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Variational principles for spectral analysis of one Sturm-Liouville problem with transmission conditions

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Abstract

We study certain spectral aspects of the Sturm-Liouville problem with a finite number of interior singularities. First, for self-adjoint realization of the considered problem, we introduce a new inner product in the direct sum of the L_2 spaces of functions defined on each of the separate intervals. Then we define some special solutions and construct the Green function in terms of them. Based on the Green function, we establish an eigenfunction expansion theorem. By applying the obtained results we extend and generalize such important spectral properties as the Parseval and Carleman equations, Rayleigh quotient, and Rayleigh-Ritz formula (minimization principle) for the considered problem.

Keywords: Sturm-Liouville problems; boundary-transmission conditions; transmission conditions; expansions theorem; Rayleigh-Ritz formula; Parseval equality; Carleman equation

1 Introduction

The Sturm-Liouville differential equations are a class of differential equations often encountered in solving PDEs using the method of separation of variables. Their solutions define many well-known special functions, such as Bessel functions, Legendre polynomials, Chebyshev polynomials, or various hypergeometric functions arising in engineering and science applications. The solutions of many problems in mathematical physics are involved in investigation of a spectral problem, that is, the investigation of the spectrum and the expansion of an arbitrary function in terms of eigenfunctions of a differential operator. The issue of expansion in eigenfunctions is a classical one going back at least to Fourier (see, *e.g.*, [1–4]). The method of Sturm expansions is widely used in calculations of the spectroscopic characteristics of atoms and molecules [5–7]. A relatively recent impact is due to the study of wave propagation in random media [8, 9], where eigenfunction expansions are an important input in the proof of localization. The use of this tool is settled by classical results in the Schrödinger operator case. But with the study of operators related to classical waves [8, 10], a need for more general results on eigenfunction expansion became apparent. An important point is that a general function can be expanded in terms of all the eigenfunctions of an operator, a so-called complete set of functions. That is, if f_n is an eigenfunction of an operator Ψ with eigenvalue μ_n (so $\Psi f_n = \mu_n f_n$), then a general function g can be expressed as the linear combination $g = c_1f_1 + c_2f_2 + \cdots$ where the c_n are coefficients, and the sum is over a complete set of functions. The advantage of expressing a general function as a linear combination of a set of eigenfunctions is that it allows us to deduce the effect of an operator on a function that is not one of its own eigenfunctions. The importance of Sturm-Liouville problems for spectral methods lies in the fact that the spectral approximation of the solution of a differential equation is usually regarded as a finite expansion of eigenfunctions of a suitable Sturm-Liouville problem. Eigenfunction expansion problems for classical Sturm-Liouville problems have been investigated by many authors (see [1, 2, 11, 12] and references therein). In this paper we investigate certain spectral problems arising in the theory of the convergence of the eigenfunction expansion for one nonclassical eigenvalue problem, which consists of the Sturm-Liouville equation

$$\mathcal{L}(y) := -a(x)y''(x) + q(x)y(x) = \lambda y(x) \tag{1}$$

on a finite number of disjoint intervals $\Omega = \bigcup_{i=1}^{n+1}(\xi_{i-1}, \xi_i)$, where $0 = \xi_0 < \xi_1 < \cdots < \xi_{n+1} = \pi$, together with boundary conditions (BCs) at the endpoints $x = 0, \pi$

$$\mathcal{L}_\alpha(y) := \alpha_1 y(0) + \alpha_2 y'(0) = 0, \tag{2}$$

$$\mathcal{L}_\beta(y) := \beta_1 y(\pi) + \beta_2 y'(\pi) = 0 \tag{3}$$

and transmission conditions at the interior points $\xi_k \in (0, \pi)$, $k = 1, 2, \ldots, n$,

$$\begin{aligned}\mathcal{L}_{2k-1}(y) = {}& \delta'_{2k-1}y'(\xi_k + 0) + \delta_{2k-1}y(\xi_k + 0) + \gamma'_{2k-1}y'(\xi_k - 0)\\ &+ \gamma_{2k-1}y(\xi_k - 0) = 0,\end{aligned} \tag{4}$$

$$\mathcal{L}_{2k}(y) = \delta'_{2k}y'(\xi_k + 0) + \delta_{2k}y(\xi_k + 0) + \gamma'_{2k}y'(\xi_k - 0) + \gamma_{2k}y(\xi_k - 0) = 0, \tag{5}$$

