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Existence and uniqueness of positive solutions for a fractional differential equation with integral boundary conditions

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Abstract

We study a class of boundary value problems for a fractional differential equation with integral boundary conditions. By means of u_0-positive operator we obtain results on the existence and uniqueness of positive solutions for the boundary value problem.

Keywords: u_0-positive operator; integral boundary condition; positive solution; fractional differential equation

1 Introduction

Nowadays, fractional differential equations become more and more important. They play an important role in engineering, science,economics, and so on. More and more people pay attention to the study of theory and applications of fractional differential equations [1–8]. Many efforts have also been made to develop the theory of fractional evolution equations: we refer the readers to [9–11]. A lot of papers are devoted to the positive solutions of boundary value problem for fractional differential equations, such as [12–14].

For example, Jiang and Yuan [12] investigated the existence and multiplicity of positive solutions of the Dirichlet-type boundary value problem for the nonlinear fractional differential equation

$$\begin{cases} D_{0^+}^{p} x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \\ x(0) = x(1) = 0. \end{cases} \tag{1.1}$$

Zhang, Liu, and Wu [13] studied the existence of multiple positive solutions for the boundary value problem

$$\begin{cases} -D_{0^+}^{p} x(t) + p(t) f(t, x(t)) - q(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \qquad x(1) = 0, \end{cases} \tag{1.2}$$

where $2 < p \leq 3$ is a real number, $q : (0,1) \to [0, +\infty]$ is a Lebesgue-integrable function nonvanishing identically on any subinterval of $(0,1)$. The authors obtained existence results by Krasnoselskii's fixed point theorem in a cone.

Cui [14] considered the uniqueness of the fractional differential equations with easy boundary conditions

$$\begin{cases} D_{0^+}^p x(t) + p(t)f(t,x(t)) + q(t) = 0, & t \in (0,1), \\ x(0) = x'(0) = 0, & x(1) = 0, \end{cases} \tag{1.3}$$

where $2 < p \leq 3$ is a real number, and $f(t,x)$ is a continuous function. The author studied the existence of solutions for (1.3) by using a u_0-positive operator.

In this paper, we discuss the existence and uniqueness of a positive solution for the following boundary value problems with integral boundary conditions:

$$\begin{cases} D_{0^+}^p x(t) + p(t)f(t,x(t)) + q(t) = 0, & t \in (0,1), \\ x(0) = x'(0) = 0, & x(1) = \int_0^1 l(s)x(s)\,ds, \end{cases} \tag{1.4}$$

where $2 < p \leq 3$ is a real number, and $D_{0^+}^p$ is the standard Riemann-Liouville derivative. By means of a new u_0-positive operator we get the existence and uniqueness of positive solutions for (1.4).

Throughout this paper, we always assume that the following conditions are satisfied:

(A_1) $p : (0,1) \to [0,+\infty)$ is a continuous function nonvanishing identically on any subinterval of $(0,1)$ with

$$\int_0^1 p(s)\,ds < +\infty.$$

(A_2) $f : [0,1] \times \mathbb{R} \to [0,+\infty)$ is continuous, and $q : (0,1) \to [0,+\infty)$ is continuous and Lebesgue integrable.

(A_3) $l : (0,1) \to [0,+\infty)$ is continuous, and $0 \leq \int_0^1 l(t)t^{p-1}\,dt < 1$.

2 Preliminaries and relevant lemmas

In order to obtain the main results of this work, we present some necessary definitions and several fundamental lemmas.

Definition 2.1 ([2, 15]) The Riemann-Liouville fractional integral of order $p > 0$ of a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^p u(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([2, 15]) The Riemann-Liouville fractional derivative of order $p > 0$ of a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^p u(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-p-1} u(s)\,ds,$$

where $n - 1 \leq p < n$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.3 ([16, 17]) Let E be a Banach space, and P a cone in E. We say that a bounded linear operator $S : E \to E$ is u_0-positive on the cone P if there exists $u_0 \in P \setminus \{\theta\}$ such that for every $x \in P \setminus \{\theta\}$, there exist a natural number n and positive constants $\alpha(x)$, $\beta(x)$ such that

$$\alpha(x)u_0 \leq S^n x \leq \beta(x)u_0.$$

Lemma 2.1 ([16, 17]) *Let E be a Banach space. Suppose that $S : E \to E$ is a completely continuous linear operator and $S(P) \subset P$. If there exist $\psi \in E \setminus \{-P\}$ and a constant $c > 0$ such that $cS\psi \geq \psi$, then the spectral radius $r(S) \neq 0$, and S has a positive eigenfunction φ corresponding to its first eigenvalue $\lambda_1 = (r(S))^{-1}$, that is, $\varphi = \lambda_1 S\varphi$.*

Let $E = C[0, 1]$, which is a Banach space with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Set $P = \{x \in E | x(t) \geq 0, \forall t \in [0, 1]\}$. In the rest of this paper, the partial ordering in $C[0, 1]$ is always given by P.

