Wei et al. *Advances in Difference Equations* (2016) 2016:65
DOI 10.1186/s13662-016-0781-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

pth moment exponential stability of hybrid stochastic fourth-order parabolic equations

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Abstract

We are concerned with a class of hybrid stochastic fourth-order parabolic equations with Markov switching in an infinite state space. By employing the fixed point theory we study the existence, uniqueness and pth moment exponential stability of the mild solution. Finally, we provide two examples to verify the effectiveness of our results.

Keywords: stochastic fourth-order parabolic equation; exponential stability; Markov chain; fixed point theory; mild solution

1 Introduction

In recent years, the hybrid stochastic differential equations (SDEs) and hybrid stochastic partial differential equations (SPDEs), for example, hybrid stochastic heat equations, have been paid much attention owing to their wide applications in natural science, engineering, biology, finance, and other areas. Researchers are interested in the existence and uniqueness of the solutions and their stability for SDEs and SPDEs. As is well known, the Lyapunov method provides a powerful implement in studying the stability of SDEs, and there are many methods to study the stability of SPDEs including hybrid stochastic heat equations, such as the Lyapunov function method [1], successive approximation approach [2], large deviation technique [3, 4], fixed pointed theory [5–7], and so on. It should be mentioned that the fixed point theory, which is introduced by Burton, is a very important method to discuss the stability of both deterministic and stochastic differential equations. In fact, many authors have used this method to study the stability of SDEs and SPDEs; lots of difficulties that arise by using the Lyapunov method vanish on applying fixed point theory, especially those problems with Markov chains and Poisson jumps. For instance, Yang and Zhu [5] studied the pth moment exponential stability of nonlinear hybrid stochastic heat equations, Luo [6] successfully gave some conditions for ensuring heat stochastic Volterra-Levin equations to be stable in mean square and also almost surely exponentially stable, Guo and Zhu [7] studied the stability of stochastic Volterra-Levin equations with Possion jumps, and so on.

However, to the best of the authors' knowledge, there are no published papers on the stability analysis for a class of stochastic fourth-order parabolic equations. In fact, fourth-order parabolic equations without noise disturbances have received extensive attention and were widely studied in the past few years. The most famous fourth-order parabolic equation is the Cahn-Hilliard equation, which was first proposed by Caln and Hilliard in 1958 when they studied the diffusion phenomena in phase transition (*e.g.*, alloy, polymer, *etc.*). Later, similar mathematical models were proposed in the study of many diffusion phenomena such as competition and exclusion of biological groups, moving process of a river basin, diffusion of an oil film over a solid suffice. As a model to describe these phenomena, the Cahn-Hilliard equation has intrigued many mathematicians' interests, and many good results [8–11] were obtained, such as the global existence, the asymptotic behavior, the stability of the solution of the Cahn-Hilliard equation, and so on. However, these results are all about deterministic equations. In fact, the real situations may exhibit indefinite stochastic factors, and they may exhibit sudden changes or go to different cases in different periods resulting in parameter transitions and probably changes in branch structure, so that we need to study the hybrid stochastic fourth-order parabolic equations involving Markov chains.

Inspired by the method of fixed point theory, which is widely used in the discussion of hybrid stochastic heat equations, in this paper, we are concerned with the stability problem for a class of hybrid stochastic fourth-order parabolic equations. Based on the fixed point theory, we not only obtain the existence and uniqueness of the mild solution, but also the pth moment exponential stability of the solution.

The rest of this paper is organized as follows. In Section 2, we introduce the notation and the model of hybrid stochastic fourth-order parabolic equations along with some necessary assumptions. In Section 3, based on the basic solution of the definite homogeneous fourth-order parabolic equation, by applying the fixed point theory we prove the existence, uniqueness, and pth moment exponential stability of hybrid stochastic fourth-order parabolic equations. In Section 4, we provide two examples to verify the effectiveness of the obtained results with some general remarks.

2 Preliminaries

In this section, we introduce some preliminaries and common notation for a more detailed description and then give the model that we will deal with.

