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New results on higher-order Daehee and Bernoulli numbers and polynomials

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Abstract
We derive a new matrix representation for higher-order Daehee numbers and polynomials, higher-order λ-Daehee numbers and polynomials, and twisted λ-Daehee numbers and polynomials of order *k*. This helps us to obtain simple and short proofs of many previous results on higher-order Daehee numbers and polynomials. Moreover, we obtain recurrence relations, explicit formulas, and some new results for these numbers and polynomials. Furthermore, we investigate the relation between these numbers and polynomials and Stirling, Nörlund, and Bernoulli numbers of higher-order. Some numerical results and program are introduced using Mathcad for generating higher-order Daehee numbers and polynomials. The results of this article generalize the results derived very recently by El-Desouky and Mustafa (Appl. Math. Sci. 9(73):3593-3610, 2015).

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1 Introduction

For $\alpha \in \mathbb{N}$, the Bernoulli polynomials of order α are defined by (see [1–15])

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1}$$

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are the Bernoulli numbers of order α defined by

$$\left(\frac{t}{e^t-1}\right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)}\frac{t^n}{n!}. \tag{2}$$

The Daehee polynomials are defined by (see [11, 12] and [15])

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}. \tag{3}$$

In the special case $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers defined by

$$\left(\frac{\log(1+t)}{t}\right) = \sum_{n=0}^{\infty} D_n\frac{t^n}{n!}. \tag{4}$$

The Stirling numbers of the first and second kind are defined, respectively, by

$$(x)_n = \prod_{i=0}^{n-1}(x-i) = \sum_{l=0}^{n} s_1(n,l)x^l, \tag{5}$$

where $s_1(n,0)=\delta_{n,0}$, $s_1(n,k)=0$, for $k>n$, and

$$x^n = \sum_{k=0}^{n} s_2(n,k)(x)_k, \tag{6}$$

where $s_2(n,0)=\delta_{n,0}$, $s_2(n,k)=0$ for $k>n$, and $\delta_{n,k}$ is the Kronecker delta.

The Stirling numbers of the second kind have the generating function (see [2, 3, 5, 6] and [7])

$$\left(e^t-1\right)^m = m!\sum_{l=m}^{\infty} s_2(l,m)\frac{t^l}{l!}. \tag{7}$$

2 Higher-order Daehee numbers and polynomials

In this section, we derive explicit formulas and recurrence relations for the higher-order Daehee numbers and polynomials of the first and second kinds. Also, we give a relation between these numbers and Nörlund numbers. Furthermore, we introduce the matrix representation of some results for higher-order Daehee numbers and polynomials obtained by Kim *et al.* [8] in terms of Stirling numbers, Nörlund numbers, and Bernoulli numbers of higher order and give simple and short proofs of these results.

Kim *et al.* [8] defined the Daehee numbers of the first kind of order k by the generating function

$$\sum_{n=0}^{\infty} D_n^{(k)}\frac{t^n}{n!} = \left(\frac{\log(1+t)}{t}\right)^k. \tag{8}$$

An explicit formula for $D_n^{(k)}$ is given by the following theorem.

Theorem 1 *For $n\in\mathbb{Z}$, $k\in\mathbb{N}$, we have*

$$D_n^{(k)} = n!\sum_{l_1+l_2+\cdots+l_k=n+k}\frac{(-1)^n}{l_1l_2\cdots l_k}. \tag{9}$$

Proof From Eq. (8) we have

$$\sum_{n=0}^{\infty} D_n^{(k)}\frac{t^{n+k}}{n!} = \left(\log(1+t)\right)^k = \left(\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l}t^l\right)^k.$$

Using the Cauchy rule for a product of series, we obtain

$$\sum_{n=0}^{\infty} D_n^{(k)}\frac{t^{n+k}}{n!} = \sum_{r=k}^{\infty}\sum_{l_1+l_2+\cdots+l_k=r}\frac{(-1)^{r-k}}{l_1l_2\cdots l_k}t^r.$$

Letting $r - k = n$ in the right-hand side, we have

$$\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^{n+k}}{n!} = \sum_{n=0}^{\infty} \sum_{l_1+l_2+\cdots+l_k=n+k}^{\infty} \frac{(-1)^n}{l_1 l_2 \cdots l_k} t^{n+k}.$$

Equating the coefficients of t^{n+k} on both sides yields (9). This completes the proof. □

Remark 1 It is worth noting that setting $k = 1$ in (9), we get Eq. (2.2) of [1] as a particular case.

Kim *et al.* ([8], Theorem 1) proved that (see [16]), for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$D_n^{(k)} = \frac{s_1(n+k,k)}{\binom{n+k}{k}}. \tag{10}$$

We can represent the Daehee numbers of the first kind of order k by an $(n+1) \times (k+1)$ matrix, $0 \leq k \leq n$, as follows:

$$\mathbf{D}^{(k)} = \begin{pmatrix} D_0^{(0)} & D_0^{(1)} & D_0^{(2)} & \cdots & D_0^{(k)} \\ D_1^{(0)} & D_1^{(1)} & D_1^{(2)} & \cdots & D_1^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_n^{(0)} & D_n^{(1)} & D_n^{(2)} & \cdots & D_n^{(k)} \end{pmatrix}.$$

The following theorem gives a recurrence relation for Daehee numbers of the first kind of order k.

Theorem 2 *For $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$D_{n+1}^{(k)} = \frac{1}{n+k+1}\left[kD_{n+1}^{(k-1)} - (n+1)(n+k)D_n^{(k)}\right], \tag{11}$$

where $D_0^{(k)} = 1$, $D_n^{(0)} = 0$ for $n \geq 1$, $k = 0, 1, \ldots, n$.

Proof The recurrence relation for the Stirling numbers of the first kind is given by

$$s_1(n+1,k) = s_1(n,k-1) - ns_1(n,k).$$

Replacing n by $n+k$, we get

$$s_1(n+k+1,k) = s_1(n+k,k-1) - (n+k)s_1(n+k,k).$$

By using relation (10) we have

$$\binom{n+k+1}{k} D_{n+1}^{(k)} = \binom{n+k}{k-1} D_{n+1}^{(k-1)} - (n+k)\binom{n+k}{k} D_n^{(k)};$$

hence,

$$(n+k+1)D_{n+1}^{(k)} = kD_{n+1}^{(k-1)} - (n+k)(n+1)D_n^{(k)}.$$

This completes the proof. □

A Mathcad program is written and executed to generate the higher-order Daehee numbers using the recurrence relation (11); see the Appendix.

For example, if $0 \le n \le 3$ and $0 \le k \le n$, then we have

$$\mathbf{D}^{(k)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 2/3 & 11/6 & 7/2 \\ 0 & -3/2 & -5 & -45/4 \end{pmatrix}.$$

Kim *et al.* ([8], Theorem 4) proved the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$B_n^{(k)} = \sum_{m=0}^{n} D_m^{(k)} s_2(n,m). \tag{12}$$

Remark 2 We can write this relation in the matrix form as follows:

$$\mathbf{B}^{(k)} = \mathbf{S}_2 \mathbf{D}^{(k)}, \tag{13}$$

where $\mathbf{D}^{(k)}$ is an $(n+1) \times (k+1)$ matrix $(0 \le k \le n)$ for the Daehee numbers of the first kind of order k, and $\mathbf{S}_2$ is an $(n+1) \times (n+1)$ lower triangular matrix for the Stirling numbers of the second kind, and $\mathbf{B}^{(k)}$ is an $(n+1) \times (k+1)$ matrix for the Bernoulli numbers of order k.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (13), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 2/3 & 11/6 & 7/2 \\ 0 & -3/2 & -5 & -45/4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 1/6 & 5/6 & 2 \\ 0 & 0 & -1/2 & -9/4 \end{pmatrix}.$$