Lemma 2.2 ([13]) *Let $p > 0$, and let $u(t)$ be an integrable function. Then*

$$I_{0^+}^p D_{0^+}^p u(t) = u(t) + c_1 t^{p-1} + c_2 t^{p-2} + \cdots + c_n t^{p-n},$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$), and n is the smallest integer greater than or equal to p.

Lemma 2.3 *Let $\sigma \in C(0, 1) \cap L(0, 1)$, $2 < p \leq 3$. Then the unique solution of*

$$\begin{cases} D_{0^+}^p x(t) + \sigma(t) = 0, & t \in (0, 1), \\ x(0) = x'(0) = 0, & x(1) = \int_0^1 l(s)x(s)\, ds, \end{cases} \tag{2.1}$$

is given by

$$x(t) = \int_0^1 G(t, s)\sigma(s)\, ds - \frac{\int_0^1 \int_\tau^1 l(s)(s - \tau)^{p-1}\sigma(\tau)\, ds\, d\tau}{1 - \int_0^1 l(s)s^{p-1}\, ds} t^{p-1},$$

where

$$G(t, s) = \begin{cases} \frac{1}{\Gamma(p)} \{\frac{t^{p-1}(1-s)^{p-1}}{1-\int_0^1 l(t)t^{p-1}\, dt} - (t - s)^{p-1}\}, & 0 \leq s \leq t \leq 1, \\ \frac{1}{\Gamma(p)} \{\frac{t^{p-1}(1-s)^{p-1}}{1-\int_0^1 l(t)t^{p-1}\, dt}\}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Proof From Lemma 2.2 it follows that

$$x(t) = -I_{0^+}^p \sigma(t) + c_1 t^{p-1} + c_2 t^{p-2} + c_3 t^{p-3}.$$

So, the solution of (2.1) is

$$x(t) = -\frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1}\sigma(s)\, ds + c_1 t^{p-1} + c_2 t^{p-2} + c_3 t^{p-3}.$$

Since $x(0) = x'(0) = 0$, we have that $c_2 = c_3 = 0$.

In addition,

$$\begin{aligned} x(1) &= -\frac{1}{\Gamma(p)}\int_0^1 (1-s)^{p-1}\sigma(s)\,ds + c_1 \\ &= \int_0^1 l(s)\left[-\frac{1}{\Gamma(p)}\int_0^s (s-\tau)^{p-1}\sigma(\tau)\,d\tau + c_1 s^{p-1}\right]ds \end{aligned}$$

yields

$$c_1 = \frac{\frac{1}{\Gamma(p)}[\int_0^1 (1-s)^{p-1}\sigma(s)\,ds - \int_0^1\int_\tau^1 l(s)(s-\tau)^{p-1}\sigma(\tau)\,ds\,d\tau]}{1-\int_0^1 l(s)s^{p-1}\,ds}.$$

Therefore, the solution of (2.1) is

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(p)}\left\{\int_0^t \left[\frac{t^{p-1}(1-s)^{p-1}}{1-\int_0^1 l(t)t^{p-1}\,dt} - (t-s)^{p-1}\right]\sigma(s)\,ds + \int_t^1 \frac{t^{p-1}(1-s)^{p-1}}{1-\int_0^1 l(t)t^{p-1}\,dt}\sigma(s)\,ds\right\} \\ &\quad - \frac{\int_0^1\int_\tau^1 l(s)(s-\tau)^{p-1}\sigma(\tau)\,ds\,d\tau}{1-\int_0^1 l(s)s^{p-1}\,ds}t^{p-1} \\ &= \int_0^1 G(t,s)\sigma(s)\,ds - \frac{\int_0^1\int_\tau^1 l(s)(s-\tau)^{p-1}\sigma(\tau)\,ds\,d\tau}{1-\int_0^1 l(s)s^{p-1}\,ds}\cdot t^{p-1}. \end{aligned}$$

This finishes the proof. □

Let $L = \int_0^1 l(t)t^{p-1}\,dt$ and $Q(s) = \frac{\int_s^1 l(u)(u-s)^{p-1}\,du}{1-L}$. By ($A_3$) we know that $0 \le L < 1$.