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with a filtration satisfying the usual conditions. Let $\{B(t), t \geq 0\}$ be a real-valued Brownian motion defined on this probability space. Let $\Theta \subset \mathbb{R}^n$ be a bounded domain equipped with C^∞ boundary $\partial\Theta$. Let $L^p(\Theta)$ denote the family of all real-valued integrable functions equipped with the usual norm $\|f\|_p := (\int_\Theta f^p(x)\,dx)^{1/p}$, $f \in L^p(\Theta)$. In particular, when $p = 2$, we denote $\|f\| := \|f\|_2 = (\int_\Theta f^2(x)\,dx)^{1/2}$ and the inner product $\langle f, g\rangle = \int_\Theta f(x)g(x)\,dx$ for $f, g \in L^2(\Theta)$. Also, we can define $H^m(\Theta)$, $m = 1, 2, 3, 4$, by $H^m(\Theta) := \{u \in L^2(\Theta) | D^\alpha u \in L^2(\Theta), |\alpha| \leq m\}$ and $H_0^m(\Theta) := \{u \in H^m(\Theta) | u = 0 \text{ on } \partial\Theta\}$. Let $A := \sum_{i=1}^n \partial^2/\partial x_i^2$ be the Laplace operator with domain $\mathcal{D}(A) := H_0^m(\Theta) \cap H^m(\Theta)$, which generates a strongly continuous semigroup e^{tA}. We denote the Fourier transform of u as $\hat{u}$ with $\hat{u}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\,dx$, and the inverse Fourier transform of u as $\check{u}$ with $\check{u}(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} u(\xi)\,d\xi$. Furthermore, let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain that takes values in a listed state space $S = \{1, 2, \ldots, N\}$, where N is some positive integer, or arrives at ∞. Moreover, we assume that the Markov chain $\{r(t), t \geq 0\}$ is independent of the Brownian motion $\{B(t), t \geq 0\}$.

In this paper, we consider the following hybrid stochastic fourth-order parabolic equation:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = Au - A^2u + \alpha(r(t))u(t,x) + \beta(r(t))u(t,x)\dot{B}(t), & x \in \Theta, t > 0, \\ u(t,x) = 0, & x \in \partial\Theta, t > 0, \\ u(0,x) = u_0(x), & x \in \Theta. \end{cases} \tag{2.1}$$

Here the initial value u_0 is a $\mathcal{D}(A)$-valued $\mathcal{F}_0$-measurable random variable, independent of $r(\cdot)$ and $B(\cdot)$, such that $E\|u_0\|^p < \infty$ for any $p > 0$. Moreover, α, β are the mappings from $S \to R$, and in this paper we take simply $\alpha_i := \alpha(i)$, $\beta_i := \beta(i)$.

Next, we will give the definitions of a mild solution to Eq. (2.1) and pth moment exponential stability, together with the well-known BDG inequality, which plays an important role in the proofs of our results. For convenience, we will take $u(t) := u(t,\cdot)$ and $u := \{u(t,\cdot)\}_{t\geq 0}$ such that u is an $L^2(\Theta)$-valued stochastic process since $u(t)$ is an $L^2(\Theta)$-valued random variable.

Definition 1 An $L^2(\Theta)$-valued stochastic process $u = \{u(t,\cdot)\}_{t\in[0,T]}$ is a mild solution to (2.1) if the following conditions are satisfied:

(i) $u \in C([0,T]; L^2(\Theta))$ for any $t \in [0,T]$, and $u(t)$ is adapted to $\mathcal{F}_t$ with

$$P\left\{\omega : \int_0^t \|u(s)\|^p\,ds < \infty\right\} = 1;$$

(ii) u satisfies the stochastic integral equation

$$u(t,x) = e^{(A-A^2)t}u_0 + \int_0^t e^{(A-A^2)(t-s)}\alpha(r(s))u(s,x)\,ds + \int_0^t e^{(A-A^2)(t-s)}\beta(r(s))u(s,x)\,dB(s) \tag{2.2}$$

a.s. for any $t \in [0,T]$ and $x \in \Theta$.

Definition 2 Equation (2.1) is called exponentially stable in the pth moment if there exist a pair of constants $\delta > 0$ and $K > 0$ such that $\mathbf{E}\|u(t)\|^p \leq K\mathbf{E}\|u_0\|^p e^{-\delta t}$, $t \geq 0$.

Lemma 1 [12] (Bukholder-Davis-Gundy inequality) *For every* $0 < p < \infty$, *there exists a universal constant* J_p *such that for every continuous local martingale* M *vanishing at zero and any stopping time* T,

$$\mathbf{E}(\|M_T\|^p) \leq \mathbf{E}\Big(\sup_{0\leq s\leq T}\|M_s\|^p\Big) \leq J_p\mathbf{E}(\langle M,M\rangle_T)^{\frac{p}{2}}, \tag{2.3}$$

where $\langle M,M\rangle$ *is the cross-variation of* M.

