating function of the Laguerre polynomials. Then from these differential equations we obtained interesting new identities for those polynomials.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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References
1. Kim, T: Identities involving Laguerre polynomials derived from umbral calculus. Russ. J. Math. Phys. **21**(1), 36-45 (2014)
2. Zill, DG, Cullen, MR: Advanced Engineering Mathematics. Jones & Bartlett, Boston (2005)
3. Abramowitz, M, Stegun, IA: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards Applied Mathematics Series, vol. 55. U.S. Government Printing Office, Washington (1964)
4. Arfken, G, Weber, H: Mathematical Methods for Physicists. Academic Press, San Diego (2000)
5. Bhrawy, AH, Alghamdi, MA: The operational matrix of Caputo fractional derivatives of modified generalized Laguerre polynomials and its applications. Adv. Differ. Equ. **2013**, Article ID 307 (2013)
6. Srivastava, HM, Lin, S-D, Liu, S-J, Lu, H-C: Integral representations for the Lagrange polynomials, Shively's pseudo-Laguerre polynomials, and the generalized Bessel polynomials. Russ. J. Math. Phys. **19**(1), 121-130 (2012)
7. Uspensky, JV: On the development of arbitrary functions in series of Hermite's and Laguerre's polynomials. Ann. Math. (2) **28**(1-4), 593-619 (1926/1927)
8. Watson, GN: An integral equation for the square of a Laguerre polynomial. J. Lond. Math. Soc. **S1-11**(4), 256 (1936)
9. Karaseva, IA: Fast calculation of signal delay in RC-circuits based on Laguerre functions. Russ. J. Numer. Anal. Math. Model. **26**(3), 295-301 (2011)
10. Carlitz, L: Some generating functions for Laguerre polynomials. Duke Math. J. **35**, 825-827 (1968)
11. Carlitz, L: The product of several Hermite or Laguerre polynomials. Monatshefte Math. **66**, 393-396 (1962)
12. Baleanu, D, Bhrawy, AH, Taha, TM: Two efficient generalized Laguerre spectral algorithms for fractional initial value problems. Abstr. Appl. Anal. **2013**, Article ID 546502 (2013)
13. Bhrawy, AH, Abdelkawy, MA, Alzahrani, AA, Baleanu, D, Alzahrani, EO: A Chebyshev-Laguerre-Gauss-Radau collocation scheme for solving a time fractional sub-diffusion equation on a semi-infinite domain. Proc. Rom. Acad., Ser. A : Math. Phys. Tech. Sci. Inf. Sci. **16**, 490-498 (2015)
14. Bhrawy, AH, Alhamed, YA, Baleanu, D, Al-Zahrani, AA: New spectral techniques for systems of fractional differential equations using fractional-order generalized Laguerre orthogonal functions. Fract. Calc. Appl. Anal. **17**, 1137-1157 (2014)
15. Bhrawy, AH, Alghamdi, MM, Taha, TM: A new modified generalized Laguerre operational matrix of fractional integration for solving fractional differential equations on the half line. Adv. Differ. Equ. **2012**, Article ID 179 (2012)
16. Bhrawy, AH, Hafez, RM, Alzahrani, EO, Baleanu, D, Alzahrani, AA: Generalized Laguerre-Gauss-Radau scheme for first order hyperbolic equations on semi-infinite domains. Rom. J. Phys. **60**, 918-934 (2015)
17. Bhrawy, AH, Taha, TM, Alzahrani, EO, Baleanu, D, Alzahrani, AA: New operational matrices for solving fractional differential equations on the half-line. PLoS ONE **10**(5), e0126620 (2015). doi:10.1371/journal.pone.0126620
18. Chaurasia, VBL, Kumar, D: On the solutions of integral equations of Fredholm type with special functions. Tamsui Oxf. J. Inf. Math. Sci. **28**, 49-61 (2012)
19. Chaurasia, VBL, Kumar, D: The integration of certain product involving special functions. Scientia, Ser. A, Math. Sci. **19**, 7-12 (2010)
20. Chen, Y, Griffin, J: Deformed q^{-1}-Laguerre polynomials, recurrence coefficients, and non-linear difference equations. Acta Phys. Pol. A **46**(9), 1871-1881 (2015)
21. Hegazi, AS, Mansour, M: Generalized q-modified Laguerre functions. Int. J. Theor. Phys. **41**(9), 1803-1813 (2002)

22. Kim, T: Identities involving Frobenius-Euler polynomials arising from non-linear differential equations. J. Number Theory **132**(12), 2854-2865 (2012)
23. Kim, T, Kim, DS: Extended Laguerre polynomials associated with Hermite, Bernoulli, and Euler numbers and polynomials. Abstr. Appl. Anal. **2012**, Article ID 957350 (2012)
24. Kim, T, Rim, S-H, Dolgy, DV, Lee, S-H: Some identities on Bernoulli and Euler polynomials arising from the orthogonality of Laguerre polynomials. Adv. Differ. Equ. **2012**, Article ID 201 (2012)
25. Koepf, W: Identities for families of orthogonal polynomials and special functions. Integral Transforms Spec. Funct. **5**, 69-102 (1997)
26. Filipuk, G, Smet, C: On the recurrence coefficients for generalized q-Laguerre polynomials. J. Nonlinear Math. Phys. **20**(Suppl. 1), 48-56 (2013)
27. Molano, LAM: An electrostatic model for zeros of classical Laguerre polynomials perturbed by a rational factor. Math. Sci. **8**(2), Article ID 120 (2014)
28. Singh, J, Kumar, D: On the distribution of mixed sum of independent random variables one of them associated with Srivastava's polynomials and H-function. J. Appl. Math. Stat. Inform. **10**, 53-62 (2014)
29. Spain, B, Smith, MG: Functions of Mathematical Physics. Van Nostrand Reinhold Company, London (1970). Chapter 10 deals with Laguerre polynomials
30. Spencer, VE: Asymptotic expressions for the zeros of generalized Laguerre polynomials and Weber functions. Duke Math. J. **3**(4), 667-675 (1937)