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A reduction formula for a q-beta integral

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Abstract
In this paper, we give a reduction formula for a specific q-integral. Our formula is expressed as a three term recurrence relations for basic hypergeometric ${}_3\phi_2$ series. This is a q-analog of work by Watson and by Bailey of 1953.

Keywords: q-beta integral; basic hypergeometric series; contiguous relations

1 Introduction and preliminaries

In [1], Watson constructed a reduction formula for the integral

$$I_n = \int_0^1 x^\alpha (1-x)^\beta \frac{d^n\{x^\gamma (1-x)^\delta\}}{dx^n}\,dx.$$

In fact, he proved that

$$I_n = \frac{\Gamma(\alpha+\gamma-n+1)\Gamma(\beta+\delta-n+1)}{\Gamma(2\sigma-n+2)} H_n,$$

where

$$H_n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} (-\gamma)_{n-r}(\beta+\delta-n+1)_{n-r}(-\delta)_r(\alpha+\gamma-n+1)_r,$$

and the notation $(x)_r$ denotes the product

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_r \equiv x(x+1)\cdots(x+r-1), \quad r \geq 1.$$

Furthermore, Watson proved that H_n satisfies the following three term recurrence relation:

$$(2\sigma-n)(\sigma-n)H_{n+2} - Q_n H_{n+1} + S_n H_n = 0, \tag{1.1}$$

where

$$S_n = (n+1)(\sigma-n-1)(\alpha+\beta-n)(\gamma+\delta-n)(\beta+\delta-n),$$

$$Q_n = (2\sigma-2n-1)\big\{(\beta\gamma-\alpha\delta)(\sigma+1) + \theta_3(n+1)(2\sigma-n)\big\},$$

and $2\sigma = \alpha + \beta + \gamma + \delta$, $2\theta_3 = \alpha - \beta - \gamma + \delta$. Also, he remarked H_n can be expressed in terms of a hypergeometric series of the type ${}_3F_2$ with last element unity which implies that (1.1) gives a three term contiguous relation for terminating ${}_3F_2$ series.

The proof introduced by Watson depends on constructing a second order linear differential equation satisfied by the integrand of I_n. On the other hand, in [2], Bailey derived relations between contiguous hypergeometric functions of the type ${}_3F_2(1)$, and by using these relations, he obtained another proof of Watson's reduction formula.

In this paper, we introduce a q-analog of the integral I_n by

$$I_{n,q} = \int_0^1 x^\alpha (qx;q)_\beta D^n_{q^{-1}} \bigl[x^\gamma \bigl(q^{\beta+1}x;q\bigr)_\delta\bigr]\, d_qx, \quad n = 0,1,2,\ldots,$$

where α, β, γ, and δ are complex numbers, and q is a positive number less than one. Our aim to obtain a reduction formula for $I_{n,q}$.

It turns out to us that Watson technique for introducing (1.1) is too hard to applied to our work. Therefore, we follow Bailey's approach for deriving the reduction formula.

We recall the following definitions (see, *e.g.*, [3–5]):

The q-shifted fractional is defined by

$$(a;q)_\infty = \prod_{j=0}^{\infty}\bigl(1 - aq^j\bigr), \quad \text{and} \quad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad \text{for } n \in \mathbb{Z}, a \in \mathbb{C}.$$

The q-derivative D_qf of an arbitrary function f is given by

$$(D_qf)(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.$$

We follow Gasper and Rahman [6] for the definitions of Jackson q-integrals, and the q-gamma and q-beta functions (see also [7–9]).

The q-integration by parts rule (see [3]) is

$$\int_0^a f(qt) D_q g(t)\, d_qt = f(a)g(a) - \lim_{n\to\infty} f\bigl(q^n\bigr)g\bigl(q^n\bigr) - \int_0^a D_qf(t) g(t)\, d_qt.$$

Let $a_1, \ldots, a_r, b_1, \ldots, b_s$ be complex numbers, the q-hypergeometric series ${}_r\phi_s$ defined by

$${}_r\phi_s(a_1,\ldots,a_r,b_1,\ldots,b_s;q,z) = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n} z^n \bigl(-q^{(n-1)/2}\bigr)^{n(s-r+1)}$$

The series representation of the function ${}_r\phi_s$ converges absolutely for all $z \in \mathbb{C}$ if $r \leq s$, and converges only for $|z| < 1$ if $r = s+1$ (for more details and results see [10–14] and [15]).

