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Periodic solutions of p-Laplacian equations with singularities

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Abstract
In this paper, the problem of existence of periodic solution is studied for p-Laplacian Liénard equations with singular at $x = 0$ and $x = +\infty$. By using the topological degree theory, some new results are obtained, and an example is given to illustrate the effectiveness of our results. Our research enriches the contents of second order differential equations with singularity.

Keywords: Liénard equation; topological degree; singularity; periodic solution

1 Introduction

The problem of periodic solution for ordinary differential equations with singularities has attracted much attention of many researchers because of its background in the applied sciences [1–6]. Lazer and Solimini in [7] considered in 1987 problems of periodic solutions for the equation with a singularity suggested by the two fundamental examples

$$x'' + \frac{1}{x^\alpha} = h(t) \tag{1.1}$$

(attractive restoring force) and

$$x'' - \frac{1}{x^\alpha} = h(t) \tag{1.2}$$

(repulsive restoring force), where $\alpha > 0$ is a constant and $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function. A necessary condition for the existence of a positive T-periodic solution of equation (1.1) is that $\bar{h} > 0$, and a necessary condition for the existence of a positive T-periodic solution for equation (1.2) is that $\bar{h} < 0$, as shown by integrating both members of the equations from 0 to T. By using the techniques of upper and lower solutions in equation (1.1) and the methods of Schauder fixed point theory in equation (1.2), respectively, they have shown that those conditions are also sufficient if, in equation (1.2), one assumes in addition that $\alpha \geq 1$. Jebelean and Mawhin in [8] considered the problems of a p-Laplacian Liénard equation of the form

$$\left(\left|x'\right|^{p-2}x'\right)' + f(x)x' + g(x) = h(t) \tag{1.3}$$

and

$$\left(\left|x'\right|^{p-2}x'\right)' + f(x)x' - g(x) = h(t), \tag{1.4}$$

where $p > 1$ is a constant, $f : [0, +\infty) \to \mathbb{R}$ is an arbitrary continuous function, $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic function with $h \in L^\infty[0, T]$. They extend the result of Lazer and Solimini in [7] to p-Laplacian-Liénard equations. We notice that all the restoring force terms in the equations studied by [7, 8] are not singular at $x = +\infty$. So far, to the best of the authors' knowledge, there are few results for the problem of equation with singular at $x = +\infty$. For example, Zhang in [9] studied the problem of periodic solutions of the Liénard equation with a repulsive singularity at $x = 0$ and a small singular force condition at $x = +\infty$,

$$x'' + f(x)x' + g(t, x) = 0, \quad 0 < t < T.$$

By using Mawhin's continuation theorem of the coincidence degree theory [10], some results on the existence of periodic solutions were obtained. In [11], Wang further studied the existence of positive periodic solutions for a delay Liénard equation with a repulsive singularity at $x = 0$ and a small singular force condition at $x = +\infty$,

$$x'' + f(x)x' + g\big(t, x(t - \tau)\big) = 0, \quad 0 < t < T.$$

In [12–14], the problem of existence of periodic solutions for some p-Laplacian Liénard equations were studied. However, the restoring forces term in these equations are all independent of variable t.

Motivated by the above mentioned work, in this paper, we study the existence of positive T-periodic solutions for p-Laplacian-like operators with singularity of the form

$$\big(|x'|^{p-2}x'\big)' + f(x)x' + g_1(x) + g_2(t, x) = h(t) \tag{1.5}$$

and

$$\big(|x'|^{p-2}x'\big)' + f(x)x' - g_1(x) + g_2(t, x) = h(t), \tag{1.6}$$

where $p > 1$ is a constant, $f : [0, \infty) \to \mathbb{R}$ is an arbitrary continuous function, $g_2 : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is a continuous function with $g_2(t + T, x) = g_2(t, x)$ for all $(t, x) \in \mathbb{R} \times [0, +\infty)$, $g_1 \in C((0, +\infty), (0, +\infty))$ and $\lim_{x\to 0+} g_1(x) = +\infty$, $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic function with $h \in L^1([0, T], \mathbb{R})$. From the corresponding definitions in [3, 7–9], we see that equation (1.5) and equation (1.6) are all singular at $x = 0$ and equation (1.5) is of attractive type and equation (1.6) is of repulsive type.