where $a(x) = a_i^2 > 0$ for $x \in \Omega_i := (\xi_{i-1}, \xi_i)$, $i = 1, 2, \ldots, n + 1$, the potential $q(x)$ is a real-valued function that is continuous in each of the intervals (ξ_{i-1}, ξ_i) and has finite limits $q(0 + 0)$, $q(\pi - 0)$, and $q(\xi_i \mp 0)$, $i = 1, 2, \ldots, n$, λ is a complex spectral parameter, and δ_k, δ'_k, γ_k, and γ'_k ($k = 1, 2, \ldots, 2n$) are real numbers. The conditions are imposed on the left and right limits of solutions and their derivatives at the interior points and are often called 'transmission conditions' or 'interface conditions.' Such type problems often arise in varies physical transfer problems (see [13]). Some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (*i.e.*, a plate composed by materials with different characteristics piled in the thickness; see [14]). Similar problems with point interactions are also studied in [15, 16], *et cetera*. Since the solutions of equation (1) may have discontinuities at the interior points of the interval and since the values of the solutions and their derivatives at the interior points ξ_i are not defined, an important question is how to introduce a new Hilbert space in such a way that the considered problem can be interpreted as a self-adjoint problem in this space. The purpose of this paper is to extend and generalize important spectral properties such as the Rayleigh quotient, eigenfunction expansion, Rayleigh-Ritz formula (minimization principle), Parseval equality, and Carleman equality for Sturm-Liouville problems with interior singularities. The 'Rayleigh quotient' is the basis of an important approximation method that is used in solid mechanics and quantum mechanics. In the latter, it is used in the estimation of energy eigenvalues of nonsolvable quantum systems, for example, many-electron atoms and molecules. We note that spectral problems for ordinary differential operators without singularities were investigated in many works (see the monographs [4, 12, 17–22] and the references therein). Some aspects of spectral problems for differential equations having singularities with classical boundary conditions at the endpoints were studied, among others, in [15, 16, 23–32], where further references can be found.

2 Some preliminary results in according Hilbert space

We denote by θ_{ijk} $(1 \le j < k \le 4)$ the determinant of the jth and kth columns of the matrix

$$T_i = \begin{bmatrix} \delta'_{2i-1} & \delta_{2i-1} & \gamma'_{2i-1} & \gamma_{2i-1} \\ \delta'_{2i} & \delta_{2i} & \gamma'_{2i} & \gamma_{2i} \end{bmatrix}, \quad i = 1, 2, \ldots, n.$$

Note that throughout this study we shall assume that $\theta_{ijk} > 0$ for all i, j, k. In the direct sum space $\mathcal{H} = \bigoplus_{i=1}^{n+1} L_2(\Omega_i)$ we define the new inner product associated with the considered BVTP (1)-(5) by

$$\langle y, z\rangle_{\mathcal{H}} := \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} y(x)\overline{z(x)}\,dx \tag{6}$$

for $y = y(x)$, $z = z(x) \in \mathcal{H}$. Here we let $\theta_{034} = \theta_{(n+1)12} = 1$. Let us introduce the linear operator $(Ay)(x) = -a(x)y''(x) + q(x)y(x)$ in the Hilbert space $\mathcal{H}$ with domain of definition $D(\mathcal{A})$ consisting of all functions $y \in \mathcal{H}$ satisfying the following conditions:

(i) y and y' are absolutely continuous in each interval Ω_i $(i = 1, 2, \ldots, n + 1)$ and has finite limits $y(\xi_0 + 0)$, $y'(\xi_0 + 0)$, $y(\xi_{n+1} - 0)$, $y'(\xi_{n+1} - 0)$, $y(\xi_k \mp 0)$, and $y'(\xi_k \mp 0)$ for $k = 1, 2, \ldots, n$;

(ii) $\mathcal{L}y(x) \in \mathcal{H}$, $\mathcal{L}_\alpha y(x) = \mathcal{L}_\beta y(x) = \mathcal{L}_{2k-1}y(x) = \mathcal{L}_{2k}y(x) = 0$, $k = 1, 2, \ldots, n$. Then problem (1)-(5) is reduced to the operator equation $\mathcal{A}y = \lambda y$ in the Hilbert space $\mathcal{H}$.

Theorem 2.1 *For all $y, z \in D(\mathcal{A})$, we have the equality $\langle \mathcal{A}y, z\rangle_{\mathcal{H}} = \langle y, \mathcal{A}z\rangle_{\mathcal{H}}$.*

Proof From the definition of Hilbert space $\mathcal{H}$ it follows that

$$\begin{aligned}
\langle \mathcal{A}y, z\rangle_{\mathcal{H}} &= \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} \mathcal{L}y(x)\overline{z(x)}\,dx \\
&= \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} y(x)\overline{\mathcal{L}z(x)}\,dx \\
&\quad + \theta_{112}\theta_{212}\cdots\theta_{n12}\bigl(W(y,\overline{z};\xi_1-) - W(y,\overline{z};0)\bigr) \\
&\quad + \theta_{134}\theta_{212}\cdots\theta_{n12}\bigl(W(y,\overline{z};\xi_2-) - W(y,\overline{z};\xi_1+)\bigr) \\
&\quad + \cdots + \theta_{134}\theta_{234}\cdots\theta_{n34}\bigl(W(y,\overline{z};\pi) - W(y,\overline{z};\xi_n+)\bigr) \\
&= \langle y, \mathcal{A}z\rangle + \theta_{112}\theta_{212}\cdots\theta_{n12}\bigl(W(y,\overline{z};\xi_1-) - W(y,\overline{z};0)\bigr) \\
&\quad + \theta_{134}\theta_{212}\cdots\theta_{n12}\bigl(W(y,\overline{z};\xi_2-) - W(y,\overline{z};\xi_1+)\bigr) \\
&\quad + \cdots + \theta_{134}\theta_{234}\cdots\theta_{n34}\bigl(W(y,\overline{z};\pi) - W(z,\overline{z};\xi_n+)\bigr),
\end{aligned} \tag{7}$$