Observe that

$$\begin{aligned} I_{0,q} &= \int_0^1 x^{\alpha+\gamma}(qx;q)_{\beta+\delta}\, d_qx = B_q(\alpha+\gamma+1, \beta+\delta+1) \\ &= \frac{\Gamma_q(\alpha+\gamma+1)\Gamma_q(\beta+\delta+1)}{\Gamma_q(\alpha+\gamma+\beta+\delta+2)}, \end{aligned}$$

and, by using the q-integration by parts, one can verify that

$$\begin{aligned} I_{1,q} &= \int_0^1 x^\alpha (qx;q)_\beta D_{q^{-1}}\left[x^\gamma \left(q^{\beta+1}x;q\right)_\delta\right] d_q x \\ &= q^{1-\gamma} \frac{\Gamma_q(\alpha+\gamma)\Gamma_q(\beta+\delta)}{\Gamma_q(\alpha+\gamma+\beta+\delta+1)}\left[[\gamma][\beta+\delta] - q^\beta[\delta][\alpha+\gamma]\right], \end{aligned}$$

where the notation $[z]$ is defined by

$$[z] := \frac{1-q^z}{1-q}.$$

Note that the above values of the integrals $I_{0,q}$ and $I_{1,q}$ coincide with I_0 and I_1, respectively, which are given by Watson in the limit $q \to 1$.

This paper is organized as follows. In Section 2, we derive three term contiguous relations for the basic hypergeometric function ${}_3\phi_2(a,b,c;d,e;q,q)$. In Section 3, we show that $I_{n,q}$ can be represented as ${}_3\phi_2(q)$ and a direct substation in the derived contiguous relation yields the result of this paper.

2 Contiguous relations of ${}_3\phi_2$

Throughout this section, we simply used a to denotes the value q^{-n} where n is an arbitrary nonnegative integer. We denote by ϕ the function

$${}_3\phi_2(a,b,c;d,e;q,q),$$

and by $\phi(a^+)$, $\phi(a^-)$ the same function when a is changed to aq, a/q, respectively. We use a similar notation when the other parameters are so changed. Also, let ϕ_+, ϕ_- be the functions defined by

$$\phi_+ = {}_3\phi_2(aq,bq,cq;dq,eq;q,q) \quad \text{and} \quad \phi_- = {}_3\phi_2(a/q,b/q,c/q;d/q,e/q;q,q).$$

By the definition of ${}_3\phi_2$, one can verify the following:

$$\phi\left(a^+\right) - \phi = qa\frac{(1-b)(1-c)}{(1-d)(1-e)}\phi_+, \tag{2.1}$$

$$\phi\left(a^-\right) - \phi = -a\frac{(1-b)(1-c)}{(1-d)(1-e)}\phi_+\left(a^-\right), \tag{2.2}$$

$$\phi\left(d^+\right) - \phi = -qd\frac{(1-a)(1-b)(1-c)}{(1-d)(1-qd)(1-e)}\phi_+\left(d^+\right), \tag{2.3}$$

$$\phi\left(d^-\right) - \phi = d\frac{(1-a)(1-b)(1-c)}{(1-d/q)(1-d)(1-e)}\phi_+. \tag{2.4}$$

