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# Connections between Hyers-Ulam stability and uniform exponential stability of discrete evolution families of bounded linear operators over Banach spaces

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## Abstract

In this article, we prove that the  $\omega$ -periodic discrete evolution family  $\Gamma := \{\rho(n, k) : n, k \in \mathbb{Z}_+, n \geq k\}$  of bounded linear operators is Hyers-Ulam stable if and only if it is uniformly exponentially stable under certain conditions. More precisely, we prove that if for each real number  $\gamma$  and each sequence  $(\xi(n))$  taken from some Banach space, the approximate solution of the nonautonomous  $\omega$ -periodic discrete system  $\theta_{n+1} = \Lambda_n \theta_n$ ,  $n \in \mathbb{Z}_+$  is represented by  $\phi_{n+1} = \Lambda_n \phi_n + e^{i\gamma(n+1)} \xi(n+1)$ ,  $n \in \mathbb{Z}_+$ ;  $\phi_0 = \theta_0$ , then the Hyers-Ulam stability of the nonautonomous  $\omega$ -periodic discrete system  $\theta_{n+1} = \Lambda_n \theta_n$ ,  $n \in \mathbb{Z}_+$  is equivalent to its uniform exponential stability.

**MSC:** 39A30

**Keywords:** Hyers-Ulam stability; uniform exponential stability; discrete evolution family of bounded linear operators; periodic sequence

## 1 Introduction

The stability theory is an important research area of the qualitative analysis of differential equations and difference equations. Ulam [1] proposed a question regarding the stability of functional equations for homomorphism as follows: *when can an approximate homomorphism from a group  $G_1$  to a metric group  $G_2$  be approximated by an exact homomorphism?* Assuming that  $G_1$  and  $G_2$  are Banach spaces, Hyers [2] brilliantly gave the first result to this question. Aoki [3] and Rassias [4] generalized this result. In particular, Rassias [4] improved the condition for the bound of the norm of the Cauchy difference  $f(x+y) - f(x) - f(y)$ . Obłozja [5] established the connections between Hyers and Lyapunov stability of ordinary differential equations. Later on, Alsina and Ger [6] investigated the stability of the differential equation  $y'(x) = y(x)$ , which was then extended to the Banach space-valued differential equation  $y'(x) = \lambda y(x)$  by Takahasi *et al.* [7]. We also refer the reader to [8–11] regarding the study of Hyers-Ulam stability of differential equations and differential operators.

The stability and nonstability of different classes of recurrences were studied by Brzdęk *et al.* [12–14] and Popa [15, 16]. Jung [17] proved the Hyers-Ulam stability of a first-order linear homogeneous matrix difference equation. Note that the investigation of the differ-

ence equations  $\theta_{n+1} = \Lambda_n \theta_n$  or  $\theta_{n+1} = \Lambda_n \theta_n + \xi_n$  leads to the idea of discrete evolution family. The main interest in this area is the asymptotic properties and stability of solutions to such systems. In recent years, the exponential stability of such systems has received a great deal of attention since it has been widely applied in research of control theory and engineering; see, e.g., [18–25] and the references cited therein.

In this paper, we are concerned with the first-order linear system

$$\theta_{n+1} = \Lambda_n \theta_n, \quad n \in \mathbb{Z}_+, \tag{1.1}$$

where  $\mathbb{Z}_+$  is the set of all nonnegative integers and  $(\Lambda_n)$  is an  $\omega$ -periodic sequence of bounded linear operators on Banach space  $\Omega$ . We proved that system (1.1) is Hyers-Ulam stable if and only if it is uniformly exponentially stable under certain conditions.

### 2 Notation and preliminaries

Throughout,  $\mathbb{R}$  stands for the set of all real numbers,  $\Omega$  denotes a real or complex Banach space,  $\mathcal{L}(\Omega)$  is the Banach algebra of all linear and bounded operators over  $\Omega$ ,  $\mathcal{L}(\mathbb{Z}_+, \Omega)$  is the space of all  $\Omega$ -valued bounded sequences endowed with the sup norm denoted by  $\|\cdot\|_\infty$ ,  $P_0^\omega(\mathbb{Z}_+, \Omega)$  stands for the space of all  $\omega$ -periodic bounded sequences  $(\xi(n))$  satisfying  $\xi(0) = 0$ , and we denote by  $\|\cdot\|$  the norms in  $\Omega$  and  $\mathcal{L}(\Omega)$ .