The interesting thing is that the main results in this paper can be applied to any damping forces term $f(x)x'$ without imposing more conditions on it than that of $f \in C([0, +\infty), \mathbb{R})$, and we not only consider equation (1.6) with a repulsive singularity at $x = 0$, but we also consider equation (1.5) with a attractive singularity at $x = 0$. Furthermore, for equation (1.5) and equation (1.6), besides $g_1(x)$ being singular at $x = 0$, we allow $g_2(t, x)$ to be singular at $x = +\infty$. Of course, a further growing restriction on $g_2(t, x)$ with respect to variable x will be needed.

2 Preliminary lemmas

The following two lemmas (Lemma 2.1 and Lemma 2.2) are all consequences of Theorem 3.1 in [15].

Lemma 2.1 *Assume that there exist constants* $0 < M_0 < M_1, M_2 > 0$, *such that the following conditions hold.*

1. *For each* $\lambda \in (0,1]$, *each possible positive* T*-periodic solution* x *to the equation*

$$\left(\left|u'\right|^{p-2}u'\right)' + \lambda f(u)u' + \lambda g_1(u) + \lambda g_2(t,u) = \lambda h(t)$$

   satisfies the inequalities $M_0 < u(t) < M_1$ *and* $|u'(t)| < M_2$ *for all* $t \in [0,T]$.
2. *Each possible solution* c *to the equation*

$$g_1(c) + \frac{1}{T}\int_0^T g_2(t,c)\,dt - \bar{h} = 0$$

   satisfies the inequality $M_0 < c < M_1$.
3. *We have*

$$\left(g_1(M_0) + \frac{1}{T}\int_0^T g_2(t,M_0)\,dt - \bar{h}\right)\left(g_1(M_1) + \frac{1}{T}\int_0^T g_2(t,M_1)\,dt - \bar{h}\right) < 0.$$

Then equation (1.5) *has at least one* T*-periodic solution* u *such that* $M_0 < u(t) < M_1$ *for all* $t \in [0,T]$.

Lemma 2.2 *Assume that there exist constants* $0 < M_0 < M_1, M_2 > 0$, *such that the following conditions hold.*

1. *For each* $\lambda \in (0,1]$, *each possible positive* T*-periodic solution* x *to the equation*

$$\left(\left|u'\right|^{p-2}u'\right)' + \lambda f(u)u' - \lambda g_1(u) + \lambda g_2(t,u) = \lambda h(t)$$

   satisfies the inequalities $M_0 < u(t) < M_1$ *and* $|u'(t)| < M_2$ *for all* $t \in [0,T]$.
2. *Each possible solution* c *to the equation*

$$g_1(c) - \frac{1}{T}\int_0^T g_2(t,c)\,dt + \bar{h} = 0$$

   satisfies the inequality $M_0 < c < M_1$.
3. *We have*

$$\left(g_1(M_0) - \frac{1}{T}\int_0^T g_2(t,M_0)\,dt - \bar{h}\right)\left(g_1(M_1) - \frac{1}{T}\int_0^T g_2(t,M_1)\,dt - \bar{h}\right) < 0.$$

Then equation (1.6) *has at least one* T*-periodic solution* u *such that* $M_0 < u(t) < m_1$ *for all* $t \in [0,T]$.

Lemma 2.3 [5] *Let* u *be an arbitrary function in* $W^{1,p}([0,T],R^n)$ *with* $u(0) = u(T) = 0$, *then*

$$\left(\int_0^T \left|u(t)\right|^p dt\right)^{1/p} \le \frac{\pi_p}{T}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{1/p},$$

where $\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\frac{\pi}{p})}$, $p \in (1,+\infty)$.

In order to study the existence of positive periodic solutions to equation (1.5) and equation (1.6), we list the following assumptions:

(H_1) $\liminf_{u\to 0^+}[g_1(u)+g_2(t,u)-\bar{h}]>0$ uniformly for all $t\in[0,T]$;
(H_2) $\limsup_{u\to +\infty}[g_1(u)+g_2(t,u)-\bar{h}]<0$ uniformly for all $t\in[0,T]$;
(H_3) $\liminf_{u\to 0^+}[g_1(u)-g_2(t,u)+\bar{h}]>0$ uniformly for all $t\in[0,T]$;
(H_4) $\limsup_{u\to +\infty}[g_1(u)-g_2(t,u)+\bar{h}]<0$ uniformly for all $t\in[0,T]$.