These equations, and the symmetries of the ${}_3\phi_2$, give us

$$c(1-a)\left\{\phi\left(a^+\right) - \phi\right\} = a(1-c)\left\{\phi\left(c^+\right) - \phi\right\}, \tag{2.5}$$

$$a(1-b)\left\{\phi\left(b^+\right) - \phi\right\} = b(1-a)\left\{\phi\left(a^+\right) - \phi\right\}, \tag{2.6}$$

$$qa(1-d/q)\left\{\phi\left(d^-\right) - \phi\right\} = d(1-a)\left\{\phi\left(a^+\right) - \phi\right\}, \tag{2.7}$$

$$d(1-e/q)\{\phi(e^-)-\phi\} = e(1-d/q)\{\phi(d^-)-\phi\}. \tag{2.8}$$

Now, applying the transformation (see [6])

$$_3\phi_2(a,b,c;d,e;q,q) = \frac{(de/bc;q)_n}{(e;q)_n}(bc/d)^n\, {}_3\phi_2(a,d/b,d/c;d,de/bc;q,q), \tag{2.9}$$

to $\psi = {}_3\phi_2(a,d/b,d/c;qd,de/bc;q,q)$ yields the following relations:

$$\begin{aligned}
\phi &= \frac{(de/bc;q)_n}{(e;q)_n}(bc/d)^n\psi(d^-),\\
\phi_+ &= \frac{(de/bc;q)_{n-1}}{(qe;q)_{n-1}}(qbc/d)^{n-1}\psi(a^+),\\
\phi_+(a^-) &= \frac{(de/bc;q)_n}{(qe;q)_n}(qbc/d)^n\psi.
\end{aligned} \tag{2.10}$$

Thus, from (2.7), changing ϕ into ψ, and using (2.10) we get

$$\begin{aligned}
&d(1-a)\left\{\frac{(qe;q)_{n-1}}{(de/bc;q)_{n-1}}(d/qbc)^{n-1}\phi_+ - \frac{(qe;q)_n}{(de/bc;q)_n}(d/qbc)^n\phi_+(a^-)\right\}\\
&\quad - a(1-d)\left\{\frac{(e;q)_n}{(de/bc;q)_n}(d/bc)^n\phi - \frac{(qe;q)_n}{(de/bc;q)_n}(d/qbc)^n\phi_+(a^-)\right\} = 0.
\end{aligned}$$

After some simplification this yields

$$a(1-d)(1-e)\phi - (a-e)(a-d)\phi_+(a^-) - (1-a)(qabc-de)\phi_+ = 0. \tag{2.11}$$

From the symmetries of the $_3\phi_2$, we have

$$b(1-d)(1-e)\phi - (b-e)(b-d)\phi_+(b^-) - (1-b)(qabc-de)\phi_+ = 0. \tag{2.12}$$

Using (2.11), (2.12), and (2.1), we obtain the following contiguous relations:

$$\begin{aligned}
&ab(a-b)(1-c)\phi + b(a-e)(a-d)\{\phi(a^-)-\phi\}\\
&\quad - a(b-e)(b-d)\{\phi(b^-)-\phi\} = 0,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&qa^2(1-b)(1-c)\phi + q(a-e)(a-d)\{\phi(a^-)-\phi\}\\
&\quad - (1-a)(qabc-de)\{\phi(a^+)-\phi\} = 0,
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
&qb^2(1-b)(1-c)\phi + q(b-e)(b-d)\{\phi(b^-)-\phi\}\\
&\quad - (1-b)(qabc-de)\{\phi(b^+)-\phi\} = 0.
\end{aligned} \tag{2.15}$$

Now, replacing b by b/q in (2.6) and b by bq in (2.13) we get

$$(b-aq)\{\phi(b^-)-\phi\} + b(1-a)\phi - b(1-a)\phi(a^+,b^-) = 0, \tag{2.16}$$

$$\begin{aligned}
&(a-qb)(ed-qabc)\{\phi(b^+)-\phi\} + bq(a-e)(a-d)\phi(a^-,b^+)\\
&\quad + \{(a-qb)(ed-qabc) - a(qb-e)(qb-d)\}\phi = 0.
\end{aligned} \tag{2.17}$$

Hence, combining (2.16) and (2.17) yields the three term contiguous relation

$$\begin{aligned}
&(a-e)(a-d)(1-b)(b-qa)\phi\big(a^-,b^+\big)\\
&\quad-(1-a)(b-e)(b-d)(a-qb)\phi\big(a^+,b^-\big)\\
&\quad+\big\{d(1-a)\big[a(e-b)+qb^2\big]+[2]ab(e-c)(a-b)-[2]b(ed+qba)\\
&\quad+qba^2\big([2]-a-d\big)+q(b-a)\big(qbac+e(b+a)\big)\\
&\quad+b^3(qa-c)+qa^2(ac-d)+ed\big(qa+b^2\big)\big\}\phi=0. \qquad (2.18)
\end{aligned}$$

3 The reduction formula

In this section, we state and prove the reduction formula for the q-integral $I_{n,q}$ stated in the introduction. We start with the following result.