Remark 2.1 If $G(t,s)$ is defined by (2.2), then $G(t,s) \ge 0$ for $t,s \in (0,1)$.

Define the operators T and A as

$$(Tx)(t) = \int_0^1 \left[G(t,s) + Q(s)t^{p-1}\right]p(s)x(s)\,ds, \quad t \in [0,1], x \in E, \tag{2.3}$$

$$(Ax)(t) = \int_0^1 \left[G(t,s) + Q(s)t^{p-1}\right]\left[p(s)f\left(s,x(s)\right) + q(s)\right]ds, \quad t \in [0,1], x \in E. \tag{2.4}$$

It is easy to show that $T: E \to E$ is a linear completely continuous operator and $T(P) \subset P$. It is not difficult to see that the solution for (1.4) is, equivalently, a fixed point of A in E.

Lemma 2.4 *The operator $A : E \to E$ defined in* (2.4) *is completely continuous, and $A(P) \subset P$.*

Proof Obviously, $A : E \to E$, and $A(P) \subset P$. The continuity of A in P is obvious. For any bounded set $D \subset P$, $A(D)$ is bounded, so that the functions in $A(D)$ are uniformly bounded. It is easy to prove that A is equicontinuous. By the Ascoli-Arzelà theorem A is completely continuous. □

Lemma 2.5 *T is a u_0-positive operator, and $u_0(t) = t^{p-1}$.*

Proof For any $x \in P \setminus \{\theta\}$, it follows from the definition of T that

$$\begin{aligned}
(Tx)(t) &= \int_0^1 \bigl[G(t,s) + Q(s)t^{p-1}\bigr]p(s)x(s)\,ds \\
&= \int_0^t \left\{ \frac{1}{\Gamma(p)} \left[\frac{t^{p-1}(1-s)^{p-1}}{1-L} - (t-s)^{p-1} \right] + Q(s)t^{p-1} \right\} p(s)x(s)\,ds \\
&\quad + \int_t^1 \left[\frac{1}{\Gamma(p)} \frac{t^{p-1}(1-s)^{p-1}}{1-L} + Q(s)t^{p-1} \right] p(s)x(s)\,ds \\
&\le \int_0^1 \frac{1}{\Gamma(p)} \frac{t^{p-1}(1-s)^{p-1}}{1-L} p(s)x(s)\,ds + \int_0^1 Q(s)t^{p-1}p(s)x(s)\,ds \\
&= \int_0^1 \left[\frac{1}{\Gamma(p)} \frac{(1-s)^{p-1}}{1-L} + Q(s) \right] p(s)x(s)\,ds \cdot t^{p-1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 \bigl[G(t,s) + Q(s)t^{p-1}\bigr]p(s)x(s)\,ds \\
&= \int_0^t \left\{ \frac{1}{\Gamma(p)} \left[\frac{t^{p-1}(1-s)^{p-1}}{1-L} - (t-s)^{p-1} \right] + Q(s)t^{p-1} \right\} p(s)x(s)\,ds \\
&\quad + \int_t^1 \left[\frac{1}{\Gamma(p)} \frac{t^{p-1}(1-s)^{p-1}}{1-L} + Q(s)t^{p-1} \right] p(s)x(s)\,ds \\
&\ge \int_0^t \left\{ \frac{1}{\Gamma(p)} \left[\frac{t^{p-1}(1-s)^{p-1}}{1-L} - (t-ts)^{p-1} \right] + Q(s)t^{p-1} \right\} p(s)x(s)\,ds \\
&\quad + \int_t^1 \left[\frac{1}{\Gamma(p)} \frac{t^{p-1}(1-s)^{p-1}}{1-L} + Q(s)t^{p-1} \right] p(s)x(s)\,ds \\
&= \left\{ \int_0^1 \frac{1}{\Gamma(p)(1-L)} (1-s)^{p-1} p(s)x(s)\,ds - \int_0^t \frac{(1-s)^{p-1}}{\Gamma(p)} p(s)x(s)\,ds \right. \\
&\quad \left. + \int_0^1 Q(s)p(s)x(s)\,ds \right\} t^{p-1} \\
&\ge \left\{ \frac{L}{\Gamma(p)(1-L)} \int_0^1 (1-s)^{p-1}p(s)x(s)\,ds + \int_0^1 Q(s)p(s)x(s)\,ds \right\} t^{p-1} \\
&= \left\{ \int_0^1 \left[\frac{L}{\Gamma(p)(1-L)} (1-s)^{p-1} + Q(s) \right] p(s)x(s)\,ds \right\} t^{p-1}.
\end{aligned}$$