where, as usual, $W(y,\overline{z};x)$ denotes the Wronskian of the functions y and $\overline{z}$. From the boundary conditions (2)-(3) it follows that

$$W(y,\overline{z};0) = 0 \quad \text{and} \quad W(y,\overline{z};\pi) = 0. \tag{8}$$

The transmission conditions (4)-(5) lead to

$$\theta_{i12} W(f,\overline{g};\xi_i-) = \theta_{i34} W(f,\overline{g};\xi_i+), \quad i = 1,2,\dots,n. \tag{9}$$

Substituting (8) and (9) into (7), we obtain the needed equality. □

Lemma 2.2 *The linear operator $\mathcal{A}$ is densely defined in $\mathcal{H}$.*

Proof It suffices to prove that if $z \in \mathcal{H}$ is orthogonal to all $y \in D(\mathcal{A})$, then $z = 0$. Suppose that $\langle y,z\rangle_{\mathcal{H}} = 0$ for all $y \in D(\mathcal{A})$. Denote by $\bigoplus_{i=1}^{n+1} C_0^\infty(\Omega_i)$ the set of all infinitely differentiable functions in Ω vanishing on some neighborhoods of the points $x = \xi_k$, $k = 0,1,2,\dots,n+1$. Taking into account that $C_0^\infty(\xi_k,\xi_{k+1})$ is dense in $L_2(\xi_k,\xi_{k+1})$ ($k = 0,1,2,\dots,n+1$), we have that the function $z(x)$ vanishes on Ω. The proof is complete. □

Corollary 2.3 *$\mathcal{A}$ is symmetric linear operator in the Hilbert space $\mathcal{H}$.*

Corollary 2.4 *All eigenvalues of problem* (1)-(3) *are real, and two eigenfunctions corresponding to the distinct eigenvalues are orthogonal in the sense of the following equality*:

$$\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} y(x)z(x)\,dx = 0. \tag{10}$$

Remark 2.5 In fact, as in our previous work [31], we can prove that the operator $\mathcal{A}$ is self-adjoint in the Hilbert space $\mathcal{H}$. Moreover, the resolvent operator $(A-\lambda I)^{-1}$ is compact in this space.

Now we define two solutions $\upsilon(x,\lambda)$ and $\vartheta(x,\lambda)$ of equation (1) on the whole $\Omega = \bigcup_{i=1}^{n+1}(\xi_{i-1},\xi_i)$ by $\upsilon(x,\lambda) = \upsilon_i(x,\lambda)$ for $x \in \Omega_i$ and $\vartheta(x,\lambda) = \vartheta_i(x,\lambda)$ for $x \in \Omega_i$ ($i = 1,2,\dots,n+1$), where $\upsilon_i(x,\lambda)$ and $\vartheta_i(x,\lambda)$ are defined recurrently by the following procedure. Let $\upsilon_1(x,\lambda)$ and $\vartheta_{n+1}(x,\lambda)$ be solutions of equation (1) on $(0,\xi_1)$ and (ξ_n,π) satisfying the initial conditions

$$y(0,\lambda) = \alpha_2, \qquad y'(0,\lambda) = -\alpha_1 \tag{11}$$

and

$$y(\pi,\lambda) = -\beta_2, \qquad y'(\pi,\lambda) = \beta_1, \tag{12}$$

respectively. In terms of these solutions, we define recurrently the other solutions $\upsilon_{i+1}(x,\lambda)$ and $\vartheta_i(x,\lambda)$ by the initial conditions

$$\upsilon_{i+1}(\xi_i+,\lambda) = \frac{1}{\theta_{i12}}\left(\theta_{i23}\upsilon_i(\xi_i-,\lambda) + \theta_{i24}\frac{\partial\upsilon_i(\xi_i-,\lambda)}{\partial x}\right), \tag{13}$$

$$\frac{\partial \upsilon_{i+1}(\xi_i+,\lambda)}{\partial x} = \frac{-1}{\theta_{i12}}\left(\theta_{i13}\upsilon_i(\xi_i-,\lambda) + \theta_{i14}\frac{\partial \upsilon_i(\xi_i-,\lambda)}{\partial x}\right) \quad \text{and} \tag{14}$$