Now, we embed equation (1.5) and equation (1.6) into the following two equations family with a parameter $\lambda\in(0,1)$, respectively,

$$\bigl(\bigl|x'\bigr|^{p-2}x'\bigr)'+\lambda f(x)x'+\lambda g_1(x)+\lambda g_2(t,x)=\lambda h(t),\quad \lambda\in(0,1] \tag{2.1}$$

and

$$\bigl(\bigl|x'\bigr|^{p-2}x'\bigr)'+\lambda f(x)x'-\lambda g_1(x)+\lambda g_2(t,x)=\lambda h(t),\quad \lambda\in(0,1]. \tag{2.2}$$

Lemma 2.4 *Assume that assumptions* (H_1) *and* (H_2) *hold, then there exist constants* D_1 *and* D_2 *with* $0<D_1<D_2$ *such that*

(1) *for each possible positive* T*-periodic solution* $u(t)$ *of equation* (2.1), *there exist* $t_0,t_1\in[0,T]$ *such that*

$$u(t_0)>D_1\quad \textit{and}\quad u(t_1)<D_2;$$

(2) $g_1(u)+\frac{1}{T}\int_0^T g_2(t,u)\,dt-\bar{h}>0$ *for all* $u\in(0,D_1]$, *and* $g_1(u)+\frac{1}{T}\int_0^T g_2(t,u)\,dt-\bar{h}<0$ *for all* $u\in[D_2,+\infty)$.

Proof Assumption (H_1) implies the existence of some $D_1>0$ such that

$$g_1(u)+g_2(t,u)-\bar{h}>0, \tag{2.3}$$

whenever $(t,u)\in[0,T]\times(0,D_1]$. Consequently,

$$g_1(u)+\frac{1}{T}\int_0^T g_2(t,u)\,dt-\bar{h}>0\quad \text{for all } u\in(0,D_1]. \tag{2.4}$$

Let $u(t)$ be a positive T-periodic solution to equation (2.1). If $0<u(t)\le D_1$ for all $t\in[0,T]$, it follows from (2.3) that

$$g_1\bigl(u(t)\bigr)+g_2\bigl(t,u(t)\bigr)-\bar{h}>0,\quad \forall t\in[0,T]$$

and hence

$$\frac{1}{T}\int_0^T\bigl[g_1\bigl(u(t)\bigr)+g_2\bigl(t,u(t)\bigr)-\bar{h}\bigr]\,dt>0. \tag{2.5}$$

But, by integrating equation (2.1) over $[0,T]$ and using the periodic condition, we have

$$0=\frac{1}{T}\int_0^T\bigl[g_1\bigl(u(t)\bigr)+g_2\bigl(t,u(t)\bigr)-h(t)\bigr]\,dt=\frac{1}{T}\int_0^T\bigl[g_1(u)+g_2(t,u)-\bar{h}\bigr]\,dt, \tag{2.6}$$

which contradicts (2.5). This contradiction implies that there is a $t_0 \in [0,T]$ such that

$$u(t_0) > D_1. \tag{2.7}$$

On the other hand, assumption (H_2) implies the existence of some $D_2 > D_1$ such that

$$g_1(u) + g_2(t,u) - \bar{h} < 0, \tag{2.8}$$

whenever $(t,u) \in [0,T] \times (D_2,+\infty)$ and then

$$g_1(u) + \frac{1}{T}\int_0^T g_2(t,u)\,dt - \bar{h} > 0 \quad \text{for all } u \in (D_2,+\infty). \tag{2.9}$$

Let $u(t)$ be an arbitrary positive T-periodic solution to equation (2.1). If $u(t) \geq D_2$ for all $t \in [0,T]$, then by (2.8) we have

$$\begin{aligned} &\frac{1}{T}\int_0^T \bigl[g_1\bigl(u(t)\bigr) + g_2\bigl(t,u(t)\bigr) - h(t)\bigr]\,dt \\ &\quad = \frac{1}{T}\int_0^T \bigl[g_1\bigl(u(t)\bigr) + g_2\bigl(t,u(t)\bigr) - \bar{h}\bigr]\,dt < 0. \end{aligned} \tag{2.10}$$