Let  $\mathcal{H}$  belong to  $\mathcal{L}(\Omega)$  and  $\sigma(\mathcal{H})$  be its spectrum. The spectral radius of  $\mathcal{H}$  is denoted by  $r(\mathcal{H}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{H})\} = \lim_{n \rightarrow \infty} \|\mathcal{H}^n\|^{\frac{1}{n}}$ .

**Definition 2.1** The operator  $\mathcal{H}$  is said to be power bounded if there exists a positive constant  $M$  such that  $\|\mathcal{H}^n\| \leq M$  for all  $n \in \mathbb{Z}_+$ .

We need the following auxiliary lemma.

**Lemma 2.2** (See [21]) *If  $\mathcal{H} \in \mathcal{L}(\Omega)$  and*

$$\sup_{\gamma \in \mathbb{R}} \sup_{n \in \mathbb{Z}_+} \left\| \sum_{k=0}^n e^{i\gamma k} \mathcal{H}^k \right\| < \infty,$$

*then  $r(\mathcal{H}) < 1$ .*

The family  $\Gamma := \{\rho(n, m) : n, m \in \mathbb{Z}_+, n \geq m\}$  of bounded linear operators is called an  $\omega$ -periodic discrete evolution family for a fixed integer  $\omega \in \{2, 3, \dots\}$  if it satisfies the following properties:

- $\rho(n, n) = I$  for all  $n \in \mathbb{Z}_+$ .
- $\rho(n, m)\rho(m, r) = \rho(n, r)$  for all  $n \geq m \geq r, n, m, r \in \mathbb{Z}_+$ .
- $\rho(n + \omega, m + \omega) = \rho(n, m)$  for all  $n \geq m, n, m \in \mathbb{Z}_+$ .

It is well known that any  $\omega$ -periodic discrete evolution family  $\Gamma$  is exponentially bounded, that is, there exist a  $\tau \in \mathbb{R}$  and an  $M_\tau \geq 0$  such that

$$\|\rho(n, m)\| \leq M_\tau e^{\tau(n-m)} \quad \text{for all } n \geq m, n, m \in \mathbb{Z}_+. \tag{2.1}$$

When the family  $\Gamma$  is exponentially bounded, its growth bound  $\tau_0(\Gamma)$  is the infimum of all  $\tau \in \mathbb{R}$  for which there exists an  $M_\tau \geq 1$  such that inequality (2.1) is fulfilled. It is well known

that

$$\tau_0(\Gamma) = \lim_{n \rightarrow \infty} \frac{\ln \|\rho(n, 0)\|}{n} = \frac{1}{\omega} \ln(r(\rho(\omega, 0))).$$

The family  $\Gamma$  is uniformly exponentially stable if  $\tau_0(\Gamma)$  is negative or, equivalently, there exist an  $M > 0$  and a  $\tau > 0$  such that  $\|\rho(n, m)\| \leq Me^{-\tau(n-m)}$  for all  $n \geq m \in \mathbb{Z}_+$ . Thus, we have the following lemma.

**Lemma 2.3** *The discrete evolution family  $\Gamma$  is uniformly exponentially stable if and only if  $r(\rho(\omega, 0)) < 1$ .*

The map  $\rho(\omega, 0)$  is also called the Poincaré map of the evolution family  $\Gamma$ . Consider the following discrete Cauchy problem:

$$\begin{cases} \phi_{n+1} = \Lambda_n \phi_n + e^{i\gamma(n+1)} \xi(n+1), & n \in \mathbb{Z}_+, \\ \phi_0 = \theta_0, \end{cases} \quad (\Lambda_n, \gamma, \theta_0)$$

where  $\gamma \in \mathbb{R}$ , the sequence  $(\Lambda_n)$  is  $\omega$ -periodic, i.e.,  $\Lambda(n + \omega) = \Lambda(n) = \Lambda_n$  for all  $n \in \mathbb{Z}_+$  and a fixed  $\omega \in \{2, 3, \dots\}$ . Let

$$\rho(n, k) := \begin{cases} \Lambda_{n-1} \Lambda_{n-2} \cdots \Lambda_k & \text{if } k \leq n-1, \\ I & \text{if } k = n. \end{cases}$$

Then the family  $\{\rho(n, k)\}_{n \geq k \geq 0}$  is a discrete  $\omega$ -periodic evolution family. Over finite dimensional spaces, the uniform exponential stability of the Cauchy problem  $(\Lambda_n, \gamma, \theta_0)$  in discrete and continuous autonomous cases has been investigated in [20, 23].