$$\vartheta_i(\xi_i-,\lambda) = \frac{-1}{\theta_{i34}}\left(\theta_{i14}\vartheta_{i+1}(\xi_i+,\lambda) + \theta_{i24}\frac{\partial \vartheta_{i+1}(\xi_i+,\lambda)}{\partial x}\right), \tag{15}$$

$$\frac{\partial \vartheta_i(\xi_i-,\lambda)}{\partial x}) = \frac{1}{\theta_{i34}}\left(\theta_{i13}\vartheta_{i+1}(\xi_i+,\lambda) + \theta_{i23}\frac{\partial \vartheta_{i+1}(\xi_i+,\lambda)}{\partial x}\right), \tag{16}$$

respectively, where $i = 1, 2, \ldots$. The existence and uniqueness of these solutions follow from the well-known theorem of ordinary differential equation theory. Moreover, by applying the method of [16] we can prove that all these solutions are entire functions of parameter $\lambda \in \mathbb{C}$ for each fixed x. Taking into account (13)-(16) and the fact that the Wronskians $\omega_i(\lambda) := W[\upsilon_i(x,\lambda), \vartheta_i(x,\lambda)]$ $(i = 1, 2, \ldots, n+1)$ are independent of the variable x, we have

$$\begin{aligned}
\omega_{i+1}(\lambda) &= \upsilon_{i+1}(\xi_i+,\lambda)\frac{\partial \vartheta_{i+1}(\xi_i+,\lambda)}{\partial x} - \frac{\partial \upsilon_{i+1}(\xi_i+,\lambda)}{\partial x}\vartheta_{i+1}(\xi_i+,\lambda) \\
&= \frac{\theta_{i34}}{\theta_{i12}}\left(\upsilon_i(\xi_i-,\lambda)\frac{\partial \vartheta_i(\xi_i,\lambda)}{\partial x} - \frac{\partial \upsilon_i(\xi_i-,\lambda)}{\partial x}\vartheta_i(\xi_i-,\lambda)\right) \\
&= \frac{\theta_{i34}}{\theta_{i12}}\omega_i(\lambda) = \prod_{j=1}^{i}\frac{\theta_{j34}}{\theta_{j12}}\omega_1(\lambda) \quad (i = 1, 2, \ldots, n).
\end{aligned}$$

It is convenient to define the characteristic function $\omega(\lambda)$ for our problem (1)-(3) as

$$\omega(\lambda) := \omega_1(\lambda) = \prod_{j=1}^{i}\frac{\theta_{j12}}{\theta_{i34}}\omega_{i+1}(\lambda) \quad (i = 1, 2, \ldots, n).$$

Remark 2.6 Obviously, $\omega(\lambda)$ is an entire function. By applying the technique of [29] we can prove that there are infinitely many eigenvalues λ_k, $k = 1, 2, \ldots$, of problem (1)-(5), which coincide with the zeros of the characteristic function $\omega(\lambda)$.

3 Eigenfunction expansion based on the Green function. Modified Parseval equality

We can show that the Green function for problem (1)-(5) is of the form

$$G(x,s;\lambda) = \begin{cases} \frac{\upsilon(s,\lambda)\vartheta(x,\lambda)}{\omega(\lambda)}, & 0 < s \leq x < \pi x, s \neq \xi_i, i = 1, 2, \ldots, n+1, \\ \frac{\upsilon(x,\lambda)\vartheta(s,\lambda)}{\omega(\lambda)}, & 0 < x \leq s < \pi x, s \neq \xi_i, i = 1, 2, \ldots, n+1, \end{cases} \tag{17}$$

for $x, s \in \Omega$ (see, *e.g.*, [26]). It is symmetric with respect to x and s and is real-valued for real λ. Let us show that the function

$$y(x,\lambda) = \sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0} G(x,s;\lambda)f(s)\,ds, \tag{18}$$

called a resolvent, is a solution of the equation

$$a(x)y'' + \{\lambda - q(x)\}y = f(x) \tag{19}$$

(where $f(x) \neq 0$ is a continuous function in each Ω_i with finite one-hand limits at the endpoints of these intervals) satisfying the boundary-transmission conditions (2)-(5). Without loss of generality, we can assume that $\lambda = 0$ is not an eigenvalue. Otherwise, we take a fixed real number η and consider the boundary-value-transmission problem for the differential equation

$$a(x)y''(x,\lambda) + \{(\lambda + \eta) - q(x)\}y(x,\lambda) = 0 \tag{20}$$

together with the same boundary-transmission conditions (2)-(5) and the same eigenfunctions as for problem (1)-(5). All the eigenvalues are shifted through η to the right. It is evident that η can be selected so that $\lambda = 0$ is not an eigenvalue of the new problem. Let $G(x,s;0) = G(x,s)$. Then the function

$$y(x,\lambda) = \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} G(x,s)f(s)\,ds \tag{21}$$

is a solution of the equation $a(x)y'' - q(x)y = f(x)$ satisfying the boundary-transmission conditions (2)-(5). We rewrite (19) in the form

$$a(x)y'' - q(x)y = f(x) - \lambda y. \tag{22}$$

Thus, the homogeneous problem ($f(x) \equiv 0$) is equivalent to the integral equation

$$y(x,\lambda) + \lambda \left\{ \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} G(x,s)y(s)\,ds \right\} = 0. \tag{23}$$