Finally, we present a necessary assumption to close this section.

Assumption A

(A_1) $\|e^{tA}\| \leq Me^{-\gamma t}$ for some constant M and $\gamma > 0$;

(A_2) $\sup_{1\leq i\leq N}|\alpha_i| < \infty$, $\sup_{1\leq i\leq N}|\beta_i| < \infty$.

3 *p*th moment exponential stability

In this section, we will first deduce the exponential stability of the basic solution of the homogeneous equation of (2.1), and then we will use the fixed point principle to discuss the existence and uniqueness of the mild solution to (2.1) and prove the exponential stability results.

Theorem 1 *Let $p \geq 2$ and suppose that Assumption* A *holds. Then* (2.1) *is exponentially stable in the pth moment if*

$$\frac{2^{p-1}\tilde{M}^p}{\gamma^p}\left|\alpha^*\right|^p + \frac{2^{p-1}\tilde{M}^p J_p}{(2\gamma)^{\frac{p}{2}}}\left|\beta^*\right|^p \in (0,1), \tag{3.1}$$

where $J_p = (p^{p+1}/2(p-1)^{p-1})^{\frac{p}{2}}$, $|\alpha^*| = \sup_{1\leq i\leq N}|\alpha_i|$, $|\beta^*| = \sup_{1\leq i\leq N}|\beta_i|$, *and* $\tilde{M}$ *is some constant.*

Proof Let H be the Banach space of all $\mathcal{F}_t$-adapted continuous processes consisting of functions $u(t,x)$ such that $\mathbf{E}\|u(t,x)\|^p \leq M^*\mathbf{E}\|u_0\|^p e^{-\eta t}$, $t \geq 0$, where $M^* > 0$, $0 < \eta < \gamma$. We denote the norm in H by $\|u(t,x)\|_H := \sup_{t\geq 0}\mathbf{E}\|u(t,x)\|^p$. Next, we divide the proof into two parts.

In part 1, we deduce the exponential stability of the basic solution of the homogeneous equation of (2.1), and in part 2, we employ the fixed point principle to discuss the case of the nonhomogeneous equation.

Part 1: Analysis of the basic solution.

The mild solution of (2.1) can be expressed as follows:

$$\begin{aligned} u(t,x) &= e^{(A-A^2)t}u_0 + \int_0^t e^{(A-A^2)(t-s)}\alpha\bigl(r(s)\bigr)u(s,x)\,ds \\ &\quad + \int_0^t e^{(A-A^2)(t-s)}\beta\bigl(r(s)\bigr)u(s,x)\,dB(s), \end{aligned} \tag{3.2}$$

where $e^{(A-A^2)t}u_0 = \int_\Theta G(x-y)u_0(y)\,dy$ is the solution of

$$u_t = Au - A^2u \tag{3.3}$$

with the initial value u_0, and $G(t,x)$ is the basic solution of (3.3).

Taking the Fourier transform of (3.3), we obtain

$$\hat{u}_t + \bigl(|\xi|^2 + |\xi|^4\bigr)\hat{u} = 0. \tag{3.4}$$

Hence, we have

$$\hat{G}(t)(\xi) = e^{-(|\xi|^2+|\xi|^4)t}, \tag{3.5}$$

and then

$$G(t,x) = F^{-1}\bigl(e^{-(|\xi|^2+|\xi|^4)t}\bigr) = G_1(t,x) * G_2(t,x). \tag{3.6}$$

Here F^{-1} denotes the inverse Fourier transform, '$*$' denotes the convolution of $G_1(t,x)$ and $G_2(t,x)$, $G_1(t,x)$ is the basic solution of

$$u_t - Au = 0 \tag{3.7}$$

with $\hat{G}_1(t)(\xi) = e^{-|\xi|^2 t}$, whereas $G_2(t,x)$ is the basic solution of

$$u_t + A^2 u = 0 \tag{3.8}$$

with $\hat{G}_2(t)(\xi) = e^{-|\xi|^4 t}$. By [11] we see that the decay behavior of solution for (3.8) is as follows:

$$\|G_2(t,x)\|_p \le C(p)t^{-\frac{1}{4}(1-\frac{1}{p})} \quad (p \ge 1). \tag{3.9}$$