Comparing (2.6) with (2.10), we see that there exists some $t_1 \in [0,T]$ such that

$$u(t_1) < D_2. \tag{2.11}$$

Clearly, (2.7) and (2.11) ensure that conclusion (1) of Lemma 2.4 holds, and conclusion (2) of Lemma 2.4 follows from (2.4) and (2.9). □

By a similar arguing to the proof of Lemma 2.4, we obtain the following result.

Lemma 2.5 *Assume that assumptions* (H_3) *and* (H_4) *hold, then there exist constant* $0 < D_3 < D_4$ *such that*

(1) *for each possible positive T-periodic solution $u(t)$ of equation* (2.2) *there exist* $t_0, t_1 \in [0,T]$ *such that*

$$u(t_0) > D_3 \quad \text{and} \quad u(t_1) < D_4;$$

(2) $g_1(u) - \frac{1}{T}\int_0^T g_2(t,u)\,dt + \bar{h} > 0$ *for all* $u \in (0,D_3]$, *and* $g_1(u) - \frac{1}{T}\int_0^T g_2(t,u)\,dt + \bar{h} < 0$ *for all* $u \in [D_4,+\infty)$.

3 Main results

Theorem 3.1 *Assume that* (H_1) *and* (H_2), *together with the following assumptions, hold*:

(H_5) $\int_0^1 g_1(u)\,du = +\infty$;

(H_6) *there are constants* $a \geq 0$ *and* $b > 0$ *such that* $|g_2(t,u)| \leq au^{p-1} + b$ *for all* $(t,u) \in [0,T] \times (0,+\infty)$;

(H_7) $(2aT)^{\frac{1}{p}}(\frac{\pi_p}{T})^{\frac{p-1}{p}} < 1$, *where* π_p *is a positive constant which is determined by Lemma* 2.3.

Then equation (1.5) *has at least one positive T-periodic solution*.

Proof First of all, we will show that there exist M_1, M_2 with $M_1 > D_1$ and $M_2 > 0$ such that each positive T-periodic solution $u(t)$ of equation (2.1) satisfies the inequalities

$$u(t) < M_1, \qquad \left|u'(t)\right| < M_2. \tag{3.1}$$

In fact, if u is a positive T-periodic solution of equation (2.1), then

$$\left(\left|u'\right|^{p-2}u'\right)' + \lambda f(u)u' + \lambda g_1(u) + \lambda g_2(t,u) = \lambda h(t), \quad \lambda \in (0,1]. \tag{3.2}$$

Integrating (3.2) over the interval $[0,T]$, we have

$$\int_0^T g_1\big(u(t)\big)\,dt + \int_0^T g_2\big(t,u(t)\big)\,dt = \int_0^T h(t)\,dt. \tag{3.3}$$

Multiply (3.3) with $u(t)$ and integrating it over the interval $[0,T]$, we have

$$\int_0^T \left|u'(t)\right|^p dt = \lambda \int_0^T g_1\big(u(t)\big)u(t)\,dt + \lambda \int_0^T g_2\big(t,u(t)\big)u(t)\,dt - \lambda \int_0^T h(t)u(t)\,dt,$$

which together with (3.3) yields

$$\begin{aligned}
\int_0^T \left|u'(t)\right|^p &\le \lambda|u|_\infty \int_0^T g_1\big(u(t)\big)\,dt + \lambda|u|_\infty \int_0^T \left|g_2\big(t,u(t)\big)\right|dt \\
&\quad + \lambda|u|_\infty \int_0^T \left|h(t)\right|dt \\
&\le \lambda|u|_\infty \int_0^T \left[h(t) - g_2\big(t,u(t)\big)\right]dt + \lambda|u|_\infty \int_0^T \left|g_2\big(t,u(t)\big)\right|dt \\
&\quad + \lambda|u|_\infty \int_0^T \left|h(t)\right|dt \\
&\le 2\lambda|u|_\infty \int_0^T \left|g_2\big(t,u(t)\big)\right|dt + 2\lambda|u|_\infty \int_0^T \left|h(t)\right|dt.
\end{aligned} \tag{3.4}$$