Denoting the collection of all the eigenvalues of problem (1)-(4) by $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n, \ldots$ and the corresponding normalized eigenfunctions by $y_0, y_1, y_2, \ldots, y_n, \ldots$, consider the series

$$Y(x,\xi) = \sum_{n=0}^{\infty} \frac{y_n(x)y_n(\xi)}{\lambda_n}.$$

We can show that $\lambda_n = O(n^2)$. From this asymptotic formula for the eigenvalues it follows that the series for $Y(x,\xi)$ converges absolutely and uniformly; therefore, $Y(x,\xi)$ is continuous in Ω. Consider the kernel

$$K(x,\xi) = G(x,\xi) + Y(x,\xi) = G(x,\xi) + \sum_{n=0}^{\infty} \frac{y_n(x)y_n(\xi)}{\lambda_n},$$

which is continuous and symmetric. By a familiar theorem in the theory of integral equations, any symmetric kernel $K(x,\xi)$ that is not identically zero has at least one eigenfunction [33], that is, there are a number μ and a function $\psi(x) \neq 0$ satisfying the equation

$$\psi(x) + \mu \left\{ \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} K(x,\xi)\psi(\xi)\,d\xi \right\} = 0. \tag{24}$$

Thus, if we show that the kernel $K(x,\xi)$ has no eigenfunctions, we obtain $K(x,\xi)\equiv 0$, that is,

$$G(x,\xi) = -\sum_{n=0}^{\infty}\frac{y_n(x)y_n(\xi)}{\lambda_n}. \tag{25}$$

It follows from equation (23) that

$$\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}G(x,\xi)\psi_n(\xi)\,d\xi = -\lambda_n^{-1}\psi_n(x). \tag{26}$$

Therefore,

$$\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}K(x,\xi)\psi_n(\xi)\,d\xi = 0, \tag{27}$$

that is, the kernel $K(x,\xi)$ is orthogonal to all eigenfunctions of the boundary-value-transmission problem (1)-(5). Let $y(x)$ be a solution of the integral equation (24). Let us show that $y(x)$ is orthogonal to all $\psi_n(x)$. In fact, it follows from (24) that

$$\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}y(x)\psi_n(x) = 0.$$

Therefore,

$$\begin{aligned} &y(x,\lambda) + \lambda_0\left\{\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}K(x,\xi)y(\xi)\,d\xi\right\} \\ &\quad = y(x,\lambda) + \lambda_0\left\{\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}G(x,\xi)y(\xi)\,d\xi\right\} = 0, \end{aligned}$$

that is, $y(x,\lambda)$ is an eigenfunction of the boundary-value-transmission problem (1)-(5). Since it is orthogonal to all $\psi_n(x)$, it is also orthogonal to itself, and, as a consequence, $y(x,\lambda) = 0$ and $K(x,\xi) = 0$. Formula (25) is thus proved.

Theorem 3.1 (Expansion theorem) *If $f(x)$ has a continuous second derivative in each Ω_i* ($i = 1,2,\dots,n+1$), *and satisfies the boundary-transmission conditions* (2)-(5), *then $f(x)$ can be expanded into an absolutely and uniformly convergent series of eigenfunctions of the boundary-value-transmission problem* (1)-(5) *on Ω, that is,*

$$f(x) = \sum_{m=0}^{\infty} r_m\psi_m(x), \tag{28}$$

where $r_m = r_m(f)$ are the Fourier coefficients of f given by

$$r_m = \sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}f(x)\psi_m(x)\,dx. \tag{29}$$

Proof Put $g(x) = a(x)f'' - q(x)f$. Then, relying on (18) and (25), we have

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} G(x,\xi)g(\xi)\,d\xi \\ &= -\sum_{m=0}^{\infty} \frac{\psi_m(x)}{\lambda_m} \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} \psi_m(\xi)g(\xi)\,d\xi \\ &\equiv \sum_{m=0}^{\infty} r_m \psi_m(x). \end{aligned} \tag{30}$$

From the orthogonality and normalization of the functions $\psi_m(x)$ we obtain (29). □

Theorem 3.2 (Modified Parseval equality) *For any function $f \in \bigoplus_{i=1}^{n+1} L_2(\Omega_i)$, we have the Parseval equality*

$$\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} f^2(x)\,dx = \sum_{m=0}^{\infty} r_m^2(f). \tag{31}$$