**Definition 2.4** System (1.1) is said to be Hyers-Ulam stable if

$$\|\phi_{n+1} - \Lambda_n \phi_n\| \leq \epsilon \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \epsilon > 0,$$

and there exist an exact solution  $\theta_n$  of (1.1) and a constant  $L \geq 0$  such that

$$\|\phi_n - \theta_n\| \leq L\epsilon \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \epsilon > 0.$$

**Remark 2.5** If  $\phi_n$  is an approximate solution of (1.1), then  $\phi_{n+1} \approx \Lambda_n \phi_n$ . Therefore, letting  $(\xi(n))$  be an error sequence, then  $\phi_n$  is the exact solution of  $\phi_{n+1} = \Lambda_n \phi_n + \xi(n)$ .

With the help of Remark 2.5, Definition 2.4 can be modified as follows.

**Definition 2.6** System (1.1) is termed Hyers-Ulam stable if  $\|\xi(n)\| \leq \epsilon$  holds for any  $n \in \mathbb{Z}_+$  and  $\epsilon > 0$ , and there exist an exact solution  $\theta_n$  of (1.1) and a constant  $L \geq 0$  such that

$$\|\phi_n - \theta_n\| \leq L\epsilon \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \epsilon > 0.$$

### 3 Main results

Consider the Cauchy problem  $(\Lambda_n, \gamma, \theta_0)$ . The solution of the Cauchy problem  $(\Lambda_n, \gamma, \theta_0)$  is given by

$$\phi_n = \rho(n, 0)\theta_0 + \sum_{k=0}^n e^{i\gamma k} \rho(n, k)\xi(k).$$

Let us divide  $n$  by  $\omega$ , i.e.,  $n = l\omega + r$  for some  $l \in \mathbb{Z}_+$ , where  $r \in \{0, 1, \dots, \omega - 1\}$ . We consider the following sets, which will be useful in this paper:

$$\mathcal{A}_j := \{1 + j\omega, 2 + j\omega, \dots, (j + 1)\omega - 1\} \quad \text{for all } j \in \mathbb{Z}_+.$$

If  $r \in \{1, 2, \dots, \omega - 1\}$ , then define

$$B_l := \{l\omega + 1, l\omega + 2, \dots, l\omega + r\}$$

and

$$C := \{0, \omega, 2\omega, \dots, l\omega\}.$$

It is clear that

$$\bigcup_{j=0}^{l-1} \mathcal{A}_j \cup B_l \cup C = \{0, 1, 2, \dots, n\}. \tag{3.1}$$

On the basis of partition (3.1), we construct the space  $\mathcal{W}$ , which consists of all the sequences of the form

$$\xi(k) := \begin{cases} (k - j\omega)[(1 + j)\omega - k]\rho(k - j\omega, 0) & \text{if } k \in \mathcal{A}_j, \\ k(\omega - k)\rho(k, 0) & \text{if } k \in B_l, \\ 0 & \text{if } k \in C. \end{cases} \tag{3.2}$$

That is,

$$\mathcal{W} := \{(\xi(n)) : \xi(n) \text{ satisfies (3.2)}\}.$$

Obviously,  $\mathcal{W}$  is the subspace of  $P_0^\omega(\mathbb{Z}_+, \Omega)$ .

Now, we state and prove the main results.

**Theorem 3.1** *Let  $\Gamma := \{\rho(n, k) : n \geq k \in \mathbb{Z}_+\}$  be the  $\omega$ -periodic discrete evolution family on  $\Omega$  and let  $\phi_{n+1} = \Lambda_n \phi_n + e^{i\gamma(n+1)}\xi(n + 1)$ ,  $\phi_0 = \theta_0$  be the approximate solution of (1.1) with the error term  $e^{i\gamma(n+1)}\xi(n + 1)$ , where  $\gamma \in \mathbb{R}$  and  $(\xi(n)) \in P_0^\omega(\mathbb{Z}_+, \Omega)$ . Then the following statements are true.*