It follows from (H_6) that

$$\int_0^T \left|u'(t)\right|^p \le 2|u|_\infty\left(a\int_0^T \left|u(t)\right|^{p-1} + bT + \|h\|_{L_1}\right).$$

With t_1 given by Lemma 2.4,

$$u(t) = u(t_1) + \int_{t_1}^t u'(s)\,ds,$$

and hence, by the Hölder inequality, we get

$$u(t) < D_2 + T^{\frac{1}{q}}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}} \tag{3.5}$$

for all $t \in [0, T]$ ($\frac{1}{p} + \frac{1}{q} = 1$). This together with (3.4) gives

$$\begin{aligned}
\int_0^T \left|u'(t)\right|^p dt &\le 2\left[D_2 + T^{\frac{1}{q}}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}}\right]\left[a\int_0^T \left|u(t)\right|^{p-1} dt + bT + \|h\|_{L_1}\right] \\
&\le 2\left[D_2 + T^{\frac{1}{q}}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}}\right] \\
&\quad \times \left[aT^{\frac{1}{p}}\left(\int_0^T \left|u(t)\right|^p dt\right)^{\frac{p-1}{p}} + bT + \|h\|_{L_1}\right] \\
&\le 2aD_2T^{\frac{1}{p}}\left(\int_0^T \left|u(t)\right|^p dt\right)^{\frac{p-1}{p}} \\
&\quad + 2aT\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^T \left|u(t)\right|^p dt\right)^{\frac{p-1}{p}} \\
&\quad + 2T^{\frac{1}{q}}\left(bT + \|h\|_{L_1}\right)\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}} + 2D_2\left(bT + \|h\|_{L_1}\right).
\end{aligned} \tag{3.6}$$

Let $v(t) = u(t) - u(t_1)$, then $v(t_1) = 0 = v(t_1 + T)$. By using Lemma 2.3, we have

$$\left(\int_0^T \left|u(t) - u(t_1)\right|^p dt\right)^{\frac{1}{p}} \le \frac{\pi_p}{T}\left(\int_0^T \left|u'(t)\right|^p\right)^{\frac{1}{p}};$$

and then

$$\begin{aligned}
\left(\int_0^T \left|u(t)\right|^p dt\right)^{\frac{1}{p}} &= \left(\int_0^T \left|u(t) - u(t_1) + u(t_1)\right|^p dt\right)^{\frac{1}{p}} \\
&\le \left(\int_0^T \left|u(t) - u(t_1)\right|^p dt\right)^{\frac{1}{p}} + \left(\int_0^T \left|u(t_1)\right|^p dt\right)^{\frac{1}{p}} \\
&\le \frac{\pi_p}{T}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}} + D_2T^{\frac{1}{p}}.
\end{aligned}$$