We denote $e^{At} := G_1(x,t)$, $e^{-A^2 t} := G_2(x,t)$, and $e^{(A-A^2)t} := G(x,t)$. Then it follows from (3.6), (3.9), Assumption (A_1), and the Young inequality with convolution that

$$\begin{aligned} \|e^{(A-A^2)t}\| &= \|G(t,\cdot)\| = \|G_1(t,\cdot) * G_2(t,\cdot)\| \\ &\le \|G_1(t,\cdot)\| \|G_2(t,\cdot)\|_1 \le C\|G_1(t,\cdot)\| \\ &= C\|e^{tA}\| \le \tilde{M}e^{-\gamma t}. \end{aligned} \tag{3.10}$$

Hence, we get the exponential stability of the basic solution $e^{(A-A^2)t}$ of (3.3).

Part 2: We will discuss the existence, uniqueness, and exponential stability of the mild solution to (2.1).

We derive the operator $\phi : H \to H$ as follows:

$$\begin{cases} \phi(u)(t) = e^{(A-A^2)t}u_0 + \int_0^t e^{(A-A^2)(t-s)}\alpha(r(s))u(s,x)\,ds \\ \qquad + \int_0^t e^{(A-A^2)(t-s)}\beta(r(s))u(s,x)\,dB(s), \\ \phi(u)(0) = u_0. \end{cases}$$

It is easy to prove that the following holds by the C_p inequality:

$$\begin{aligned} \mathbf{E}\|\phi(u)(t)\|^p &\le 3^{p-1}\mathbf{E}\|e^{(A-A^2)t}u_0\|^p + 3^{p-1}\mathbf{E}\left\|\int_0^t e^{(A-A^2)(t-s)}\alpha\bigl(r(s)\bigr)u(s,x)\,ds\right\|^p \\ &\quad + 3^{p-1}\mathbf{E}\left\|\int_0^t e^{(A-A^2)(t-s)}\beta\bigl(r(s)\bigr)u(s,x)\,dB(s)\right\|^p \\ &:= 3^{p-1}\sum_{i=1}^3 \mathbf{E}\|I_i(t)\|^p. \end{aligned} \tag{3.11}$$

Next, we will divide the proof into three steps.

Claim 1: ϕ is continuous in the pth moment on $[0,+\infty)$.

Proof of Claim 1: Let $u \in H$, $t_1 \ge 0$, and $|r|$ be sufficiently small. Then we get

$$\begin{aligned} \mathbf{E}\|I_1(t_1+r) - I_1(t_1)\|^p &= \mathbf{E}\|e^{(A-A^2)(t_1+r)}u_0 - e^{(A-A^2)t_1}u_0\|^p \\ &\le \mathbf{E}\|e^{(A-A^2)t_1}u_0\bigl(e^{(A-A^2)r} - 1\bigr)\|^p \to 0 \quad \bigl(|r| \to 0\bigr); \end{aligned} \tag{3.12}$$

$$\begin{aligned}
\mathbf{E}\big\|I_2(t_1+r)-I_2(t_1)\big\|^p &= \mathbf{E}\bigg\|\int_0^{t_1+r} e^{(A-A^2)(t_1+r-s)}\alpha\big(r(s)\big)u(s,x)\,ds \\
&\quad - \int_0^{t_1} e^{(A-A^2)(t_1-s)}\alpha\big(r(s)\big)u(s,x)\,ds\bigg\|^p \\
&= \mathbf{E}\bigg\|\int_0^{t_1} e^{(A-A^2)(t_1+r-s)}\alpha\big(r(s)\big)u(s,x)\,ds \\
&\quad + \int_{t_1}^{t_1+r} e^{(A-A^2)(t_1+r-s)}\alpha\big(r(s)\big)u(s,x)\,ds \\
&\quad - \int_0^{t_1} e^{(A-A^2)(t_1-s)}\alpha\big(r(s)\big)u(s,x)\,ds\bigg\|^p \\
&\leq 2^{p-1}\mathbf{E}\bigg\|\int_0^{t_1} e^{(A-A^2)(t_1-s)}\alpha\big(r(s)\big)u(s,x)\big(e^{(A-A^2)r}-1\big)\,ds\bigg\|^p \\
&\quad + 2^{p-1}\mathbf{E}\bigg\|\int_{t_1}^{t_1+r} e^{(A-A^2)(t_1+r-s)}\alpha\big(r(s)\big)u(s,x)\,ds\bigg\|^p \\
&\to 0 \quad \big(|r|\to 0\big).
\end{aligned} \tag{3.13}$$