Proposition 3.1 *The q-integral $I_{n,q}$ can be represented in terms of basic hypergeometric series ${}_3\phi_2$, that is,*

$$I_{n,q}=S_n{}_3\phi_2\big(q^{-n},q^{-\beta-\delta-\alpha-\gamma+n-1},q^{-\beta};q^{-\beta-\delta},q^{-\alpha-\beta};q,q\big),$$

where

$$S_n=q^{\frac{n}{2}(2\beta+5-n)}\frac{\Gamma_q(\beta+\delta+1)\Gamma_q(\alpha+\gamma+1)}{\Gamma_q(\alpha+\gamma+\beta+\delta-n+2)}\frac{(q^{-\gamma};q)_n(q^{-\alpha-\beta};q)_n}{(q^{-\alpha-\gamma};q)_n(q^{1-n+\gamma};q)_n}.$$

Proof Calculating $D^n_{q^{-1}}[x^\gamma(q^{\beta+1}x;q)_\delta]$ by using a q^{-1}-type Leibiniz rule (see [6], p.27) gives

$$I_{n,q}=\int_0^1 x^\alpha(qx;q)_\beta\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_{q^{-1}}\big(D^{n-k}_{q^{-1}}f\big)\big(q^{-k}x\big)D^k_{q^{-1}}g(x)\,d_qx,$$

where

$$f(x)=x^\gamma\quad\text{and}\quad g(x)=\big(q^{\beta+1}x,q\big)_\delta.$$

Note that

$$\begin{aligned}
&\big\{D^{n-k}_{q^{-1}}(\cdot)^\gamma\big\}\big(q^{-k}x\big)=\frac{(-q)^{n-k}(q^{-\gamma};q)_{n-k}}{(1-q)^{n-k}}q^{-k^2+nk-\gamma k}x^{\gamma-n+k},\\
&D^k_{q^{-1}}\big(q^{\beta+1}x,q\big)_\delta=q^{(\beta+\delta+1)k}\frac{q^{-\frac{1}{2}k(k-1)}}{(1-q)^k}\big(q^{\beta+1}x,q\big)_{\delta-k}\big(q^{-\delta},q\big)_k.
\end{aligned}$$

This implies

$$\begin{aligned}
I_{n,q}={}&\sum_{k=0}^n\begin{bmatrix}n\\k\end{bmatrix}_{q^{-1}}\frac{(-q)^{n-k}(q^{-\gamma};q)_{n-k}}{(1-q)^n}\big(q^{-\delta};q\big)_k q^{-\frac{1}{2}k(k-1)}\\
&\times q^{(\beta+\delta+1)k-(k-n+\gamma)k}\int_0^1 x^{\alpha+\gamma-n+k}(qx;q)_\beta\big(q^{\beta+1}x;q\big)_{\delta-k}\,d_qx. \qquad (3.1)
\end{aligned}$$

Denoting the q-integral in the left-hand-side of (3.1) by J_n, we obtain

$$\begin{aligned} J_n &= \int_0^1 x^{\alpha+\gamma-n+k}(qx;q)_{\beta+\delta-k}\,d_qx \\ &= B_q(\alpha+\gamma-n+k+1,\beta+\delta-k+1) \\ &= \frac{\Gamma_q(\alpha+\gamma-n+k+1)\Gamma_q(\beta+\delta-k+1)}{\Gamma_q(\alpha+\gamma+\beta+\delta-n+2)}. \end{aligned}$$