Proof If $f(x)$ satisfies the conditions of Theorem 3.1, then (31) follows immediately from the uniform convergence of the series (28). Indeed,

$$\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} f^2(x)\,dx = \sum_{m=0}^{\infty} r_m^2(f). \tag{32}$$

Now, suppose that $f(x)$ is an arbitrary square-integrable function on the intervals Ω_i ($i = 1, 2, \ldots, n+1$). Slightly modifying the familiar theorem in the theory of real analysis, we can show that there exists a sequence of infinitely differentiable functions $f_k(x)$, converging in mean square to $f(x)$, such that each function $f_k(x)$ is identically zero in some neighborhoods of the points ξ_i ($i = 0, 1, \ldots, n+1$). From (32) it follows that

$$\begin{aligned} &\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} \bigl[f_s(x) - f_t(x)\bigr]^2\,dx \\ &\quad = \sum_{m=0}^{\infty} \bigl[r_m(f_s) - r_m(f_t)\bigr]^2, \end{aligned} \tag{33}$$

where $r_m(f_s)$ are, as usual, the Fourier coefficients in (29). Since the left-hand side (33) tends to zero as $s, t \to \infty$, the right-hand side also tends to zero. By applying the Cauchy-Schwarz inequality we obtain

$$\bigl|r_m(f) - r_m(f_s)\bigr| \le \left\{ \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} \bigl[f(x) - f_s(x)\bigr]^2\,dx \right\}^{\frac{1}{2}}.$$

On the other hand, from the convergence in the mean of $f_s(x)$ to $f(x)$ it follows that

$$\lim_{s\to\infty} r_m(f_s) = r_m(f), \quad m = 0, 1, 2, \ldots.$$

It follows from (33) that

$$\sum_{n=0}^{N}\bigl[r_m(f_s)-r_m(f_t)\bigr]^2 \le \sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}\bigl[f_s(x)-f_t(x)\bigr]^2\,dx$$

for an arbitrary integer N. Passing to the limit as $s\to\infty$, we obtain

$$\sum_{n=0}^{N}\bigl[r_m(f)-r_m(f_t)\bigr]^2 \le \sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+0}^{\xi_{k+1}-0}\bigl[f(x)-f_t(x)\bigr]^2\,dx.$$

Now letting $N\to\infty$ gives

$$\sum_{n=0}^{\infty}\bigl[r_m(f)-r_m(f_t)\bigr]^2 \le \sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+}^{\xi_{k+1}-}\bigl[f(x)-f_t(x)\bigr]^2\,dx.$$

Taking into account the Minkowski inequality, we see that the series $\sum_{m=0}^{\infty} r_m^2(f)$ converges. Since

$$\begin{aligned}
&\left|\sum_{m=0}^{\infty}\bigl(r_m(f)\bigr)^2-\sum_{m=0}^{\infty}\bigl(r_m(f_t)\bigr)^2\right|\\
&\quad=\left|\sum_{m=0}^{\infty}\bigl[r_m(f)-r_m(f_t)\bigr]\bigl[r_m(f)+r_m(f_t)\bigr]\right|\\
&\quad\le\left(\sum_{m=0}^{\infty}\bigl|r_m(f)-r_m(f_t)\bigr|^2\right)^{\frac12}\left(\sum_{m=0}^{\infty}\bigl|r_m(f)+r_m(f_t)\bigr|^2\right)^{\frac12},
\end{aligned}$$

we deduce that $\sum_{m=0}^{\infty}\{r_m(f_t)\}^2\to\sum_{m=0}^{\infty} r_m^2(f)$ as $t\to\infty$. Moreover, from the convergence in the mean of $f_t(x)$ to $f(x)$ we derive that

$$\begin{aligned}
&\lim_{t\to\infty}\left(\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+}^{\xi_{k+1}-}f_t^2(x)\,dx\right)\\
&\quad=\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+}^{\xi_{k+1}-}f^2(x)\,dx.
\end{aligned}$$

Finally, letting $t\to\infty$ in the equality

$$\sum_{k=0}^{n}\frac{1}{a_{k+1}^2}\prod_{i=0}^{k}\theta_{i34}\prod_{i=k+1}^{n+1}\theta_{i12}\int_{\xi_k+}^{\xi_{k+1}-}f_t^2(x)\,dx=\sum_{m=0}^{\infty}\bigl(r_m(f_t)\bigr)^2,$$

we obtain (31) for arbitrary $f\in\bigoplus_{i=1}^{n+1}L_2(\Omega_i)$. The proof is complete. □

4 Modified Carleman equality

We now return to formula (18), whose right-hand side has been called the resolvent. Let

$$y(x,\lambda)=\sum_{n=0}^{\infty}t_n(\lambda)\psi_n(x). \tag{34}$$

Then, from (22) we have

$$\begin{aligned} r_m(f) &= \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+}^{\xi_{k+1}-} \left(a(x)y(x)'' + \bigl(\lambda - q(x)y(x)\bigr)\psi_m(x)\,dx\right. \\ &= -\lambda_m t_m(\lambda) + t_m(\lambda). \end{aligned} \tag{35}$$