Kim *et al.* ([8], Theorem 3) introduced the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$D_n^{(k)} = \sum_{m=0}^{n} s_1(n,m) B_m^{(k)}. \tag{14}$$

We can write this relation in the matrix form as follows:

$$\mathbf{D}^{(k)} = \mathbf{S}_1 \mathbf{B}^{(k)}, \tag{15}$$

where $\mathbf{S}_1$ is an $(n+1) \times (n+1)$ lower triangular matrix for the Stirling numbers of the first kind.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (15), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 1/6 & 5/6 & 2 \\ 0 & 0 & -1/2 & -9/4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 2/3 & 11/6 & 7/2 \\ 0 & -3/2 & -5 & -45/4 \end{pmatrix}.$$

Remark 3 Using the matrix form (15), we easily derive a short proof of Theorem 4 in Kim *et al.* [8]. Multiplying both sides by the Stirling number of second kind, we get

$$\mathbf{S}_2\mathbf{D}^{(k)} = \mathbf{S}_2\mathbf{S}_1\mathbf{B}^{(k)} = \mathbf{I}\mathbf{B}^{(k)} = \mathbf{B}^{(k)},$$

where $\mathbf{I}$ is the identity matrix of order $(n+1)$.

Kim *et al.* [8] defined the Daehee polynomials of order k by the generating function as follows:

$$\sum_{n=0}^{\infty} D_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{\log(1+t)}{t}\right)^k (1+t)^x. \tag{16}$$

Liu and Srivastava [14] defined the Nörlund numbers of the second kind $b_n^{(x)}$ as follows:

$$\left(\frac{t}{\log(1+t)}\right)^x = \sum_{n=0}^{\infty} b_n^{(x)} t^n. \tag{17}$$

Next, we give a relation between the Daehee polynomials of order k and the Nörlund numbers of the second kind $b_n^{(x)}$ in the following theorem.

Theorem 3 *For $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$D_m^{(k)}(z) = m!\sum_{n=0}^{m}\binom{z}{m-n} b_n^{(-k)}. \tag{18}$$

Proof From Eq. (17) by multiplying both sides by $(1+t)^z$ we have

$$\begin{aligned}\left(\frac{t}{\log(1+t)}\right)^x (1+t)^z &= \sum_{n=0}^{\infty} b_n^{(x)} t^n (1+t)^z = \sum_{n=0}^{\infty} b_n^{(x)} t^n \sum_{i=0}^{\infty}\binom{z}{i} t^i \\ &= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} b_n^{(x)}\binom{z}{m-n} t^m = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{z}{m-n} b_n^{(x)} t^m. \end{aligned} \tag{19}$$

Replacing x by $-k$ in (19), we have

$$\begin{aligned}\left(\frac{\log(1+t)}{t}\right)^k (1+t)^z &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{z}{m-n} b_n^{(-k)} t^m \\ &= \sum_{m=0}^{\infty} m!\sum_{n=0}^{m}\binom{z}{m-n} b_n^{(-k)}\frac{t^m}{m!}. \end{aligned} \tag{20}$$

From (16) and (20) we have (18). This completes the proof. □

Corollary 1 *Setting $k = 1$ in* (18), *we have*

$$D_m(z) = m!\sum_{n=0}^{m}\binom{z}{m-n} b_n^{(-1)}. \tag{21}$$

Setting $z = 0$ in (18), we have the following relation between Daehee numbers of higher order and Nörlund numbers of the second kind.

Corollary 2 *For* $k \in \mathbb{N}$, *by setting* $z = 0$ *in* (18) *we obtain*

$$D_m^{(k)} = m! b_m^{(-k)}. \tag{22}$$

The following relation between Bernoulli numbers and Bernoulli polynomials of order k is given by Kimura [13]:

$$B_n^{(k)}(x) = \sum_{j=0}^{n} \binom{n}{j} B_j^{(k)} x^{n-j}. \tag{23}$$

Therefore, we can represent (23) in the matrix form

$$\mathbf{B}^{(k)}(x) = \mathbf{P}(x)\mathbf{B}^{(k)}, \tag{24}$$

where $\mathbf{B}^{(k)}(x)$ is an $(n+1) \times (k+1)$ matrix $(0 \le k \le n)$ for Bernoulli polynomials of order k,

$$\mathbf{B}^{(k)}(x) = \begin{pmatrix} B_0^{(0)}(x) & B_0^{(1)}(x) & B_0^{(2)}(x) & \cdots & B_0^{(k)}(x) \\ B_1^{(0)}(x) & B_1^{(1)}(x) & B_1^{(2)}(x) & \cdots & B_1^{(k)}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_n^{(0)}(x) & B_n^{(1)}(x) & B_n^{(2)}(x) & \cdots & B_n^{(k)}(x) \end{pmatrix},$$

where the column k represents the Bernoulli polynomials of order k, $\mathbf{B}^{(k)}$ is an $(n+1) \times (k+1)$ matrix $(0 \le k \le n)$ for Bernoulli numbers of order k, and the matrix $\mathbf{P}(x)$, the Pascal matrix, is the $(n+1) \times (n+1)$ lower triangular matrix defined by

$$\bigl(\mathbf{P}(x)\bigr)_{ij} = \begin{cases} \binom{i}{j} x^{i-j}, & i \ge j, \\ 0 & \text{otherwise,} \end{cases} \qquad i,j = 0,1,\dots,n.$$

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (24), we have

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 1/6 & 5/6 & 2 \\ 0 & 0 & -1/2 & -9/4 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x & x - \frac{1}{2} & x - 1 & x - \frac{3}{2} \\ x^2 & x^2 - x + \frac{1}{6} & x^2 - 2x + \frac{5}{6} & x^2 - 3x + 2 \\ x^3 & x^3 - \frac{3}{2}x^2 + \frac{1}{2}x & x^3 - 3x^2 + \frac{5}{2}x - \frac{1}{2} & x^3 - \frac{9}{2}x^2 + 6x - \frac{9}{4} \end{pmatrix}. \end{aligned}$$

Kim *et al.* ([8], Theorem 5) introduced the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$D_n^{(k)}(x) = \sum_{m=0}^{n} s_1(n,m) B_m^{(k)}(x). \tag{25}$$

We can write this relation in the matrix form as follows:

$$\mathbf{D}^{(k)}(x) = \mathbf{S}_1\mathbf{B}^{(k)}(x), \tag{26}$$

where $\mathbf{D}^{(k)}(x)$ is the $(n+1)\times(k+1)$ matrix for the Daehee polynomials of the first kind with order k, and $\mathbf{B}^{(k)}(x)$ is the $(n+1)\times(k+1)$ matrix for the Bernoulli polynomials of order k.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (26), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ x & x-\frac{1}{2} & x-1 & x-\frac{3}{2} \\ x^2 & x^2-x+\frac{1}{6} & x^2-2x+\frac{5}{6} & x^2-3x+2 \\ x^3 & x^3-\frac{3}{2}x^2+\frac{1}{2}x & x^3-3x^2+\frac{5}{2}x-\frac{1}{2} & x^3-\frac{9}{2}x^2+6x-\frac{9}{4} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ x & x-\frac{1}{2} & x-1 & x-\frac{3}{2} \\ x^2-x & x^2-2x+\frac{2}{3} & x^2-3x+\frac{11}{6} & x^2-4x+\frac{7}{2} \\ x^3-3x^2+2x & x^3-\frac{9}{2}x^2+\frac{11}{2}x-\frac{3}{2} & x^3-6x^2+\frac{21}{2}x-5 & x^3-\frac{15}{2}x^2+17x-\frac{45}{4} \end{pmatrix}.$$