- (1) *If system (1.1) is uniformly exponentially stable, then it is Hyers-Ulam stable.*
- (2) *If system (1.1) is Hyers-Ulam stable for each  $\gamma \in \mathbb{R}$  and each  $\omega$ -periodic sequence  $(\xi(n)) \in \mathcal{W} \subset P_0^\omega(\mathbb{Z}_+, \Omega)$ , then system (1.1) is uniformly exponentially stable, i.e.,  $\Gamma$  is uniformly exponentially stable.*

*Proof* (1) Let  $\epsilon > 0$  and  $\phi_n$  be the approximate solution of (1.1) such that  $\sup_{n \in \mathbb{Z}_+} \|\phi_{n+1} - \Lambda_n \phi_n\| = \sup_{n \in \mathbb{Z}_+} \|e^{i\gamma(n+1)} \xi(n+1)\|$ ,  $\phi_0 = \theta_0$ , and  $\sup_{n \in \mathbb{Z}_+} \|\xi(n)\| \leq \epsilon$ , and let  $\theta_n$  be the exact solution of (1.1). Taking into account that  $\Gamma$  is uniformly exponentially stable, we conclude that there exist two positive constants  $M$  and  $\nu$  such that

$$\begin{aligned} \sup_{n \in \mathbb{Z}_+} \|\phi_n - \theta_n\| &= \sup_{n \in \mathbb{Z}_+} \left\| \rho(n, 0)\theta_0 + \sum_{k=0}^n e^{i\gamma k} \rho(n, k)\xi(k) - \rho(n, 0)\theta_0 \right\| \\ &= \sup_{n \in \mathbb{Z}_+} \left\| \sum_{k=0}^n e^{i\gamma k} \rho(n, k)\xi(k) \right\| \\ &\leq \sum_{k=0}^n \sup_{n \in \mathbb{Z}_+} \|e^{i\gamma k} \rho(n, k)\xi(k)\| \\ &= \sum_{k=0}^n \|e^{i\gamma k}\| \sup_{n \in \mathbb{Z}_+} \|\rho(n, k)\| \|\xi(k)\| \\ &= \sum_{k=0}^n \sup_{n \in \mathbb{Z}_+} \|\rho(n, k)\| \|\xi(k)\| \\ &\leq \sum_{k=0}^n M e^{-\nu(n-k)} \epsilon \\ &= M e^{-\nu n} \sum_{k=0}^n e^{\nu k} \epsilon \\ &= M e^{-\nu n} \left( \frac{1 - e^{(n+1)\nu}}{1 - e^\nu} \right) \epsilon \\ &= L \epsilon, \end{aligned}$$

where  $L := M e^{-\nu n} (1 - e^{(n+1)\nu}) / (1 - e^\nu)$ . Thus, system (1.1) is Hyers-Ulam stable.

(2) Let  $(\xi(n)) \in \mathcal{W}$ . Then

$$\begin{aligned} \sum_{k=1}^n e^{i\gamma k} \rho(n, k)\xi(k) &= \sum_{k=1}^{l\omega+r} e^{i\gamma k} \rho(l\omega+r, k)\xi(k) \\ &= \sum_{k \in \bigcup_{j=0}^{l-1} A_j \cup B_l \cup C} e^{i\gamma k} \rho(l\omega+r, k)\xi(k) \\ &= \sum_{k \in \bigcup_{j=0}^{l-1} A_j} e^{i\gamma k} \rho(l\omega+r, k)\xi(k) \\ &\quad + \sum_{k \in B_l} e^{i\gamma k} \rho(l\omega+r, k)\xi(k) \\ &\quad + \sum_{k \in C} e^{i\gamma k} \rho(l\omega+r, k)\xi(k) \\ &= \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} \rho(l\omega+r, k)\xi(k) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} \rho(l\omega+r, k) \xi(k) \\
 & + \sum_{k \in C} e^{i\gamma k} \rho(l\omega+r, k) \xi(k).
 \end{aligned}$$

By virtue of (3.2), we have

$$\begin{aligned}
 \sum_{k=1}^n e^{i\gamma k} \rho(n, k) \xi(k) &= \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} \rho(l\omega+r, k) (k-j\omega) [(1+j)\omega-k] \rho(k-j\omega, 0) \\
 & + \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} \rho(l\omega+r, k) k(\omega-k) \rho(k, 0) \\
 &= \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} \rho(l\omega+r, k) (k-j\omega) [(1+j)\omega-k] \rho(k-j\omega, 0) \\
 & + \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} \rho(l\omega+r, k) k(\omega-k) \rho(k, 0) \\
 &= L_1 + L_2,
 \end{aligned}$$

where

$$L_1 := \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} \rho(l\omega+r, k) (k-j\omega) [(1+j)\omega-k] \rho(k-j\omega, 0)$$

and

$$L_2 := \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} \rho(l\omega+r, k) k(\omega-k) \rho(k, 0).$$