Hence, $t_m(\lambda) = \frac{r_m}{\lambda-\lambda_m}$, and the expansion of the resolvent is

$$\begin{aligned} y(x,\lambda) &= \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} G(x,s;\lambda) f(s)\,ds \\ &= \sum_{m=0}^{\infty} \frac{r_m \psi_m(x)}{\lambda - \lambda_m}. \end{aligned} \tag{36}$$

From this an important formula can now be derived. Substituting equality (29) into the right-hand side of (36), we find that

$$\begin{aligned} &\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} G(x,s;\lambda) f(s)\,ds \\ &\quad = \sum_{m=0}^{\infty} \frac{\psi_m(x)}{\lambda - \lambda_m} \left\{ \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} f(s)\psi_m(s)\,dx \right\}. \end{aligned} \tag{37}$$

Since $f(s)$ is arbitrary,

$$G(x,s;\mu) = \sum_{m=0}^{\infty} \frac{\psi_m(x)\psi_m(s)}{\mu - \lambda_m}. \tag{38}$$

Thus, we obtain

$$\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} G(x,x;\mu)\,dx = \sum_{m=0}^{\infty} \frac{1}{\mu - \lambda_m}. \tag{39}$$

Denoting by $S(\lambda)$ the number of eigenvalues λ_n less than λ, from (39) we get the modified Carleman equation for our problem (1)-(5)

$$\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} G(x,x;\mu)\,dx = \int_0^{\infty} \frac{dS(\lambda)}{\mu - \lambda}. \tag{40}$$

5 The Rayleigh quotient and minimization principle for problem (1)-(5)

Let (λ,ψ) be an eigen-pair for linear operator $\mathcal{A}$ in the Hilbert space $\mathcal{H}$, that is, $\mathcal{A}\psi = \lambda\psi$. From this equality it follows that

$$\lambda = \frac{\langle \mathcal{A}\psi, \psi\rangle_{\mathcal{H}}}{\|\psi\|_{\mathcal{H}}^2}.$$

This expression (the so-called Rayleigh quotient) enables to relate an eigenvalue λ to its eigenfunction ψ. Especially in quantum physics it is important to find the first eigenvalue. The Rayleigh quotient plays an important role in this content.

Lemma 5.1 (Rayleigh quotient) *Let* (λ, ψ) *be an eigen-pair for the Sturm-Liouville differential equation* (1). *Then the Rayleigh quotient for problem* (1)-(5) *takes the form*

$$\lambda := R(\psi) = \frac{\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \{a_k^2 (\psi \psi' |_{\xi_k+0}^{\xi_{k+1}-0}) + \int_{\xi_k+0}^{\xi_{k+1}-0} (a_k^2 (\psi')^2 + q\psi^2)\, dx\}}{\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} \psi^2\, dx}. \quad (41)$$

Proof The needed Rayleigh quotient (41) can be derived from the Sturm-Liouville equation

$$-a(x)\psi''(x) + q(x)\psi(x) = \lambda\psi(x), \quad x \in \Omega, \quad (42)$$

by multiplying by ψ and integrating over Ω. Then we have

$$\lambda = \frac{-\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \{a_k^2 \int_{\xi_k+0}^{\xi_{k+1}-0} \psi\psi''\, dx + \int_{\xi_k+0}^{\xi_{k+1}-0} q\psi^2\, dx\}}{\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} \psi^2\, dx}.$$

Integrating by parts gives equation (41). □

Equation (41) is the Rayleigh quotient for considered problem (1)-(5).

Theorem 5.2 (Minimization principle) *The infimum of the Rayleigh quotient for all nonzero continuous functions satisfying the boundary-transmission conditions* (2)-(4) *is equal to the least eigenvalue, that is,*

$$\lambda_1 := \inf R(y) = \inf \frac{-\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \{a_k^2 (\psi\psi' |_{\xi_k+0}^{\xi_{k+1}-0}) + \int_{\xi_k+0}^{\xi_{k+1}-0} (a_k^2 (y')^2 + qy^2)\, dx\}}{-\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} y^2\, dx}. \quad (43)$$