Then, by the BDG inequality (Lemma 1) and (3.10), the following holds as $|r|\to 0$:

$$\begin{aligned}
&\mathbf{E}\big\|I_3(t_1+r)-I_3(t_1)\big\|^p \\
&\quad = \mathbf{E}\bigg\|\int_0^{t_1+r} e^{(A-A^2)(t_1+r-s)}\beta\big(r(s)\big)u(s,x)\,dB(s) - \int_0^{t_1} e^{(A-A^2)(t_1-s)}\beta\big(r(s)\big)u(s,x)\,dB(s)\bigg\|^p \\
&\quad \leq 2^{p-1}\mathbf{E}\bigg\|\int_0^{t_1} e^{(A-A^2)(t_1-s)}\beta\big(r(s)\big)u(s,x)\big(e^{(A-A^2)r}-1\big)\,dB(s)\bigg\|^p \\
&\qquad + 2^{p-1}\mathbf{E}\bigg\|\int_{t_1}^{t_1+r} e^{(A-A^2)(t_1+r-s)}\beta\big(r(s)\big)u(s,x)\,dB(s)\bigg\|^p \\
&\quad \leq 2^{p-1}J_p\mathbf{E}\bigg(\int_0^{t_1} \tilde{M}^2 e^{-2(t_1-s)}\big\|\beta\big(r(s)\big)u(s,x)\big(e^{(A-A^2)r}-1\big)\big\|^2\,ds\bigg)^{\frac{p}{2}} \\
&\qquad + 2^{p-1}J_p\mathbf{E}\bigg(\int_{t_1}^{t_1+r} \tilde{M}^2 e^{-2(t_1+r-s)}\big\|\beta\big(r(s)\big)u(s,x)\big\|^2\,ds\bigg)^{\frac{p}{2}} \\
&\quad \to 0 \quad \big(|r|\to 0\big).
\end{aligned} \tag{3.14}$$

Hence, we see that ϕ is pth continuous on $[0,+\infty)$.
Claim 2: $\phi(H)$ is contained in H.
Proof of Claim 2: It follows from (3.10) and (3.11) that

$$\mathbf{E}\big\|I_1(t)\big\|^p = \mathbf{E}\big\|e^{(A-A^2)t}u_0\big\|^p \leq \tilde{M}^p e^{-p\gamma t}\mathbf{E}\|u_0\|^p \leq \tilde{M}^p e^{-\eta t}\mathbf{E}\|u_0\|^p. \tag{3.15}$$