Kim *et al.* ([8], Theorem 7) introduced the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$B_n^{(k)}(x) = \sum_{m=0}^{n} D_m^{(k)}(x)s_2(n,m). \tag{27}$$

We can write Eq. (27) in the matrix form as follows:

$$\mathbf{B}^{(k)}(x) = \mathbf{S}_2\mathbf{D}^{(k)}(x). \tag{28}$$

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (28), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ x & x-\frac{1}{2} & x-1 & x-\frac{3}{2} \\ x^2-x & x^2-2x+\frac{2}{3} & x^2-3x+\frac{11}{6} & x^2-4x+\frac{7}{2} \\ x^3-3x^2+2x & x^3-\frac{9}{2}x^2+\frac{11}{2}x-\frac{3}{2} & x^3-6x^2+\frac{21}{2}x-5 & x^3-\frac{15}{2}x^2+17x-\frac{45}{4} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ x & x-\frac{1}{2} & x-1 & x-\frac{3}{2} \\ x^2 & x^2-x+\frac{1}{6} & x^2-2x+\frac{5}{6} & x^2-3x+2 \\ x^3 & x^3-\frac{3}{2}x^2+\frac{1}{2}x & x^3-3x^2+\frac{5}{2}x-\frac{1}{2} & x^3-\frac{9}{2}x^2+6x-\frac{9}{4} \end{pmatrix}.$$

Remark 4 We can prove Theorem 7 in Kim *et al.* [8] by using the matrix form (26) as follows. Multiplying both sides of (26) by the Stirling number of second kind, we have

$$\mathbf{S}_2\mathbf{D}^{(k)}(x) = \mathbf{S}_2\mathbf{S}_1\mathbf{B}^{(k)}(x) = \mathbf{I}\mathbf{B}^{(k)}(x) = \mathbf{B}^{(k)}(x).$$

Kim *et al.* [8] defined the Daehee numbers of the second kind of order k by the generating function as follows:

$$\sum_{n=0}^{\infty} \hat{D}_n^k\big[(k)\big]\frac{t^n}{n!} = \left(\frac{(1-t)\log(1-t)}{-t}\right)^k. \tag{29}$$

Kim *et al.* ([8], Theorem 8) introduced the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\hat{D}_n^k\big[(k)\big] = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} B_l^{(k)}, \tag{30}$$

where $\left[\begin{smallmatrix} n \\ l \end{smallmatrix}\right] = (-1)^{n-l}s_1(n,l) = |s_1(n,k)| = \mathfrak{s}(n,k)$, and $\mathfrak{s}(n,k)$ are the signless Stirling numbers of the first kind; see [3] and [5, 6].

We can write this theorem in the matrix form as follows:

$$\hat{\mathbf{D}}^{(k)} = \mathfrak{S}\mathbf{B}^{(k)}, \tag{31}$$

where $\hat{\mathbf{D}}^{(k)}$ is the $(n+1) \times (k+1)$ matrix of Daehee numbers of the second kind of order k, and $\mathfrak{S}$ is the $(n+1) \times (n+1)$ lower triangular matrix for the signless Stirling numbers of the first kind.

For example, setting $0 \leq n \leq 3$ and $0 \leq k \leq n$ in (31), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 1/6 & 5/6 & 2 \\ 0 & 0 & -1/2 & -9/4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & -1/3 & -1/6 & 1/2 \\ 0 & -1/2 & 0 & 3/4 \end{pmatrix}.$$

Kim *et al.* ([8], Theorem 9) introduced the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$B_n^{(k)} = \sum_{m=0}^{n}(-1)^{n-m}s_2(n,m)\hat{D}_m^k\big[(k)\big]. \tag{32}$$

We can write Eq. (32) in the matrix form as follows:

$$\mathbf{B}^{(k)} = \tilde{\mathbf{S}}_2\hat{\mathbf{D}}^{(k)}, \tag{33}$$

where $\tilde{\mathbf{S}}_2$ is the $(n+1) \times (n+1)$ lower triangular matrix for signed Stirling numbers of the second kind defined by

$$(\tilde{\mathbf{S}}_2)_{ij} = \begin{cases} (-1)^{i-j}s_2(i,j), & i \geq j, \\ 0 & \text{otherwise}, \end{cases} \qquad i,j = 0,1,\ldots,n.$$

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (33), we have

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&1&-3&1\end{pmatrix}\begin{pmatrix}1&1&1&1\\0&-1/2&-1&-3/2\\0&-1/3&-1/6&1/2\\0&-1/2&0&3/4\end{pmatrix}=\begin{pmatrix}1&1&1&1\\0&-1/2&-1&-3/2\\0&1/6&5/6&2\\0&0&-1/2&-9/4\end{pmatrix}.$$

Remark 5 We can prove Theorem 9 in Kim *et al.* [8] by using the matrix form (31) as follows. Multiplying both sides of (31) by the matrix of sign Stirling numbers of second kind $\tilde{\mathbf{S}}_2$, we have

$$\tilde{\mathbf{S}}_2\hat{\mathbf{D}}^{(k)} = \tilde{\mathbf{S}}_2\mathfrak{S}\mathbf{B}^{(k)} = \mathbf{I}\mathbf{B}^{(k)} = \mathbf{B}^{(k)},$$

which gives Eq. (33), where we used the identity $\tilde{\mathbf{S}}_2\mathfrak{S} = \mathbf{I}$.

Kim *et al.* [8] defined the Daehee polynomials of the second kind of order k by the generating function as follows:

$$\sum_{n=0}^{\infty}\hat{D}_n^k\big[(k)\big](x)\frac{t^n}{n!} = \left(\frac{(1-t)\log(1-t)}{-t}\right)^k(1-t)^x. \tag{34}$$

Kim *et al.* ([8], Eq. (31)) introduced the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\hat{D}_n^k\big[(k)\big](x) = \sum_{m=0}^{n}(-1)^{n-m}s_1(n,m)B_m^{(k)}(-x). \tag{35}$$

Equation (35) is equivalent to

$$\hat{D}_n^k\big[(k)\big](x) = \sum_{m=0}^{n}\mathfrak{s}(n,m)B_m^{(k)}(-x). \tag{36}$$

We can write Eq. (36) in the matrix form as follows:

$$\hat{\mathbf{D}}^{(k)}(x) = \mathfrak{S}\mathbf{B}^{(k)}(-x), \tag{37}$$

where $\hat{\mathbf{D}}^{(k)}(x)$ is the $(n+1)\times(k+1)$ matrix of the Daehee polynomials of the second kind of order k, and $\mathbf{B}^{(k)}(x)$ is the $(n+1)\times(k+1)$ matrix of the Bernoulli polynomials.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (37), we have

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&2&3&1\end{pmatrix}\begin{pmatrix}1&1&1&1\\-x&-x-\frac{1}{2}&-x-1&-x-\frac{3}{2}\\x^2&x^2+x+\frac{1}{6}&x^2+2x+\frac{5}{6}&x^2+3x+2\\-x^3&-x^3-\frac{3}{2}x^2-\frac{1}{2}x&-x^3-3x^2-\frac{5}{2}x-\frac{1}{2}&-x^3-\frac{9}{2}x^2-6x-\frac{9}{4}\end{pmatrix}$$

$$=\begin{pmatrix}1&1&1&1\\-x&-x-\frac{1}{2}&-x-1&-x-\frac{3}{2}\\x^2-x&x^2-\frac{1}{3}&x^2+x-\frac{1}{6}&x^2+2x+\frac{1}{2}\\3x^2-x^3-2x&\frac{3x^2}{2}-x^3+\frac{x}{2}-\frac{1}{2}&\frac{3x}{2}-x^3&x-\frac{3x^2}{2}-x^3+\frac{3}{4}\end{pmatrix}.$$