The inequalities imply that T is a u_0-positive operator and $u_0(t) = t^{p-1}$. This proof is completed. □

Lemma 2.6 *Let T be defined in* (2.3). *Then the spectral radius $r(T) \neq 0$, and T has a positive eigenfunction $\varphi^*(t)$ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Proof Let $\psi(t) = t^{p-1}$ and $c = \{\int_0^1 [\frac{L}{\Gamma(p)(1-L)}(1-s)^{p-1} + Q(s)]p(s)\psi(s)\,ds\}^{-1} > 0$. Then from the proof of Lemma 2.5 we have $cT\psi \ge \psi$. Thereby, from Lemma 2.1 we get that $r(T) \neq 0$ and that T has a positive eigenfunction $\varphi^*(t)$ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$, that is, $\varphi^*(t) = \lambda_1 T\varphi^*$. This completes the proof. □

From Lemma 2.5 and Definition 2.3 we get the following lemma.

Lemma 2.7 *There exist $k_1(\varphi^*), k_2(\varphi^*) > 0$ such that*

$$k_1(\varphi^*)u_0 \leq T\varphi^* = \frac{1}{\lambda_1}\varphi^* \leq k_2(\varphi^*)u_0.$$

Furthermore, T is a u_0-positive operator, and $u_0(t) = \varphi^*(t)$.

3 Main results

Theorem 3.1 *Suppose that there exists $k \in [0,1)$such that*

$$\left|f(t,u) - f(t,v)\right| \leq k\lambda_1|u - v|, \quad \forall t \in [0,1], u, v \in \mathbb{R},$$

where λ_1 is the first eigenvalue of T. Then (1.4) *has a unique positive solution x^*, and for any $x_0 \in P$, the iterative sequence $x_n = Ax_{n-1}$ $(n = 1,2,\ldots)$ converges to x^*.*

Proof For any given $x_0 \in P$, let $x_n = Ax_{n-1}$ $(n = 1,2,\ldots)$. Because $A(P) \subset P$, we know that $\{x_n\} \subset P$. By Lemmas 2.5 and 2.7 there exists $\beta_1 > 0$ such that

$$\left(T|x_1 - x_0|\right)(t) \leq \beta_1\varphi^*(t), \quad t \in [0,1].$$

Notice that T is increasing on P. Then, for $n \in \mathbb{N}^+$,

$$\begin{aligned}
\left|x_{n+1}(t) - x_n(t)\right| &= \left|(Ax_n)(t) - (Ax_{n-1})(t)\right| \\
&= \left|\int_0^1 \left(G(t,s) + Q(s)t^{p-1}\right)\left[p(s)f\left(s, x_n(s)\right) + q(s)\right] ds \right. \\
&\quad \left. - \int_0^1 \left(G(t,s) + Q(s)t^{p-1}\right)\left[p(s)f\left(s, x_{n-1}(s)\right) + q(s)\right] ds\right| \\
&\leq \int_0^1 \left(G(t,s) + Q(s)t^{p-1}\right)p(s)\left|f\left(s, x_n(s)\right) - f\left(s, x_{n-1}(s)\right)\right| ds \\
&\leq k\lambda_1 \int_0^1 \left(G(t,s) + Q(s)t^{p-1}\right)p(s)\left|x_n(s) - x_{n-1}(s)\right| ds \\
&\leq k\lambda_1 T\left(|x_n - x_{n-1}|\right)(t) \leq \cdots \leq k^n\lambda_1^n T^n\left(|x_1 - x_0|\right)(t) \\
&\leq k^n\lambda_1^n T^{n-1}\left(\beta_1\varphi^*(t)\right) = k^n\beta_1\lambda_1^n\frac{1}{\lambda_1^{n-1}}\varphi^*(t) = k^n\beta_1\lambda_1\varphi^*(t).
\end{aligned}$$