By Assumptions (A_1) and (A_2), (3.10), and by the Hölder inequality we have

$$\begin{aligned}
\mathbf{E}\big\|I_2(t)\big\|^p &= \mathbf{E}\bigg\|\int_0^t e^{(A-A^2)(t-s)}\alpha\big(r(s)\big)u(s,x)\,ds\bigg\|^p \\
&\leq \tilde{M}^p\mathbf{E}\bigg(\int_0^t e^{-\gamma(t-s)}\big|\alpha\big(r(s)\big)\big|\big\|u(s,x)\big\|\,ds\bigg)^p
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{M}^p \mathbf{E}\left(\int_0^t e^{-\gamma(t-s)(1-\frac{1}{p})} e^{-\gamma(t-s)\frac{1}{p}} \left|\alpha\big(r(s)\big)\right| \left\|u(s,x)\right\| ds\right)^p \\
&\le \tilde{M}^p \mathbf{E}\left(\int_0^t e^{-\gamma(t-s)}\, ds\right)^{p-1} \int_0^t e^{-\gamma(t-s)} \left|\alpha\big(r(s)\big)\right|^p \left\|u(s,x)\right\|^p ds \\
&\le \frac{\tilde{M}^p}{\gamma^{p-1}} \mathbf{E} \int_0^t e^{-\gamma(t-s)} \left|\alpha\big(r(s)\big)\right|^p \left\|u(s,x)\right\|^p ds \\
&\le \frac{\tilde{M}^p |\alpha^*|^p}{\gamma^{p-1}} \mathbf{E} \int_0^t e^{-\gamma(t-s)} \left\|u(s,x)\right\|^p ds \\
&\le \frac{\tilde{M}^p |\alpha^*|^p}{\gamma^{p-1}} \int_0^t e^{-\gamma(t-s)} M^* \mathbf{E}\|u_0\|^p e^{-\eta t}\, ds \\
&\le \frac{\tilde{M}^p M^* |\alpha^*|^p}{\gamma^{p-1}} \mathbf{E}\|u_0\|^p \int_0^t e^{-\gamma(t-s)} e^{-\eta t}\, ds \\
&\le \frac{\tilde{M}^p M^* |\alpha^*|^p}{\gamma^{p-1}(\gamma-\eta)} \mathbf{E}\|u_0\|^p e^{-\eta t},
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
\mathbf{E}\left\|I_3(t)\right\|^p &= \mathbf{E}\left\|\int_0^t e^{(A-A^2)(t-s)} \beta\big(r(s)\big) u(s,x)\, dB(s)\right\|^p \\
&\le \tilde{M}^p J_p \mathbf{E}\left(\int_0^t e^{-2\gamma(t-s)} \left|\beta\big(r(s)\big)\right|^2 \left\|u(s,x)\right\|^2 ds\right)^{\frac{p}{2}} \\
&\le \tilde{M}^p J_p \mathbf{E}\left(\int_0^t e^{-2\gamma(t-s)(1-\frac{2}{p})} e^{-2\gamma(t-s)\frac{2}{p}} \left|\beta\big(r(s)\big)\right|^2 \left\|u(s,x)\right\|^2 ds\right)^{\frac{p}{2}} \\
&\le \tilde{M}^p J_p \mathbf{E}\left(\int_0^t e^{-2\gamma(t-s)}\, ds\right)^{\frac{p}{2}-1} \int_0^t e^{-2\gamma(t-s)} \left|\beta\big(r(s)\big)\right|^p \left\|u(s,x)\right\|^p ds \\
&\le \frac{\tilde{M}^p J_p}{(2\gamma)^{\frac{p}{2}-1}} \left|\beta^*\right|^p \int_0^t e^{-2\gamma(t-s)} \mathbf{E}\left\|u(s,x)\right\|^p ds \\
&\le \frac{\tilde{M}^p J_p}{(2\gamma)^{\frac{p}{2}-1}} \left|\beta^*\right|^p M^* \int_0^t e^{-2\gamma(t-s)} \mathbf{E}\|u_0\|^p e^{-\eta s}\, ds \\
&\le \frac{\tilde{M}^p J_p}{(2\gamma)^{\frac{p}{2}-1}} \left|\beta^*\right|^p M^* \mathbf{E}\|u_0\|^p \int_0^t e^{-2\gamma(t-s)} e^{-\eta s}\, ds \\
&\le \frac{\tilde{M}^p M^* J_p |\beta^*|^p}{(2\gamma)^{\frac{p}{2}-1}(2\gamma-\eta)} \mathbf{E}\|u_0\|^p e^{-\eta t}.
\end{aligned}
\tag{3.17}
$$

From (3.15)-(3.17) it is easy to see that

$$
\mathbf{E}\left\|\phi(u)(t)\right\|^p \le k\mathbf{E}\left\|\phi(u)(0)\right\|^p e^{-\eta t}, \tag{3.18}
$$

where $k = 3^{p-1}\tilde{M}^p(1 + \frac{M^*|\alpha^*|^p}{\gamma^{p-1}(\gamma-\eta)} + \frac{M^* J_p |\beta^*|^p}{(2\gamma-\eta)(2\gamma)^{\frac{p}{2}-1}})$, which means that $\phi(H) \subseteq H$.

Claim 3: ϕ is contractive for arbitrary $u, v \in H$ with $u(0,x) = v(0,x) = u_0(x)$.

Proof of Claim 3: Consider the following:

$$
\begin{aligned}
&\mathbf{E} \sup_{0\le t<\infty} \left\|\phi(u)(t) - \phi(v)(t)\right\|^p \\
&\quad \le 2^{p-1} \mathbf{E} \sup_{0\le t<\infty} \left\|\int_0^t e^{(A-A^2)(t-s)} \alpha\big(r(s)\big)\big(u(s,x) - v(s,x)\big)\, ds\right\|^p
\end{aligned}
$$