Kim *et al.* ([8], Theorem 11) introduced the following result: for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$B_n^{(k)}(-x) = \sum_{m=0}^{n} (-1)^{n-m} s_2(n,m) \hat{D}_m^k\big[(k)\big](x). \tag{38}$$

We can write Eq. (38) in the matrix form as follows:

$$\mathbf{B}^{(k)}(-x) = \tilde{\mathbf{S}}_2 \hat{\mathbf{D}}^{(k)}(x). \tag{39}$$

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (39), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -x & -x-\frac{1}{2} & -x-1 & -x-\frac{3}{2} \\ x^2-x & x^2-\frac{1}{3} & x^2+x-\frac{1}{6} & x^2+2x+\frac{1}{2} \\ 3x^2-x^3-2x & \frac{3x^2}{2}-x^3+\frac{x}{2}-\frac{1}{2} & \frac{3x}{2}-x^3 & x-\frac{3x^2}{2}-x^3+\frac{3}{4} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ -x & -x-\frac{1}{2} & -x-1 & -x-\frac{3}{2} \\ x^2 & x^2+x+\frac{1}{6} & x^2+2x+\frac{5}{6} & x^2+3x+2 \\ -x^3 & -x^3-\frac{3x^2}{2}-\frac{x}{2} & -x^3-3x^2-\frac{5x}{2}-\frac{1}{2} & -x^3-\frac{9x^2}{2}-6x-\frac{9}{4} \end{pmatrix}.$$

Remark 6 We can prove Eq. (39) ([8], Theorem 11), directly by using the matrix form (37) as follows. Multiplying both sides of (37) by $\tilde{\mathbf{S}}_2$, we get

$$\tilde{\mathbf{S}}_2 \hat{\mathbf{D}}^{(k)}(x) = \tilde{\mathbf{S}}_2 \mathfrak{S} \mathbf{B}^{(k)}(-x) = \mathbf{I}\mathbf{B}^{(k)}(-x) = \mathbf{B}^{(k)}(-x),$$

and, thus, we have Eq. (39).

3 The λ-Daehee numbers and polynomials of higher order

In this section we introduce the matrix representation for the λ-Daehee numbers and polynomials of higher order given by Kim *et al.* [9]. Hence, we can derive these results in matrix representation and prove these results simply by using the given matrix forms.

The λ-Daehee polynomials of the first kind of order k can be defined by the generating function

$$\left(\frac{\lambda \log(1+t)}{(1+t)^{\lambda}-1}\right)^k (1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \tag{40}$$

When $x = 0$, $D_{n,\lambda}^{(k)} = D_{n,\lambda}^{(k)}(0)$ are called the λ-Daehee numbers of order k:

$$\left(\frac{\lambda \log(1+t)}{(1+t)^{\lambda}-1}\right)^k = \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)} \frac{t^n}{n!}. \tag{41}$$

It is easy to see that $D_n^{(k)}(x) = D_{n,1}^{(k)}(x)$ and $D_{n,\lambda}(x) = D_{n,\lambda}^{(1)}(x)$.

Kim *et al.* ([9], Theorem 3) obtained the following results: for $n \ge 0$ and $k \in \mathbb{N}$,

$$D_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n} s_1(n,m) \lambda^m B_m^{(k)}\left(\frac{x}{\lambda}\right) \tag{42}$$

and

$$\lambda^n B_n^{(k)}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n} s_2(n,m) D_{m,\lambda}^{(k)}(x). \tag{43}$$

We can write these results in the following matrix forms:

$$\mathbf{D}_\lambda^{(k)}(x) = \mathbf{S}_1 \boldsymbol{\Lambda} \mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right) \tag{44}$$

and

$$\boldsymbol{\Lambda} \mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right) = \mathbf{S}_2 \mathbf{D}_\lambda^{(k)}(x), \tag{45}$$

where, $\mathbf{D}_\lambda^{(k)}(x)$ is the $(n+1)\times(k+1)$ matrix for the λ-Daehee polynomials of the first kind of order k, $\mathbf{B}^{(k)}(x/\lambda)$ is the $(n+1)\times(k+1)$ matrix for the Bernoulli polynomials of order k with $x \to x/\lambda$, and $\boldsymbol{\Lambda}$ is the $(n+1)\times(n+1)$ diagonal matrix with elements $(\boldsymbol{\Lambda})_{ii} = \lambda^i$, $i = j = 0, 1, \ldots, n$.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (44), we have

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix} \\
&\quad\times \begin{pmatrix} 1 & 1 & 1 & 1 \\
\frac{x}{\lambda} & \frac{x}{\lambda} - \frac{1}{2} & \frac{x}{\lambda} - 1 & \frac{x}{\lambda} - \frac{3}{2} \\
\frac{x^2}{\lambda^2} & \frac{x^2}{\lambda^2} - \frac{x}{\lambda} + \frac{1}{6} & \frac{x^2}{\lambda^2} - \frac{2x}{\lambda} + \frac{5}{6} & \frac{x^2}{\lambda^2} - \frac{3x}{\lambda} + 2 \\
\frac{x^3}{\lambda^3} & \frac{x}{2\lambda} - \frac{3x^2}{2\lambda^2} + \frac{x^3}{\lambda^3} & \frac{5x}{2\lambda} - \frac{3x^2}{\lambda^2} + \frac{x^3}{\lambda^3} - \frac{1}{2} & \frac{6x}{\lambda} - \frac{9x^2}{2\lambda^2} + \frac{x^3}{\lambda^3} - \frac{9}{4} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 \\
x & x - \frac{\lambda}{2} & x - \lambda & x - \frac{3\lambda}{2} \\
x(x-1) & \frac{\lambda^2}{6} - \lambda x + \frac{\lambda}{2} + x^2 - x & \frac{5\lambda^2}{6} - 2\lambda x + \lambda + x^2 - x & 2\lambda^2 - 3\lambda x + \frac{3\lambda}{2} + x^2 - x \\
D_{3,\lambda}^{(0)}(x) & D_{3,\lambda}^{(1)}(x) & D_{3,\lambda}^{(2)}(x) & D_{3,\lambda}^{(3)}(x) \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&D_{3,\lambda}^{(0)}(x) = x(x-1)(x-2), \qquad D_{3,\lambda}^{(1)}(x) = -\frac{1}{2}(\lambda - 2x + 2)\left(\lambda - 2x + x^2 - \lambda x\right), \\
&D_{3,\lambda}^{(2)}(x) = -\frac{1}{2}(\lambda - x + 1)\left(\lambda^2 - 4\lambda x + 4\lambda + 2x^2 - 4x\right), \\
&D_{3,\lambda}^{(3)}(x) = -\frac{1}{4}(3\lambda - 2x + 2)\left(3\lambda^2 - 6\lambda x + 6\lambda + 2x^2 - 4x\right).
\end{aligned}$$

Remark 7 In fact, we can prove Eq. (45) by simply by multiplying Eq. (44) by $\mathbf{S}_2$ as follows:

$$\mathbf{S}_2 \mathbf{D}_\lambda^{(k)}(x) = \mathbf{S}_2 \mathbf{S}_1 \boldsymbol{\Lambda} \mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right) = \mathbf{I} \boldsymbol{\Lambda} \mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right) = \boldsymbol{\Lambda} \mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right).$$

The following theorem gives a relation between the Daehee polynomials of higher order and λ-Daehee polynomials of higher order.