Using [6], Eq. (I.8) and Eq. (I.35), we get

$$J_n = \frac{\Gamma_q(\alpha+\gamma+1)\Gamma_q(\beta+\delta+1)}{\Gamma_q(\alpha+\gamma+\beta+\delta-n+2)} \frac{(1-q)^n q^{k^2-nk+\frac{1}{2}n(n+1)} q^{-(\beta+\delta)k-(n-k)(\alpha+\gamma)}}{(-1)^n(q^{-\beta-\delta};q)_k(q^{-\alpha-\gamma};q)_{n-k}}. \tag{3.2}$$

Using [6], Eq. (I.11), we get

$$\frac{(q^{-\gamma};q)_{n-k}}{(q^{-\alpha-\gamma};q)_{n-k}} = \frac{(q^{-\gamma};q)_n}{(q^{-\alpha-\gamma};q)_n} \frac{(q^{1-n+\alpha+\gamma};q)_k}{(q^{1-n+\gamma};q)_k} q^{-\alpha k}. \tag{3.3}$$

Substituting (3.2) into (3.1), using (3.3), yields

$$\begin{aligned} I_{n,q} &= \frac{\Gamma_q(\alpha+\gamma+1)\Gamma_q(\beta+\delta+1)}{\Gamma_q(\alpha+\gamma+\beta+\delta-n+2)} q^{n(1-\gamma-\alpha)+\frac{1}{2}n(n+1)} \frac{(q^{-\gamma};q)_n}{(q^{-\alpha-\gamma};q)_n} \\ &\quad \times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} (-1)^k q^{-\frac{1}{2}k(k-1)} \frac{(q^{1-n+\alpha+\gamma};q)_k(q^{-\delta};q)_k}{(q^{1-n+\gamma};q)_k(q^{-\delta-\beta};q)_k}. \end{aligned} \tag{3.4}$$

Now using [6], Eq. (I.42) and Eq. (I.47), we get

$$\begin{aligned} I_{n,q} &= K_n \sum_{k=0}^{n} q^k \frac{(q^{-n},q^{-\delta},q^{1-n+\alpha+\gamma};q)_k}{(q^{-\beta-\delta},q,q^{1-n+\gamma};q)_k} \\ &= K_n {}_3\phi_2\bigl(q^{-n},q^{1-n+\alpha+\gamma},q^{-\delta};q^{-\beta-\delta},q^{1-n+\gamma};q,q\bigr), \end{aligned}$$

where

$$K_n = \frac{\Gamma_q(\beta+\delta+1)\Gamma_q(\alpha+\gamma+1)}{\Gamma_q(\alpha+\gamma+\beta+\delta-n+2)} q^{n(1-\gamma-\alpha)+\frac{1}{2}n(n+1)} \frac{(q^{-\gamma};q)_n}{(q^{-\alpha-\gamma};q)_n}.$$

Using the transformation (2.9) yields the required result and completes the proof. □

Corollary 3.2 *If $\gamma = 0$ then $I_{n,q}$ vanishes for all values of n where $n-\beta-\delta$ and β are nonnegative integers.*

Proof Since

$${}_3\phi_2\bigl(a,b_1q^{m_1},b_2q^{m_2};b_1,b_2;q,a^{-1}q^{-(m_1+m_2)}\bigr) = 0, \tag{3.5}$$

where m_1, m_2 are arbitrary nonnegative integers (see [6]), the proof follows directly from Proposition 3.1 and (3.5). □

Watson remarked I_n vanishes for odd values of n in two special cases, (i) $\alpha = \gamma$ and $\beta = \delta$ and (ii) $\alpha = \beta$ and $\gamma = \delta$.

Now, we can derive the reduction formula for $I_{n,q}$.