$$
\begin{aligned}
&+ 2^{p-1}\mathbf{E} \sup_{0\le t<\infty} \left\| \int_0^t e^{(A-A^2)(t-s)}\beta\big(r(s)\big)\big(u(s,x)-v(s,x)\big)\,dB(s)\right\|^p \\
&\le 2^{p-1}\tilde{M}^p\mathbf{E} \sup_{0\le t<\infty} \left(\int_0^t e^{-\gamma(t-s)}\big|\alpha\big(r(s)\big)\big|\big\|u(s,x)-v(s,x)\big\|\,ds\right)^p \\
&\quad + 2^{p-1}\tilde{M}^p J_p\mathbf{E} \sup_{0\le t<\infty} \left(\int_0^t e^{-2\gamma(t-s)}\big|\beta\big(r(s)\big)\big|^2\big\|u(s,x)-v(s,x)\big\|^2\,ds\right)^{\frac{p}{2}} \\
&\le 2^{p-1}\frac{\tilde{M}^p|\alpha^*|^p}{\gamma^{p-1}}\mathbf{E} \sup_{0\le t<\infty} \int_0^t e^{-\gamma(t-s)}\big\|u(s,x)-v(s,x)\big\|^p\,ds \\
&\quad + 2^{p-1}\frac{\tilde{M}^p J_p|\beta^*|^p}{(2\gamma)^{\frac{p}{2}-1}}\mathbf{E} \sup_{0\le t<\infty} \int_0^t e^{-2\gamma(t-s)}\big\|u(s,x)-v(s,x)\big\|^p\,ds \\
&\le 2^{p-1}\frac{\tilde{M}^p|\alpha^*|^p}{\gamma^{p-1}}\mathbf{E} \sup_{0\le t<\infty} \big\|u(s,x)-v(s,x)\big\|^p \int_0^t e^{-\gamma(t-s)}\,ds \\
&\quad + 2^{p-1}\frac{\tilde{M}^p J_p|\beta^*|^p}{(2\gamma)^{\frac{p}{2}-1}}\mathbf{E} \sup_{0\le t<\infty} \big\|u(s,x)-v(s,x)\big\|^p \int_0^t e^{-2\gamma(t-s)}\,ds \\
&\le 2^{p-1}\frac{\tilde{M}^p|\alpha^*|^p}{\gamma^{p}}\mathbf{E} \sup_{0\le t<\infty} \big\|u(s,x)-v(s,x)\big\|^p \\
&\quad + 2^{p-1}\frac{\tilde{M}^p J_p|\beta^*|^p}{(2\gamma)^{\frac{p}{2}}}\mathbf{E} \sup_{0\le t<\infty} \big\|u(s,x)-v(s,x)\big\|^p \\
&\le \tilde{k}\mathbf{E} \sup_{0\le t<\infty} \big\|u(s,x)-v(s,x)\big\|^p,
\end{aligned} \tag{3.19}
$$

where

$$
\tilde{k} = \frac{2^{p-1}\tilde{M}^p|\alpha^*|^p}{\gamma^p} + \frac{2^{p-1}\tilde{M}^p J_p|\beta^*|^p}{(2\gamma)^{\frac{p}{2}}}. \tag{3.20}
$$

Recalling condition (3.1) and noting that $\tilde{k} \in (0,1)$, we see that ϕ is a contractive mapping. By the fixed point theory we derive that ϕ has a unique fixed point $u(t,x)$ in H, which is also exponentially stable in the pth moment from the proofs of the two parts.

Therefore, the proof of the desired assertion in Theorem 1 is completed. □

Remark 1 If $p = 2$, then it is obvious that (2.1) is mean square exponentially stable.

Remark 2 In Theorem 1, we apply the fixed point theory to obtain the existence and uniqueness of the solution for a class of linear hybrid stochastic fourth-order parabolic equations. In fact, if we add some proper assumptions, then we can also get some good results for the nonlinear case. We leave this for the future work.

4 Two examples

In this section, we provide two examples for stochastic fourth-order parabolic equations with Markovian switching as applications of our main results.