Theorem 4 *For $m \geq 0$, we have*

$$D_{m,\lambda}^{(k)}(\lambda x) = m! \sum_{n=0}^{m} \sum_{i_1+i_2+\cdots+i_n=m} \frac{D_n^{(k)}(x)}{n!} \binom{\lambda}{i_1} \binom{\lambda}{i_2} \cdots \binom{\lambda}{i_n}. \tag{46}$$

Proof From (16), replacing $(1+t)$ by $(1+t)^{\lambda}$, we have

$$\left(\frac{\lambda \log(1+t)}{(1+t)^{\lambda}-1}\right)^k (1+t)^{\lambda x} = \sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{((1+t)^{\lambda}-1)^n}{n!}.$$

Thus, from (40) we get

$$\begin{aligned} \sum_{m=0}^{\infty} D_{m,\lambda}^{(k)}(\lambda x) \frac{t^m}{m!} &= \sum_{n=0}^{\infty} \frac{D_n^{(k)}(x)}{n!} \left(\sum_{i=0}^{\lambda} \binom{\lambda}{i} t^i - 1 \right)^n \\ &= \sum_{n=0}^{\infty} \frac{D_n^{(k)}(x)}{n!} \left(\sum_{i=1}^{\lambda} \binom{\lambda}{i} t^i \right)^n. \end{aligned}$$

Using the Cauchy rule for a product of series, we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} D_{m,\lambda}^{(k)}(\lambda x) \frac{t^m}{m!} &= \sum_{n=0}^{\infty} \frac{D_n^{(k)}(x)}{n!} \sum_{m=n}^{\infty} \sum_{i_1+i_2+\cdots+i_n=m} \binom{\lambda}{i_1} \cdots \binom{\lambda}{i_n} t^m \\ &= \sum_{m=0}^{\infty} m! \sum_{n=0}^{m} \sum_{i_1+i_2+\cdots+i_n=m} \frac{D_n^{(k)}(x)}{n!} \binom{\lambda}{i_1} \cdots \binom{\lambda}{i_n} \frac{t^m}{m!}. \end{aligned}$$

Equating the coefficients of t^m on both sides yields (46). This completes the proof. □

Setting $x = 0$ in (46), we have the following corollary as a particular case.

Corollary 3 *For $m \geq 0$, we have*

$$D_{m,\lambda}^{(k)} = m! \sum_{n=0}^{m} \sum_{i_1+i_2+\cdots+i_n=m} \frac{D_n^{(k)}}{n!} \binom{\lambda}{i_1} \binom{\lambda}{i_2} \cdots \binom{\lambda}{i_n}. \tag{47}$$

Kim *et al.* [9] defined the λ-Daehee polynomials of the second kind of order k as follows:

$$\left(\frac{\lambda \log(1+t)}{(1+t)^{\lambda}-1}\right)^k (1+t)^{\lambda k + x} = \sum_{n=0}^{\infty} \hat{D}_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \tag{48}$$

Kim *et al.* ([9], Theorem 5) proved that

$$\hat{D}_{m,\lambda}^{(k)}(x) = \sum_{l=0}^{m} s_1(m,l) \lambda^l B_l^{(k)}\left(k + \frac{x}{\lambda}\right) \tag{49}$$

and

$$\lambda^m B_m^{(k)}\left(k + \frac{x}{\lambda}\right) = \sum_{n=0}^{m} s_2(m,n) \hat{D}_{n,\lambda}^{(k)}(x). \tag{50}$$

Also, Kim *et al.* ([9], Eq. (35)) introduced the following result:

$$B_n^{(k)}(k-x) = (-1)^n B_n^{(k)}(x). \tag{51}$$

Remark 8 We can write (49) and (50), respectively, in the following matrix forms:

$$\hat{\mathbf{D}}_\lambda^{(k)}(x) = \mathbf{S}_1 \mathbf{\Lambda}_1 \mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) \tag{52}$$

and

$$\mathbf{\Lambda}_1 \mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) = \mathbf{S}_2 \hat{\mathbf{D}}_\lambda^{(k)}(x), \tag{53}$$

where $\hat{\mathbf{D}}_\lambda(x)$ is the $(n+1)\times(n+1)$ matrix for the λ-Daehee polynomials of the second kind of order k, and $\mathbf{\Lambda}_1$ is the $(n+1)\times(n+1)$ diagonal matrix with elements $(\mathbf{\Lambda}_1)_{ii} = (-\lambda)^i$ for $i = j = 0, 1, \ldots, n$.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (52), we have

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & -\lambda^3 \end{pmatrix} \\
&\quad\times \begin{pmatrix} 1 & 1 & 1 & 1 \\
-\frac{x}{\lambda} & -\frac{x}{\lambda} - \frac{1}{2} & -\frac{x}{\lambda} - 1 & -\frac{x}{\lambda} - \frac{3}{2} \\
\frac{x^2}{\lambda^2} & \frac{1}{\lambda^2}(\frac{\lambda^2}{6} + \lambda x + x^2) & \frac{1}{\lambda^2}(\frac{5}{6}\lambda^2 + 2\lambda x + x^2) & \frac{1}{\lambda^2}(2\lambda + x)(\lambda + x) \\
-\frac{x^3}{\lambda^3} & \frac{x(\lambda+2x)(\lambda+x)}{2\lambda^3} & -\frac{(\lambda+x)(\lambda^2+4\lambda x+2x^2)}{2\lambda^3} & -\frac{(3\lambda+2x)(3\lambda^2+6\lambda x+2x^2)}{4\lambda^3}
\end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 \\
x & \frac{\lambda}{2} + x & \lambda + x & \frac{3\lambda}{2} + x \\
x(x-1) & \frac{\lambda^2}{6} + \lambda x - \frac{\lambda}{2} + x^2 - x & \frac{5\lambda^2}{6} + 2\lambda x - \lambda + x^2 - x & 2\lambda^2 + 3\lambda x - \frac{3\lambda}{2} + x^2 - x \\
\hat{D}_{3,\lambda}^{(0)} & \hat{D}_{3,\lambda}^{(1)}(x) & \hat{D}_{3,\lambda}^{(2)}(x) & \hat{D}_{3,\lambda}^{(3)}(x)
\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&\hat{D}_{3,\lambda}^{(0)}(x) = x(x-1)(x-2), \qquad \hat{D}_{3,\lambda}^{(1)}(x) = -\frac{1}{2}(\lambda + 2x - 2)\bigl(\lambda + 2x - x^2 - \lambda x\bigr), \\
&\hat{D}_{3,\lambda}^{(2)}(x) = \frac{1}{2}(\lambda + x - 1)\bigl(\lambda^2 + 4\lambda x - 4\lambda + 2x^2 - 4x\bigr), \\
&\hat{D}_{3,\lambda}^{(3)}(x) = \frac{1}{4}(3\lambda + 2x - 2)\bigl(3\lambda^2 + 6\lambda x - 6\lambda + 2x^2 - 4x\bigr).
\end{aligned}$$

Remark 9 We can prove Eq. (50) easily by using the matrix form, multiplying Eq. (52) by $\mathbf{S}_2$ as follows:

$$\mathbf{S}_2 \hat{\mathbf{D}}_\lambda^{(k)}(x) = \mathbf{S}_2 \mathbf{S}_1 \mathbf{\Lambda}_1 \mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) = \mathbf{I}\mathbf{\Lambda}_1 \mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) = \mathbf{\Lambda}_1 \mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right).$$

4 The twisted λ-Daehee numbers and polynomials of higher order

Kim *et al.* [10] defined the twisted λ-Daehee polynomials of the first kind of order k by the generating function

$$\left(\frac{\lambda\log(1+\xi t)}{(1+\xi t)^{\lambda}-1}\right)^{k}(1+\xi t)^{x}=\sum_{n=0}^{\infty}D_{n,\xi}^{(k)}(x|\lambda)\frac{t^{n}}{n!}. \tag{54}$$

In the special case $x=0$, $D_{n,\xi,\lambda}^{(k)}=D_{n,\xi}^{(k)}(0|\lambda)$ are called the twisted λ-Daehee numbers of the first kind of order k:

$$\left(\frac{\lambda\log(1+\xi t)}{(1+\xi t)^{\lambda}-1}\right)^{k}=\sum_{n=0}^{\infty}D_{n,\xi,\lambda}^{(k)}\frac{t^{n}}{n!}. \tag{55}$$

The twisted Bernoulli polynomials of order $r\in\mathbb{N}$ are defined by the generating function (see [4])

$$\left(\frac{t}{\xi e^{t}-1}\right)^{r}e^{xt}=\sum_{n=0}^{\infty}B_{n,\xi}^{(r)}(x)\frac{t^{n}}{n!}. \tag{56}$$

The relation between the twisted λ-Daehee polynomials and λ-Daehee polynomials of order k is given in the following corollary.