Theorem 3.3 *The reduction formula satisfies a three term recurrence relations of* $I_{n,q}$. *More precisely, the following holds*:

If

$$W_n = {}_3\phi_2\bigl(q^{-n}, q^{-\beta-\delta-\alpha-\gamma+n-1}, q^{-\beta}; q^{-\beta-\delta}, q^{-\alpha-\beta}; q, q\bigr),$$

then

$$L_n W_{n+1} - Q_n W_{n-1} + M_n W_n = 0, \tag{3.6}$$

where

$$\begin{aligned}
L_n &= \bigl(q^{-n} - q^{\theta_1}\bigr)\bigl(q^{-n} - q^{\theta_2}\bigr)\bigl(1 - q^{\theta_3+n}\bigr)\bigl(q^{\theta_3+n} - q^{1-n}\bigr), \\
Q_n &= \bigl(1 - q^{-n}\bigr)\bigl(q^{\theta_3+n} - q^{\theta_1}\bigr)\bigl(q^{\theta_3+n} - q^{\theta_2}\bigr)\bigl(q^{-n} - q^{\theta_3+n+1}\bigr), \\
M_n &= q^{\theta_2}\bigl(1 - q^{-n}\bigr)\bigl[q^{-n}\bigl(q^{\theta_1} - q^{\theta_3+n}\bigr) + q^{2(\theta_3+n)+1}\bigr] + q^{3(\theta_3+n)}\bigl(q^{1-n} - q^{-\beta}\bigr) \\
&\quad - [2]q^{\theta_3+n}\bigl(q^{\theta_1+\theta_2} + q^{1+\theta_3}\bigr) + q^{1+\theta_3-n}\bigl([2] - q^{-n} - q^{\theta_2}\bigr) + q^{1-2n}\bigl(q^{-n-\beta} - q^{\theta_2}\bigr) \\
&\quad + q\bigl(q^{\theta_3+n} - q^{-n}\bigr)\bigl(q^{1+\theta_3-\beta} + q^{\theta_1}\bigl(q^{\theta_3+n} + q^{-n}\bigr)\bigr) + q^{\theta_1+\theta_2}\bigl(q^{1-n} + q^{2(\theta_3+n)}\bigr) \\
&\quad + [2]q^{\theta_3}\bigl(q^{\theta_1} - q^{-\beta}\bigr)\bigl(q^{-n} - q^{\theta_3+n}\bigr),
\end{aligned}$$

and $\theta_1 = -\alpha - \beta$, $\theta_2 = -\delta - \beta$, $\theta_3 = -\alpha - \beta - \delta - \gamma - 1$.

Proof This result follows by applying Proposition 3.1 and using equation (2.18) with

$$b = q^{-\beta-\delta-\alpha-\gamma+n-1}, \qquad c = q^{-\beta}, \qquad d = q^{-\beta-\delta} \quad \text{and} \quad e = q^{-\alpha-\beta}. \qquad \square$$

Recall that the little q-Jacobi polynomials, see [16], are defined by

$$P_n(x; a, b; q) = {}_2\phi_1\bigl(q^{-n}, abq^{n+1}; aq; q, qx\bigr), \tag{3.7}$$

and the formula

$$P_n(x; c, d; q) = \sum_{k=0}^{n} a_{k,n} P_k(x; a, b; q)$$

holds with

$$\begin{aligned}
a_{k,n} &= C_k {}_3\phi_2\bigl(q^{k-n}, cdq^{n+k+1}, aq^{k+1}; cq^{k+1}, abq^{2k+2}; q, q\bigr), \\
C_k &= (-1)^k q \frac{(q^{-n}, aq, cdq^{n+1}; q)_k}{(q, cq, abq^{k+1}; q)_k}.
\end{aligned} \tag{3.8}$$

Remark 3.4 In (3.7), if we take $a = q^{-\beta-1}$, $b = q^{-\alpha-1}$, $c = q^{-\beta-\delta-1}$ and $d = q^{-\alpha-\gamma-1}$, we get $a_{0,n} = \frac{1}{S_n} I_{n,q}$. Thus, the little q-Jacobi polynomials and the q-integrals $I_{n,q}$ are related in the following way:

$$P_n(x; c, d; q) = \frac{1}{S_n} I_{n,q} + \sum_{k=1}^{n} a_{k,n} P_k(x; a, b; q).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have made equal and significant contributions in writing this paper. They read and approved the final manuscript.

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