Example 1 Consider the following stochastic fourth-order parabolic equation with Markovian switching:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = Au - A^2 u + \alpha(r(t))u(t,x) + \beta(r(t))u(t,x)\dot{B}(t), & x \in (0,\pi), t > 0, \\ u(t,0) = u(t,\pi) = 0, & t > 0, \\ u(0,x) = \sqrt{\frac{\pi}{2}}\cos x, & x \in (0,\pi), \end{cases} \tag{4.1}$$

where $\{r(t), t \geq 0\}$ takes values in $S = \{1,2\}$ with generator $\Gamma = (\gamma_{ij})_{2\times 2}$:

$$-\gamma_{11} = \gamma_{12} = 2, \qquad -\gamma_{22} = \gamma_{21} = q > 0, \tag{4.2}$$

which implies that the Markov chain $\{r(t), t \geq 0\}$ has a unique invariant measure $\pi = (\pi_1, \pi_2) = (q/q+2, 2/q+2)$. Take $\alpha(1) = a/\tilde{M}$, $\alpha(2) = b/\tilde{M}$, $\alpha(3) = c/\tilde{M}$, $\alpha(4) = d/\tilde{M}$, where $a,b,c,d \in R$, $\tilde{M}$ are some constants. Recall that A is an infinitesimal generator with a strongly continuous semigroup e^{tA}, $t \geq 0$, so that the eigenfunctions of $-A$ are $e_n(x) = \sqrt{\frac{\pi}{2}}\sin nx \in \mathcal{D}(A)$, $n = 1,2,\ldots$, and the relevant eigenvalues of A are $\lambda_n = n^2$, that is, $Au = \sum_{n=1}^{\infty} -n^2\langle u, e_n\rangle_H e_n$, and we can easily get $e^{tA}u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle_H e_n$, $u \in H$, and $\|e^{tA}\| \leq e^{-\pi^2 t}$, $t \geq 0$. So we have $\gamma = \pi^2$, $M = 1$. Then it is easy to compute

$$\tilde{k} = \frac{2^{p-1}}{\pi^{2p}}[a \vee b]^p + \frac{\sqrt{2}^{p-2} J_p}{\pi^p}[c \vee d]^p. \tag{4.3}$$

So we can conclude that system (4.1) is pth moment exponentially stable if $\tilde{k} < 1$.

Example 2 We consider the stochastic fourth-order parabolic equation (4.1) with the infinite state space S in which the Markov chain $\{r(t), t \geq 0\}$ takes values.

Let $\{r(t), t \geq 0\}$ in (4.1) be a right-continuous Markov chain that takes values in an infinite state space $S = \{1,2,\ldots\}$. Take $\alpha_i = \frac{1}{3\tilde{M}i}$ and $\beta_i = \frac{J_p^{-\frac{1}{p}}}{\tilde{M}i}$, where $\tilde{M}$ is some constant. Then

$$\left|\alpha^*\right| = \sup_{1\leq i\leq N}\left|\frac{1}{3\tilde{M}i}\right| = \frac{1}{3\tilde{M}}, \qquad \left|\beta^*\right| = \sup_{1\leq i\leq N}\left|\frac{J_p^{-\frac{1}{p}}}{\tilde{M}i}\right| = \frac{J_p^{-\frac{1}{p}}}{\tilde{M}}. \tag{4.4}$$

It is easy to conclude that

$$\tilde{k} = \frac{1}{3\pi^{2p}} + \frac{(\sqrt{2})^{p-2}}{\pi^p} \in (0,1). \tag{4.5}$$

Therefore, by Theorem 1 we see that (4.1) is pth moment exponentially stable.

Remark 3 The state space S in Example 2 in which Markov chain $\{r(t), t \geq 0\}$ takes values is infinite, so generally we can get the pth moment exponential stability of the solution by assuming the space to be finite.

Remark 4 If $p = 2$,then (4.1) is mean square exponentially stable.

Remark 5 It should be mentioned that Bao *et al.* [4] applied the large derivative technique to discuss the Lyapunov exponent stability of hybrid stochastic heat equation. Differently from [4], we have studied the pth moment exponential stability of hybrid stochastic fourth-order parabolic equations. Moreover, we employ the fixed point theory, which is different from the method of [4].

5 Conclusions

In this paper, we have studied the stability problem of a class of hybrid stochastic fourth-order parabolic equations. Based on the fixed point theory and Bukholder-Davis-Gundy inequality, we not only established the existence and uniqueness of the equation in an infinite state space, but also proved the pth moment exponential stability of the system. Moreover, at the end of this paper, we gave two simple examples to verify all our conditions. In the future work, we will focus ourselves on the stability of more complicated models such as nonlinear hybrid stochastic fourth-order parabolic equations with Markov chains and variable time delay.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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Acknowledgements

This work was jointly supported by the Alexander von Humboldt Foundation of Germany (Fellowship CHN/1163390), the National Natural Science Foundation of China (61374080), and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

Received: 14 October 2015 Accepted: 7 February 2016 Published online: 02 March 2016

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