Corollary 4 *For $n\geq 0$ and $k\in\mathbb{N}$, we have*

$$D_{n,\xi}^{(k)}(x|\lambda)=\xi^{n}D_{n,\lambda}^{(k)}(x). \tag{57}$$

Proof Replacing t with ξt in (40), we have

$$\left(\frac{\lambda\log(1+\xi t)}{(1+\xi t)^{\lambda}-1}\right)^{k}(1+\xi t)^{x}=\sum_{n=0}^{\infty}D_{n,\lambda}^{(k)}(x)\frac{(\xi t)^{n}}{n!}=\sum_{n=0}^{\infty}\xi^{n}D_{n,\lambda}^{(k)}(x)\frac{t^{n}}{n!}. \tag{58}$$

Hence, by virtue of (54) and (58) we get (57). This completes the proof. □

If we put $x=0$ and $\lambda=1$ in (57), then we have, respectively,

$$D_{n,\xi,\lambda}^{(k)}=\xi^{n}D_{n,\lambda}^{(k)}\quad\text{and}\quad D_{n,\xi}^{(k)}(x)=\xi^{n}D_{n}^{(k)}(x).$$

Kim *et al.* ([10], Theorem 1) proved the following relation: for $m\in\mathbb{Z}$ and $k\in\mathbb{N}$,

$$D_{m,\xi}^{(k)}(x|\lambda)=\xi^{m}\sum_{l=0}^{m}S_{1}(m,l)\lambda^{l}B_{l}^{(k)}\left(\frac{x}{\lambda}\right) \tag{59}$$

and

$$\lambda^{m}B_{m,\xi^{\lambda}}^{(k)}\left(\frac{x}{\lambda}\right)=\sum_{n=0}^{m}D_{n,\xi}^{(k)}(x|\lambda)\xi^{-n-x}s_{2}(m,n), \tag{60}$$

where $B^{(k)}_{m,\xi^\lambda}(\frac{x}{\lambda})$ is defined by Kim *et al.* ([10], Eq. (15)) as follows:

$$\left(\frac{\lambda t}{\xi^\lambda e^{\lambda t}-1}\right)^k \left(\xi e^t\right)^x = \xi^x \sum_{m=0}^{\infty} \lambda^m B^{(k)}_{m,\xi^\lambda}\left(\frac{x}{\lambda}\right)\frac{t^m}{m!}. \tag{61}$$

Remark 10 We can write (59) in the matrix form

$$\mathbf{D}^{(k)}_{\xi}(x|\lambda) = \boldsymbol{\Xi}\mathbf{S}_1\boldsymbol{\Lambda}\mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right), \tag{62}$$

where $\mathbf{D}^{(k)}_{\xi}(x|\lambda)$ is the $(n+1)\times(k+1)$ matrix for the twisted Daehee numbers of the first kind of order k, and $\boldsymbol{\Xi}$ is the $(n+1)\times(n+1)$ diagonal matrix with elements $(\boldsymbol{\Xi})_{ii} = \xi^i$ for $i = j = 0,1,\ldots,n$.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (62), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & \xi^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{x}{\lambda} & \frac{x}{\lambda}-\frac{1}{2} & \frac{x}{\lambda}-1 & \frac{x}{\lambda}-\frac{3}{2} \\ \frac{x^2}{\lambda^2} & \frac{x^2}{\lambda^2}-\frac{x}{\lambda}+\frac{1}{6} & \frac{x^2}{\lambda^2}-\frac{2x}{\lambda}+\frac{5}{6} & \frac{x^2}{\lambda^2}-\frac{3x}{\lambda}+2 \\ \frac{x^3}{\lambda^3} & \frac{x}{2\lambda}-\frac{3x^2}{2\lambda^2}+\frac{x^3}{\lambda^3} & \frac{5x}{2\lambda}-\frac{3x^2}{\lambda^2}+\frac{x^3}{\lambda^3}-\frac{1}{2} & \frac{6x}{\lambda}-\frac{9x^2}{2\lambda^2}+\frac{x^3}{\lambda^3}-\frac{9}{4} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ \xi x & -\frac{\xi}{2}(\lambda-2x) & -\xi(\lambda-x) & -\frac{\xi}{2}(3\xi-2x) \\ \xi^2 x(x-1) & \xi^2(\frac{\lambda^2}{6}-\lambda x+\frac{\lambda}{2}+x^2-x) & \xi^2(\frac{5}{6}\lambda^2-2\lambda x+\lambda+x^2-x) & \xi^2(2\lambda^2-3\lambda x+\frac{3}{2}\lambda+x^2-x) \\ D^{(0)}_{3,\xi}(x|\lambda) & D^{(1)}_{3,\xi}(x|\lambda) & D^{(2)}_{3,\xi}(x|\lambda) & D^{(3)}_{3,\xi}(x|\lambda) \end{pmatrix},$$

where

$$D^{(0)}_{3,\xi}(x|\lambda) = \xi^3 x(x-1)(x-2), \qquad D^{(1)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2}(\lambda-2x+2)\left(\lambda-2x+x^2-\lambda x\right),$$
$$D^{(2)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2}(\lambda-x+1)\left(\lambda^2-4\lambda x+4\lambda+2x^2-4x\right),$$
$$D^{(0)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{4}(3\lambda-2x+2)\left(3\lambda^2-6\lambda x+6\lambda+2x^2-4x\right).$$

Remark 11 In fact, it seems that the statement in (60) is not correct (the second equation of Kim *et al.* [10], Theorem 1). From (62), multiplying both sides by $\boldsymbol{\Xi}^{-1}$, we have

$$\boldsymbol{\Xi}^{-1}\mathbf{D}^{(k)}_{\xi}(x|\lambda) = \boldsymbol{\Xi}^{-1}\boldsymbol{\Xi}\mathbf{S}_1\boldsymbol{\Lambda}\mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right) = \mathbf{S}_1\boldsymbol{\Lambda}\mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right);$$

then multiplying both sides by $\mathbf{S}_2$, we have

$$\mathbf{S}_2\boldsymbol{\Xi}^{-1}\mathbf{D}^{(k)}_{\xi}(x|\lambda) = \mathbf{S}_2\mathbf{S}_1\boldsymbol{\Lambda}\mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right) = \boldsymbol{\Lambda}\mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right). \tag{63}$$

From (60) and (63) it is clear that there is a contradiction.

In the following theorem we obtained a corrected relation.