Thus, for any $m \in \mathbb{N}^+$,

$$\begin{aligned}
\left|x_{n+m}(t) - x_n(t)\right| &= \left|x_{n+m}(t) - x_{n+m-1}(t) + \cdots + x_{n+1}(t) - x_n(t)\right| \\
&\leq \left|x_{n+m}(t) - x_{n+m-1}(t)\right| + \cdots + \left|x_{n+1}(t) - x_n(t)\right| \\
&\leq \beta_1\lambda_1\left(k^{n+m-1} + \cdots + k^n\right)\varphi^*(t) \\
&= \beta_1\lambda_1\frac{k^n(1-k^m)}{1-k}\varphi^*(t) \leq \beta_1\lambda_1\frac{k^n}{1-k}\varphi^*(t).
\end{aligned}$$

Therefore,

$$\|x_{n+m} - x_n\| \leq \beta_1\lambda_1\frac{k^n}{1-k}\left\|\varphi^*\right\|.$$

Because $\lim_{n\to\infty}\beta_1\lambda_1\frac{k^n}{1-k}\|\varphi^*\| = 0$, $\{x_n\}$ is a Cauchy sequence.

By the completeness of E and the closeness of P there exists $x^* \in P$ such that $\lim_{n\to\infty} x_n = x^*$.

Passing to the limit in $x_{n+1} = Ax_n$, we get $x^* = Ax^*$, and it follows that x^* is a fixed point of A in P.

Next, we prove the uniqueness of a fixed point of A in P. Suppose that there exist two elements $x, y \in P$ such that $x = Ax$ and $y = Ay$. By Lemmas 2.5 and 2.7 there exists $\beta_2 > 0$ such that

$$\bigl(T\bigl(|x-y|\bigr)\bigr)(t) \le \beta_2\varphi^*(t), \quad t \in [0,1].$$

Then, for any $n \in \mathbb{N}^+$, we get

$$\begin{aligned}
\bigl|x(t)-y(t)\bigr| &= \bigl|\bigl(A^n x\bigr)(t) - \bigl(A^n y\bigr)(t)\bigr| = \bigl|\bigl[A\bigl(A^{n-1}x\bigr)\bigr](t) - \bigl[A\bigl(A^{n-1}y\bigr)\bigr](t)\bigr| \\
&= \biggl|\int_0^1 \bigl(G(t,s)+Q(s)t^{p-1}\bigr)\bigl[p(s)f\bigl(s,A^{n-1}x(s)\bigr)+q(s)\bigr]\,ds \\
&\quad - \int_0^1 \bigl(G(t,s)+Q(s)t^{p-1}\bigr)\bigl[p(s)f\bigl(s,A^{n-1}y(s)\bigr)+q(s)\bigr]\,ds\biggr| \\
&\le \int_0^1 \bigl(G(t,s)+Q(s)t^{p-1}\bigr)p(s)\bigl|f\bigl(s,A^{n-1}x(s)\bigr)-f\bigl(s,A^{n-1}y(s)\bigr)\bigr|\,ds \\
&\le k\lambda_1\int_0^1 \bigl(G(t,s)+Q(s)t^{p-1}\bigr)p(s)\bigl|A^{n-1}x(s)-A^{n-1}y(s)\bigr|\,ds \\
&\le k\lambda_1 T\bigl(\bigl|A^{n-1}x-A^{n-1}y\bigr|\bigr)(t) \le \cdots \le k^n\lambda_1^n T^n\bigl(|x-y|\bigr)(t) \\
&\le k^n\lambda_1^n T^{n-1}\bigl(\beta_2\varphi^*(t)\bigr) = k^n\beta_2\lambda_1^n\frac{1}{\lambda_1^{n-1}}\varphi^*(t) = k^n\beta_2\lambda_1\varphi^*(t).
\end{aligned}$$

From $\|x-y\| \le k^n\beta_2\lambda_1\|\varphi^*\|$ and $\lim_{n\to\infty} k^n\beta_2\lambda_1\|\varphi^*\| = 0$ we have $\|x-y\| \le 0$, and thus $x = y$. Therefore, x^* is the unique fixed point of A in P or, equivalently, x^* is the unique positive solution of (1.4). The proof is completed. □

4 Conclusions

The method of a u_0-positive operator is an important tool in boundary value problems for fractional differential equations. We established the existence of positive solutions for a fractional differential problem with integral boundary conditions by means of a u_0-positive operator.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

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