By substituting into (3.6), we get

$$\begin{aligned}
\int_0^T \left|u'\right|^p dt &\le 2aD_2T^{\frac{1}{p}}\left[\frac{\pi_p}{T}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}} + D_2T^{\frac{1}{p}}\right]^{p-1} \\
&\quad + 2aT\left(\int_0^T \left|u'\right|^p dt\right)^{\frac{1}{p}}\left[\frac{\pi_p}{T}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}} + D_2T^{\frac{1}{p}}\right]^{p-1} \\
&\quad + 2T^{\frac{1}{q}}\left(bT + \|h\|_{L_1}\right)\left(\int_0^T \left|u'\right|^p dt\right)^{\frac{1}{p}} + 2D_2\left(bT + \|h\|_{L_1}\right),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left(1 - (2aT)^{\frac{1}{p}}\left(\frac{\pi_p}{T}\right)^{\frac{p-1}{p}}\right)\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}} \\
&\quad \le \left(2aD_2T^{\frac{1}{p}}\right)^{\frac{1}{p}}\left(\frac{\pi_p}{T}\right)^{\frac{p-1}{p}}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{p-1}{p^2}}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(2aD_2T^{\frac{1}{p}}\right)^{\frac{1}{p}} D_2^{\frac{p-1}{p}} T^{\frac{p-1}{p^2}} + \left[2D_2\left(bT + \|h\|_{L_1}\right)\right]^{\frac{1}{p}} \\
&+ (2aT)^{\frac{1}{p}} \left(D_2T^{\frac{1}{p}}\right)^{\frac{p-1}{p}} \left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p^2}} \\
&+ \left(2T^{\frac{1}{q}}\right)^{\frac{1}{p}} \left(bT + \|h\|_{L_1}\right)^{\frac{1}{p}} \left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p^2}} \\
= {} & \left(2aD_2T^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(\frac{\pi_p}{T}\right)^{\frac{p-1}{p}} \left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{p-1}{p^2}} \\
&+ \left(2aD_2T^{\frac{1}{p}}\right)^{\frac{1}{p}} D_2^{\frac{p-1}{p}} T^{\frac{p-1}{p^2}} + \left[2D_2\left(bT + \|h\|_{L_1}\right)\right]^{\frac{1}{p}} \\
&+ \left[(2aT)^{\frac{1}{p}} \left(D_2T^{\frac{1}{q}}\right)^{\frac{p-1}{p}} + 2^{\frac{1}{p}} T^{\frac{1}{pq}} \left(bT + \|h\|_{L_1}\right)^{\frac{1}{p}}\right] \left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p^2}}.
\end{aligned}
$$

Since $\frac{1}{p} > \max\{\frac{1}{p^2}, \frac{p-1}{p^2}\}$, it follows from ($H_5$) that there exists a positive constant C_1 such that

$$
\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\frac{1}{p}} < C_1.
$$

Then, by (3.5), we get

$$
u(t) < D_2 + T^{\frac{1}{q}} C_1 =: M_1 \quad \text{for all } t \in \mathbb{R}. \tag{3.7}
$$

Now, if u attains its maximum over $[0, T]$ at $t_2 \in [0, T]$, then $u'(t_2) = 0$ and we deduce from (3.2) that

$$
\left|u'(t)\right|^{p-2} u'(t) = \lambda \int_{t_2}^{t} \left[-f(u)u' - g_1(u) - g_2(t,u) + h(t)\right] dt
$$

for all $t \in [0, T]$. Thus, if $F' = f$, then

$$
\begin{aligned}
\left|u'(t)\right|^{p-1} &\leq \lambda \left|F\left(u(t)\right) - F\left(u(t_2)\right)\right| \\
&\quad + \lambda \left|\int_{t_2}^{t} \left|g_2\left(s, u(s)\right)\right| ds + \lambda \int_{t_2}^{t} g_1\left(u(s)\right) ds + \lambda \int_{t_2}^{t} \left|h(s)\right| ds\right| \\
&\leq 2\lambda \max_{0 \leq u \leq R} \left|F(u)\right| + \lambda \int_0^T \left|g_2\left(s, u(s)\right)\right| ds \\
&\quad + \lambda \int_0^T \left|g_1\left(u(s)\right)\right| ds + \lambda \int_0^T \left|h(s)\right| ds.
\end{aligned} \tag{3.8}
$$

Since $g_1 \in C((0, +\infty), (0, \infty))$, it follows from (3.3) that

$$
\int_0^T \left|g_1\left(u(s)\right)\right| ds = \int_0^T g_1\left(u(s)\right) ds \leq \int_0^T \left|g_2\left(s, u(s)\right)\right| ds + \int_0^T \left|h(s)\right| ds.
$$

Substituting it into (3.8), we have

$$
\left|u'(t)\right|^{p-1} \leq 2\lambda \max_{0 \leq u \leq R} \left|F(u)\right| + 2\lambda \int_0^T \left|g_2\left(s, u(s)\right)\right| ds + 2\lambda \int_0^T \left|h(s)\right| ds.
$$

Using (H_6), we obtain

$$\left|u'(t)\right|^{p-1} \leq 2\lambda \max_{0\leq u\leq R}\left|F(u)\right| + 2\lambda\left(aM_1^{p-1} + bT + \|h\|_{L_1}\right).$$

Thus, we have

$$\left|u'(t)\right|^{p-1} < \lambda C_2, \quad \forall t \in [0,T], \tag{3.9}$$

and then

$$\left|u'(t)\right| < C_2^{\frac{1}{p-1}} := M_2, \quad \forall t \in [0,T], \tag{3.10}$$

where $C_2 = 2\max_{0\leq u\leq R}|F(u)| + 2(aM_1^{p-1} + bT + \|h\|_{L_1}) + 1$. Equations (3.7) and (3.10) ensure that (3.1) holds.