Theorem 5 *For $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\lambda^m B_m^{(k)}\left(\frac{x}{\lambda}\right) = \sum_{n=0}^{m} D_{n,\xi}^{(k)}(x|\lambda)\xi^{-n} s_2(m,n). \tag{64}$$

Proof From Eq. (54), replacing t by $(e^t - 1)/\xi$, we have

$$\left(\frac{\lambda \log(1 + \frac{\xi(e^t-1)}{\xi})}{(1 + \frac{\xi(e^t-1)}{\xi})^\lambda - 1}\right)^k \left(1 + \frac{\xi(e^t - 1)}{\xi}\right)^x = \sum_{n=0}^{\infty} D_{n,\xi}^{(k)}(x|\lambda)\frac{(e^t - 1)^n}{n!\xi^n},$$

$$\left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^k e^{tx} = \sum_{n=0}^{\infty} D_{n,\xi}^{(k)}(x|\lambda)\frac{(e^t - 1)^n}{n!\xi^n}. \tag{65}$$

Substituting (7) into (65), we have

$$\begin{aligned}\left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^k e^{\lambda t(\frac{x}{\lambda})} &= \sum_{n=0}^{\infty} D_{n,\xi}^{(k)}(x|\lambda)\xi^{-n} \sum_{m=n}^{\infty} s_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} D_{n,\xi}^{(k)}(x|\lambda)\xi^{-n} s_2(m,n)\frac{t^m}{m!}.\end{aligned} \tag{66}$$

From (1) and (66) we have

$$\sum_{m=0}^{\infty} \lambda^m B_m^{(k)}\left(\frac{x}{\lambda}\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} D_{n,\xi}^{(k)}(x|\lambda)\xi^{-n} s_2(m,n)\frac{t^m}{m!}. \tag{67}$$

Equating the coefficients of t^m on both sides gives (64). This completes the proof. □

Moreover, we can represent Eq. (64) in the following matrix form as in (63):

$$\mathbf{B}^{(k)}\left(\frac{x}{\lambda}\right) = \mathbf{\Lambda}^{-1}\mathbf{S}_2\,\mathbf{\Xi}^{-1}\mathbf{D}_\xi^{(k)}(x|\lambda). \tag{68}$$

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (68), we have

$$\begin{aligned}&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda^3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\xi} & 0 & 0 \\ 0 & 0 & \frac{1}{\xi^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\xi^3} \end{pmatrix} \\ &\quad\times \begin{pmatrix} 1 & 1 & 1 & 1 \\ \xi x & -\frac{\xi}{2}(\lambda - 2x) & -\xi(\lambda - x) & -\frac{\xi}{2}(3\xi - 2x) \\ \xi^2 x(x-1) & \xi^2(\frac{\lambda^2}{6} - \lambda x + \frac{\lambda}{2} + x^2 - x) & \xi^2(\frac{5}{6}\lambda^2 - 2\lambda x + \lambda + x^2 - x) & \xi^2(2\lambda^2 - 3\lambda x + \frac{3}{2}\lambda + x^2 - x) \\ D_{3,\xi}^{(0)}(x|\lambda) & D_{3,\xi}^{(1)}(x|\lambda) & D_{3,\xi}^{(2)}(x|\lambda) & D_{3,\xi}^{(3)}(x|\lambda) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{x}{\lambda} & \frac{x}{\lambda} - \frac{1}{2} & \frac{x}{\lambda} - 1 & \frac{x}{\lambda} - \frac{3}{2} \\ \frac{x^2}{\lambda^2} & \frac{x^2}{\lambda^2} - \frac{x}{\lambda} + \frac{1}{6} & \frac{x^2}{\lambda^2} - \frac{2x}{\lambda} + \frac{5}{6} & \frac{x^2}{\lambda^2} - \frac{3x}{\lambda} + 2 \\ \frac{x^3}{\lambda^3} & \frac{x}{2\lambda} - \frac{3x^2}{2\lambda^2} + \frac{x^3}{\lambda^3} & \frac{5x}{2\lambda} - \frac{3x^2}{\lambda^2} + \frac{x^3}{\lambda^3} - \frac{1}{2} & \frac{6x}{\lambda} - \frac{9x^2}{2\lambda^2} + \frac{x^3}{\lambda^3} - \frac{9}{4} \end{pmatrix},\end{aligned}$$

where

$$D^{(0)}_{3,\xi}(x|\lambda) = \xi^3 x(x-1)(x-2), \qquad D^{(1)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2}(\lambda-2x+2)\bigl(\lambda-2x+x^2-\lambda x\bigr),$$
$$D^{(2)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2}(\lambda-x+1)\bigl(\lambda^2-4\lambda x+4\lambda+2x^2-4x\bigr),$$
$$D^{(3)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{4}(3\lambda-2x+2)\bigl(3\lambda^2-6\lambda x+6\lambda+2x^2-4x\bigr).$$

Kim *et al.* [10] introduced the twisted λ-Daehee polynomials of the second kind of order k as follows:

$$\left(\frac{\lambda\log(1+\xi t)(1+\xi t)^\lambda}{(1+\xi t)^\lambda-1}\right)^k(1+\xi t)^x = \sum_{n=0}^{\infty}\hat{D}^{(k)}_{n,\xi}(x|\lambda)\frac{t^n}{n!}. \tag{69}$$

Setting $x=0$, $\hat{D}^{(k)}_{n,\xi,\lambda} = \hat{D}^{(k)}_{n,\xi}(0|\lambda)$, we have the twisted Daehee numbers of second kind of order k:

$$\left(\frac{\lambda\log(1+\xi t)(1+\xi t)^\lambda}{(1+\xi t)^\lambda-1}\right)^k = \sum_{n=0}^{\infty}\hat{D}^{(k)}_{n,\xi,\lambda}\frac{t^n}{n!}. \tag{70}$$

Kim *et al.* ([10], Theorem 2) proved that, for $m\in\mathbb{Z}$ and $k\in\mathbb{N}$,

$$\xi^{-m}\hat{D}_{n,\xi}(x|\lambda) = \sum_{l=0}^{m}s_1(m,l)\lambda^l B^{(k)}_l\left(k+\frac{x}{\lambda}\right) \tag{71}$$

and

$$\lambda^m B^{(k)}_{m,\xi^\lambda}\left(k+\frac{x}{\lambda}\right) = \sum_{n=0}^{m}\hat{D}^{(k)}_{n,\xi}(x|\lambda)s_2(m,n)\xi^{-n-\lambda k-x}. \tag{72}$$

Using Eq. (51), we can write (71) in the matrix form

$$\hat{\mathbf{D}}^{(k)}_{\xi}(x|\lambda) = \boldsymbol{\Xi}\,\mathbf{S}_1\boldsymbol{\Lambda}_1\mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right), \tag{73}$$

where $\hat{\mathbf{D}}^{(k)}_{\xi}(x|\lambda)$ is the $(n+1)\times(k+1)$ matrix for the twisted Daehee numbers of the second kind of order k.

For example, setting $0\le n\le 3$ and $0\le k\le n$ in (73), we have

$$\begin{pmatrix}1&0&0&0\\0&\xi&0&0\\0&0&\xi^2&0\\0&0&0&\xi^3\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&2&-3&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-\lambda&0&0\\0&0&\lambda^2&0\\0&0&0&-\lambda^3\end{pmatrix}$$
$$\times\begin{pmatrix}1&1&1&1\\-\frac{x}{\lambda}&-\frac{x}{\lambda}-\frac{1}{2}&-\frac{x}{\lambda}-1&-\frac{x}{\lambda}-\frac{3}{2}\\\frac{x^2}{\lambda^2}&\frac{x^2}{\lambda^2}+\frac{x}{\lambda}+\frac{1}{6}&\frac{x^2}{\lambda^2}+\frac{2x}{\lambda}+\frac{5}{6}&\frac{x^2}{\lambda^2}+\frac{3x}{\lambda}+2\\-\frac{x^3}{\lambda^3}&-\frac{x}{2\lambda}-\frac{3x^2}{2\lambda^2}-\frac{x^3}{\lambda^3}&-\frac{5x}{2\lambda}-\frac{3x^2}{\lambda^2}-\frac{x^3}{\lambda^3}-\frac{1}{2}&-\frac{6x}{\lambda}-\frac{9x^2}{2\lambda^2}-\frac{x^3}{\lambda^3}-\frac{9}{4}\end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ \xi x & \frac{\xi}{2}(\lambda + 2x) & \xi(\lambda + x) & \frac{\xi}{2}(3\xi + 2x) \\ \xi^2 x(x-1) & \xi^2(\frac{\lambda^2}{6} + \lambda x - \frac{\lambda}{2} + x^2 - x) & \xi^2(\frac{5}{6}\lambda^2 + 2\lambda x - \lambda + x^2 - x) & \xi^2(2\lambda^2 + 3\lambda x - \frac{3}{2}\lambda + x^2 - x) \\ \hat{D}^{(0)}_{3,\xi}(x|\lambda) & \hat{D}^{(1)}_{3,\xi}(x|\lambda) & \hat{D}^{(2)}_{3,\xi}(x|\lambda) & \hat{D}^{(3)}_{3,\xi}(x|\lambda) \end{pmatrix},$$