Proof Suppose that $\{\lambda_n\}$ is an increasing sequence of all eigenvalues of the Sturm-Liouville problem (1)-(5). Let us write the Rayleigh quotient in the form

$$R(y) = \frac{-\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} y\mathcal{L}_k y\, dx}{-\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} y^2\, dx}, \quad (44)$$

where $\mathcal{L}_k y := -a_k^2 y'' + qy$. Now, we expand the arbitrary function y in terms of the orthogonal eigenfunctions ψ_n. Denote $\Gamma := \{y \in \bigoplus_{i=1}^{n+1} C^2(\Omega_i)$: there exist finite one-hand limits $y^k(0+0), y^{(k)}(\pi-0), y^{(k)}(\xi_i \mp 0)$ for $i = \overline{1,n}, \mathcal{L}_\alpha y = \mathcal{L}_\beta y = \mathcal{L}_{2k-1} y = \mathcal{L}_{2k} y = 0, k = 1, 2, \ldots, n, y \neq 0\}$. If $y \in \Gamma$, then the series

$$y(x) = \sum_{m=0}^{\infty} r_m \psi_m(x) \quad (45)$$

converges uniformly to y, where $r_m = r_m(y)$ is the Fourier coefficient of y with respect to the orthogonal set ψ_n. By applying the standard well-known technique we can show that

$$\mathcal{L}y = \sum_{m=1}^{\infty} r_m \lambda_m \psi_m. \tag{46}$$

Now substitution of (45) and (46) into (44) gives us

$$R(y) = \frac{-\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} (\sum_{m=1}^{\infty} \sum_{s=1}^{\infty} r_m r_s \lambda_s \psi_m \psi_s)\,dx}{-\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \theta_{i34} \prod_{i=k+1}^{n+1} \theta_{i12} \int_{\xi_k+0}^{\xi_{k+1}-0} (\sum_{m=1}^{\infty} \sum_{s=1}^{\infty} r_m r_s \psi_m \psi_s)\,dx}. \tag{47}$$

Since the eigenfunctions ψ_n are orthogonal, equation (47) becomes

$$R(y) = \frac{\sum_{m=1}^{\infty} r_m^2 \lambda_m^2 \|\psi_m\|_{\mathcal{H}}^2}{\sum_{m=1}^{\infty} r_m^2 \|\psi_m\|_{\mathcal{H}}^2}. \tag{48}$$

Let λ_1 be the principal eigenvalue ($\lambda_1 < \lambda_m$ for all $m \geq 1$). Then

$$R(y) = \frac{\lambda_n \sum_{m=1}^{\infty} r_m^2 \|\psi_m\|_{\mathcal{H}}^2}{\sum_{m=0}^{\infty} r_m^2 \|\psi_m\|_{\mathcal{H}}^2} \geq \frac{\lambda_1 \sum_{m=1}^{\infty} r_m^2 \|\psi_m\|_{\mathcal{H}}^2}{\sum_{m=0}^{\infty} r_m^2 \|\psi_m\|_{\mathcal{H}}^2} = \lambda_1. \tag{49}$$

Therefore, $R(y) \geq \lambda_1$ for all $y \in \Gamma$, and thus $\inf R(y) \geq \lambda_1$. On the other hand, it is obvious that $R(y_1) = \lambda_1$, where y_1 is an eigenfunction corresponding to the least eigenvalue λ_1. The proof complete. □

Remark 5.3 In fact, it is proven that $\lambda_1 = \min R(y)$.

Corollary 5.4 *Let* $\lambda_1 < \lambda_2 < \cdots$ *be the eigenvalues of problem* (1)-(5). *Denote* $\Gamma_k := \{y \in \Gamma : \langle y, \psi_i \rangle = 0, i = 1, 2, \ldots, k\}$. *Then we have the equality*

$$\lambda_{k+1} = \min_{y \in \Gamma_k, y \neq 0} R(y) = \min_{y \in \Gamma_k, y \neq 0} \frac{\sum_{m=k+1}^{\infty} r_m^2 \lambda_m^2 \|\psi_m\|_{\mathcal{H}}^2}{\sum_{m=k+1}^{\infty} r_m^2 \|\psi_m\|_{\mathcal{H}}^2}. \tag{50}$$

Proof Consider relation (48). Let $y \in \Gamma_k$, $y \neq 0$. Then $r_j = 0$ ($j = 1, 2, \ldots, k$), and, consequently, by (47) we have

$$R(y) = \frac{\sum_{m=k+1}^{\infty} r_m^2 \lambda_m^2 \|\psi_m\|_{\mathcal{H}}^2}{\sum_{m=k+1}^{\infty} r_m^2 \|\psi_m\|_{\mathcal{H}}^2}. \tag{51}$$

Now since $\lambda_k < \lambda_m$ for $m > k + 1$, it follows that $R(y) \geq \lambda_{k+1}$, and, furthermore, the equality holds if $r_m = 0$ for $m > k + 1$ (*i.e.*, $y = r_{k+1}\psi_{k+1}$). □

Remark 5.5 By applying the Rayleigh-Ritz formula (43) it is difficult to explicitly compute the principal eigenvalues. But using the Rayleigh quotient (41) with appropriate test functions, we can obtain a good approximation for the eigenvalues. Moreover, from formula (50) it follows that $\lambda_k \leq R(z_k)$ for each test function $z_k \in \Gamma_k$. Thus, we can also find an upper bound for the kth eigenvalue.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. The authors read and approved the final manuscript.

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