Below, we will show that there exists a constant $M_0 \in (0, D_1)$, such that each positive T-periodic solution of equation (2.1) satisfies

$$u(t) > M_0 \quad \text{for all } t \in [0,T]. \tag{3.11}$$

Suppose that $u(t)$ is an arbitrary positive T-periodic solution of equation (2.1), then $u(t)$ satisfies equation (3.2), *i.e.*,

$$\left(\left|u'\right|^{p-2}u'\right)' + \lambda f(u)u' + \lambda g_1(u) + \lambda g_2(t,u) = \lambda h(t), \quad \lambda \in (0,1). \tag{3.12}$$

Let t_0 be determined in Lemma 2.4. Multiplying (3.12) by $u'(t)$ and integrating over the interval $[t_0, t]$ (or $[t, t_0]$), we get

$$\begin{aligned}
&\int_{t_0}^{t}\left(\left|u'(s)\right|^{p-2}u'(s)\right)'u'(s)\,ds + \lambda\int_{t_0}^{t} f\left(u(s)\right)\left(u'(s)\right)^2\,ds \\
&\qquad + \lambda\int_{t_0}^{t} g_1\left(u(s)\right)u'(s)\,ds + \lambda\int_{t_0}^{t} g_2\left(s,u(s)\right)u'(s)\,ds \\
&\quad = \lambda\int_{t_0}^{t} h(s)u'(s)\,ds, \quad \lambda \in (0,1).
\end{aligned} \tag{3.13}$$

Set $y(t) = |u'(t)|^{p-2}u'(t)$, then $y(t)$ is absolutely continuous and $u'(t) = |y(t)|^{q-2}y(t)$, where $q \in (1,+\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. So

$$\begin{aligned}
&\int_{t_0}^{t}\left(\left|u'(s)\right|^{p-2}u'(s)\right)'u'(s)\,ds \\
&\quad = \int_{t_0}^{t}\left|y(s)\right|^{q-2}y(s)y'(s)\,ds = \frac{|y(t)|^q}{q} - \frac{|y(t_0)|^q}{q} = \frac{|u'(t)|^p}{q} - \frac{|u'(t_0)|^p}{q}.
\end{aligned}$$

Substituting into (3.13), we get

$$\begin{aligned}
&\frac{|u'(t)|^p}{q} - \frac{|u'(t_0)|^p}{q} + \lambda\int_{t_0}^{t} f(u)\left(u'\right)^2\,dt \\
&\quad = -\lambda\int_{t_0}^{t} g_1(u)u'\,dt - \lambda\int_{t_0}^{t} g_2(t,u)u'\,dt + \lambda\int_{t_0}^{t} h(t)u'\,dt,
\end{aligned}$$

which yields the estimate

$$\lambda \int_{u(t)}^{u(t_0)} g_1(s)\,ds \le \frac{|u'(t)|^p}{q} + \frac{|u'(t_0)|^p}{q} + \lambda \int_0^T \bigl|f(u)\bigr|\bigl(u'\bigr)^2\,dt + \lambda \int_0^T \bigl|g_2(t,u)\bigr|\bigl|u'\bigr|\,dt + \lambda \int_0^T \bigl|h(t)u'\bigr|\,dt.$$