We write  $L_1$  in the form

$$\begin{aligned}
 L_1 &= \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} \rho(l\omega+r, k) (k-j\omega) [(1+j)\omega-k] \rho(k-j\omega, 0) \\
 &= \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} \rho(l\omega+r, k) (k-j\omega) [(1+j)\omega-k] \rho(k, j\omega) \\
 &= \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} \rho(l\omega+r, j\omega) (k-j\omega) [(1+j)\omega-k] \\
 &= \sum_{j=0}^{l-1} \rho(l\omega+r, j\omega) \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} (k-j\omega) [(1+j)\omega-k] \\
 &= \sum_{j=0}^{l-1} \rho(l\omega+r, j\omega) \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k} (k-j\omega) [\omega-(k-j\omega)]
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=0}^{l-1} \rho(r, 0) \rho^{l-j}(\omega, 0) e^{i\gamma j\omega} \sum_{v=1}^{\omega-1} e^{i\gamma v} v(\omega - v) \\
 &= \rho(r, 0) \sum_{v=1}^{\omega-1} e^{i\gamma v} v(\omega - v) \sum_{j=0}^{l-1} e^{i\gamma j\omega} \rho^{l-j}(\omega, 0) \\
 &= \rho(r, 0) \sum_{v=1}^{\omega-1} e^{i\gamma v} v(\omega - v) \sum_{\alpha=1}^l e^{i\gamma\omega(l-\alpha)} \rho^\alpha(\omega, 0) \\
 &= \rho(r, 0) \sum_{v=1}^{\omega-1} e^{i\gamma v} v(\omega - v) e^{i\gamma l\omega} \sum_{\alpha=1}^l e^{-i\gamma\omega\alpha} \rho^\alpha(\omega, 0) \\
 &= G(\gamma, \omega) \sum_{\alpha=1}^l e^{-i\gamma\omega\alpha} \rho^\alpha(\omega, 0),
 \end{aligned}$$

where  $G(\gamma, \omega) := \rho(r, 0) \sum_{v=1}^{\omega-1} e^{i\gamma v} v(\omega - v) e^{i\gamma l\omega} \neq 0$ . Furthermore,

$$\begin{aligned}
 L_2 &= \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} \rho(l\omega + r, k) k(\omega - k) \rho(k, 0) \\
 &= \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} \rho(l\omega + r, 0) k(\omega - k) \\
 &= \rho(l\omega + r, 0) \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} k(\omega - k).
 \end{aligned}$$

Therefore,

$$\sum_{k=0}^n e^{i\gamma k} \rho(n, k) \xi(k) = G(\gamma, \omega) \sum_{\alpha=1}^l e^{-i\gamma\omega\alpha} \rho^\alpha(\omega, 0) + \rho(l\omega + r, 0) \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k} k(\omega - k).$$

Since (1.1) is Hyers-Ulam stable,

$$\sup_{n \in \mathbb{Z}_+} \|\phi_n - \theta_n\| = \sup_{n \in \mathbb{Z}_+} \left\| \sum_{k=1}^n e^{i\gamma k} \rho(n, k) \xi(k) \right\|$$

is bounded, and so  $L_1$  is bounded, *i.e.*,

$$\sup_{l \geq 0} \left\| \sum_{\alpha=0}^l e^{-i\gamma\omega\alpha} \rho^\alpha(\omega, 0) \right\| < \infty.$$

Using Lemma 2.2, we deduce that  $r(\rho(\omega, 0)) < 1$ . Hence, by Lemma 2.3,  $\Gamma$  is uniformly exponentially stable. This completes the proof.  $\square$

On the basis of Theorem 3.1, we obtain the following corollary as the main result of this paper.

**Corollary 3.2** *Assume that for each  $\gamma \in \mathbb{R}$  and each sequence  $(\xi(n)) \in \mathcal{W}$ , the approximate solution of the nonautonomous  $\omega$ -periodic discrete system (1.1) is presented by  $\phi_{n+1} =$*

$\Lambda_n \phi_n + e^{iy(n+1)} \xi(n+1)$ ,  $n \in \mathbb{Z}_+$ ;  $\phi_0 = \theta_0$ . Then the Hyers-Ulam stability of (1.1) is equivalent to its uniform exponential stability.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

Both authors contributed equally to this work and are listed in alphabetical order. They both read and approved the final version of the manuscript.

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#### References

- Ulam, SM: A Collection of Mathematical Problems. Interscience Publishers, New York (1960)
- Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
- Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
- Rassias, THM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
- Obłozza, M: Connections between Hyers and Lyapunov stability of the ordinary differential equations. Rocznik Nauk.-Dydakt. Prace Mat. **14**, 141-146 (1997)
- Alsina, C, Ger, R: On some inequalities and stability results related to the exponential function. J. Inequal. Appl. **2**, 373-380 (1998)
- Takahasi, S-E, Miura, T, Miyajima, S: On the Hyers-Ulam stability of the Banach space-valued differential equation  $y' = \lambda y$ . Bull. Korean Math. Soc. **39**, 309-315 (2002)
- Huang, J, Li, Y: Hyers-Ulam stability of linear functional differential equations. J. Math. Anal. Appl. **426**, 1192-1200 (2015)
- Jung, S-M: Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. J. Math. Anal. Appl. **320**, 549-561 (2006)
- Miura, T, Miyajima, S, Takahasi, S-E: A characterization of Hyers-Ulam stability of first order linear differential operators. J. Math. Anal. Appl. **286**, 136-146 (2003)
- Zada, A, Shah, O, Shah, R: Hyers-Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems. Appl. Math. Comput. **271**, 512-518 (2015)
- Brzdęk, J, Popa, D, Xu, B: Remarks on stability of linear recurrence of higher order. Appl. Math. Lett. **23**, 1459-1463 (2010)
- Brzdęk, J, Popa, D, Xu, B: Note on nonstability of the linear recurrence. Abh. Math. Semin. Univ. Hamb. **76**, 183-189 (2006)
- Brzdęk, J, Popa, D, Xu, B: On nonstability of the linear recurrence of order one. J. Math. Anal. Appl. **367**, 146-153 (2010)
- Popa, D: Hyers-Ulam-Rassias stability of a linear recurrence. J. Math. Anal. Appl. **309**, 591-597 (2005)
- Popa, D: Hyers-Ulam stability of the linear recurrence with constant coefficients. Adv. Differ. Equ. **2005**, 101-107 (2005)
- Jung, S-M: Hyers-Ulam stability of the first-order matrix difference equations. Adv. Differ. Equ. **2015**, 170 (2015)
- Ahmad, N, Khalid, H, Zada, A: Uniform exponential stability of discrete semigroup and space of asymptotically almost periodic sequences. Z. Anal. Anwend. **34**, 477-484 (2015)
- Buşe, C, Khan, A, Rahmat, G, Tabassum, A: Uniform exponential stability for nonautonomous system via discrete evolution semigroups. Bull. Math. Soc. Sci. Math. Roum. **57**, 193-205 (2014)
- Buşe, C, Zada, A: Dichotomy and boundedness of solutions for some discrete Cauchy problems. In: Proceedings of IWOTA-2008. Operator Theory: Advances and Applications, vol. 203, pp. 165-174 (2010)
- Buşe, C, Zada, A: Boundedness and exponential stability for periodic time dependent systems. Electron. J. Qual. Theory Differ. Equ. **2009**, 37 (2009)
- Chicone, C, Latushkin, Y: Evolution Semigroups in Dynamical Systems and Differential Equations. Mathematical Surveys and Monographs, vol. 70. Am. Math. Soc., Providence (1999)
- Zada, A: A characterization of dichotomy in terms of boundedness of solutions for some Cauchy problems. Electron. J. Differ. Equ. **2008**, 94 (2008)
- Zada, A, Ahmad, N, Khan, IU, Khan, FM: On the exponential stability of discrete semigroups. Qual. Theory Dyn. Syst. **14**, 149-155 (2015)
- Zada, A, Arif, M, Khalid, H: Asymptotic behavior of linear and almost periodic discrete evolution systems on Banach space  $\mathcal{A}AP'_0(\mathbb{Z}_+, \mathcal{W})$ . Qual. Theory Dyn. Syst. (2015). doi:10.1007/s12346-015-0177-5