From (3.9) we get

$$\lambda \int_{u(t)}^{u(t_0)} g_1(s)\,ds \le \frac{2\lambda C_2^{\frac{p}{p-1}}}{q} + \lambda\Bigl(\max_{0\le u\le M_2}\bigl|f(u)\bigr|\Bigr)TC_2^{\frac{2}{p-1}} + \lambda\bigl(aM_1^{p-1}+bT\bigr)C_2^{\frac{1}{p-1}} + \lambda\|h\|_{L_1}C_2^{\frac{1}{p-1}},$$

which gives

$$\int_{u(t)}^{u(t_0)} g_1(s)\,ds \le C_3 \quad \text{for all } t \in [t_0, t_0+T] \tag{3.14}$$

with

$$C_3 = \frac{2C_2^{\frac{p}{p-1}}}{q} + \Bigl(\max_{0\le u\le M_2}\bigl|f(u)\bigr|\Bigr)TC_2^{\frac{2}{p-1}} + \bigl(aM_1^{p-1}+bT\bigr)C_2^{\frac{1}{p-1}} + \|h\|_{L_1}C_2^{\frac{1}{p-1}}.$$

From (H_6) there exists $M_0 \in (0, D_1)$ such that

$$\int_{\eta}^{D_1} g_2(s)\,ds > C_3 \quad \text{for all } \eta \in (0, M_0]. \tag{3.15}$$

Therefore, if there is a $t^* \in [t_0, t_0+T]$ such that $u(t^*) \le M_0$, then from (3.15) we get

$$\int_{u(t^*)}^{u(t_0)} g_1(s)\,ds \ge \int_{u(t^*)}^{D_1} g_1(s)\,ds > C_3,$$

which contradicts (3.14). This contradiction shows that $u(t) > M_0$ for all $t \in [0,T]$. So (3.11) holds. Let $m_0 \in (0, M_0)$ and $m_1 \in (M_1+D_2, +\infty)$ be two constants, then from (3.2) and (3.11), we see that each possible positive T-periodic solution u satisfies

$$m_0 < u(t) < m_1, \qquad \bigl|u'(t)\bigr| < M_2.$$

This implies that condition 1 and condition 2 of Lemma 2.1 are satisfied. Also, we can deduce from Lemma 2.4 that

$$g_1(c) + \frac{1}{T}\int_0^T g_2(t,c)\,dt - \bar{h} > 0 \quad \text{for } c \in (0, m_0]$$

and

$$g_1(c) + \frac{1}{T}\int_0^T g_2(t,c)\,dt - \bar{h} < 0 \quad \text{for } c \in [m_1, +\infty),$$

which results in

$$\left(g_1(m_0) + \frac{1}{T}\int_0^T g_2(t, m_0)\,dt - \bar{h}\right)\left(g_1(m_1) + \frac{1}{T}\int_0^T g_2(t, m_1)\,dt - \bar{h}\right) < 0.$$

So condition 3 of Lemma 2.1 holds. By using Lemma 2.1, we see that equation (1.5) has at least one positive T-periodic solution. The proof is complete. □

By using Lemma 2.5 and Lemma 2.2, we can obtain the following result.

Theorem 3.2 *Assume that* (H_3) *and* (H_4), *together with the following assumptions hold*:

(H_5) $\int_0^1 g_1(u)\,du = +\infty$;
(H_6) *there are constants* $a \geq 0$ *and* $b > 0$ *such that* $|g_2(t, u)| \leq au^{p-1} + b$ *for all* $(t, u) \in [0, T] \times (0, +\infty)$;
(H_7) $(aT)^{\frac{1}{p}}(\frac{\pi_p}{T})^{\frac{p-1}{p}} < 1$, *where* π_p *is a positive constant which is determined by Lemma* 2.3.

Then equation (1.6) *has at least one positive* T*-periodic solution*.

Example 3.1 Consider the following equation:

$$x''(t) + f\big(x(t)\big)x'(t) + \frac{1}{x^2(t)} - a\left(1 + \frac{1}{2}\sin t\right)x(t) = \cos t, \tag{3.16}$$

where f is an arbitrary continuous function, $a \in (0, \frac{1}{3\pi})$ is a constant. Corresponding to equation (1.5), we can assume that $g_1(u) = \frac{1}{u^2}$, $g_2(t, u) = a(1 + \sin t)u$, and $h(t) = \cos t$. By simple calculating, we can verify that assumptions (H_1)-(H_2), (H_5)-(H_7) are all satisfied. Thus, by using Theorem 3.1, we see that equation (3.16) has at least one positive 2π-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results are due to SL, TZ, and YG. The authors read and approved the final manuscript.

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