where

$$\hat{D}^{(0)}_{3,\xi}(x|\lambda) = \xi^3 x(x-1)(x-2),$$
$$\hat{D}^{(1)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2}(\lambda + 2x - 2)\big(\lambda + 2x - x^2 - \lambda x\big),$$
$$\hat{D}^{(2)}_{3,\xi}(x|\lambda) = \frac{\xi^3}{2}(\lambda + x - 1)\big(\lambda^2 + 4\lambda x - 4\lambda + 2x^2 - 4x\big),$$
$$\hat{D}^{(3)}_{3,\xi}(x|\lambda) = \frac{\xi^3}{4}(3\lambda + 2x - 2)\big(3\lambda^2 + 6\lambda x - 6\lambda + 2x^2 - 4x\big).$$

From (73), multiplying both sides by $\mathbf{\Xi}^{-1}$, we have

$$\mathbf{\Xi}^{-1}\hat{\mathbf{D}}^{(k)}_{\xi}(x|\lambda) = \mathbf{\Xi}^{-1}\mathbf{\Xi}\mathbf{S}_1\mathbf{\Lambda}_1\mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) = \mathbf{S}_1\mathbf{\Lambda}_1\mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right),$$

and multiplying both sides by $\mathbf{S}_2$, we have

$$\mathbf{S}_2\mathbf{\Xi}^{-1}\hat{\mathbf{D}}^{(k)}_{\xi}(x|\lambda) = \mathbf{S}_2\mathbf{S}_1\mathbf{\Lambda}_1\mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) = \mathbf{I}\mathbf{\Lambda}_1\mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) = \mathbf{\Lambda}_1\mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right). \tag{74}$$

From (72) and (74) we have a contradiction.

Remark 12 In fact, it clear that (72) is not correct (the second equation of Kim *et al.* [10], Theorem 2).

We give the correction of (72) in the following theorem.

Theorem 6 *For $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\lambda^m B^{(k)}_m\left(k + \frac{x}{\lambda}\right) = \sum_{n=0}^{m} \hat{D}^{(k)}_{n,\xi}(x|\lambda)\xi^{-n}s_2(m,n). \tag{75}$$

Proof From Eq. (69), replacing t by $(e^t - 1)/\xi$, we have

$$\left(\frac{\lambda\log(1 + \frac{\xi(e^t-1)}{\xi})(1 + \frac{\xi(e^t-1)}{\xi})^{\lambda}}{(1 + \frac{\xi(e^t-1)}{\xi})^{\lambda} - 1}\right)^k\left(1 + \frac{\xi(e^t-1)}{\xi}\right)^x = \sum_{n=0}^{\infty}\hat{D}^{(k)}_{n,\xi}(x|\lambda)\frac{(e^t-1)^n}{n!\xi^n},$$
$$\left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^k e^{(k\lambda + x)t} = \sum_{n=0}^{\infty}\hat{D}^{(k)}_{n,\xi}(x|\lambda)\frac{(e^t-1)^n}{n!\xi^n},$$
$$\left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^k e^{\lambda t(k + \frac{x}{\lambda})} = \sum_{n=0}^{\infty}\hat{D}^{(k)}_{n,\xi}(x|\lambda)\xi^{-n}\frac{(e^t-1)^n}{n!}. \tag{76}$$

Substituting Eq. (7) into Eq. (76), we have

$$\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^k e^{\lambda t(k+\frac{x}{\lambda})} = \sum_{n=0}^{\infty} \hat{D}_{n,\xi}^{(k)}(x|\lambda)\xi^{-n} \sum_{m=n}^{\infty} s_2(m,n)\frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \hat{D}_{n,\xi}^{(k)}(x|\lambda)\xi^{-n} s_2(m,n)\frac{t^m}{m!}. \tag{77}$$

From Eq. (1) and Eq. (77) we have

$$\sum_{m=0}^{\infty} \lambda^m B_m^{(k)}\left(k+\frac{x}{\lambda}\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \hat{D}_{n,\xi}^{(k)}(x|\lambda)\xi^{-n} s_2(m,n)\frac{t^m}{m!}. \tag{78}$$

Equating the coefficients of t^m on both sides gives (75). This completes the proof. □

Moreover, by using Eq. (51) we can represent Eq. (75) in the following matrix form:

$$\mathbf{B}^{(k)}\left(-\frac{x}{\lambda}\right) = \mathbf{\Lambda}_1^{-1}\mathbf{S}_2\,\mathbf{\Xi}^{-1}\hat{\mathbf{D}}_{\xi}^{(k)}(x|\lambda), \tag{79}$$

where $\mathbf{\Lambda}\mathbf{B}^{(k)}(k+\frac{x}{\lambda}) = \mathbf{\Lambda}_1\mathbf{B}^{(k)}(-\frac{x}{\lambda})$.

For example, setting $0 \le n \le 3$ and $0 \le k \le n$ in (79), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{\lambda^3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\xi} & 0 & 0 \\ 0 & 0 & \frac{1}{\xi^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\xi^3} \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 1 & 1 & 1 \\ \xi x & \frac{\xi}{2}(\lambda+2x) & \xi(\lambda+x) & \frac{\xi}{2}(3\xi+2x) \\ \xi^2 x(x-1) & \xi^2(\frac{\lambda^2}{6}+\lambda x-\frac{\lambda}{2}+x^2-x) & \xi^2(\frac{5}{6}\lambda^2+2\lambda x-\lambda+x^2-x) & \xi^2(2\lambda^2+3\lambda x-\frac{3}{2}\lambda+x^2-x) \\ \hat{D}_{3,\xi}^{(0)}(x|\lambda) & \hat{D}_{3,\xi}^{(1)}(x|\lambda) & \hat{D}_{3,\xi}^{(2)}(x|\lambda) & \hat{D}_{3,\xi}^{(3)}(x|\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{x}{\lambda} & -\frac{x}{\lambda}-\frac{1}{2} & -\frac{x}{\lambda}-1 & -\frac{x}{\lambda}-\frac{3}{2} \\ \frac{x^2}{\lambda^2} & \frac{x^2}{\lambda^2}+\frac{x}{\lambda}+\frac{1}{6} & \frac{x^2}{\lambda^2}+\frac{2x}{\lambda}+\frac{5}{6} & \frac{x^2}{\lambda^2}+\frac{3x}{\lambda}+2 \\ -\frac{x^3}{\lambda^3} & -\frac{x}{2\lambda}-\frac{3x^2}{2\lambda^2}-\frac{x^3}{\lambda^3} & -\frac{5x}{2\lambda}-\frac{3x^2}{\lambda^2}-\frac{x^3}{\lambda^3}-\frac{1}{2} & -\frac{6x}{\lambda}-\frac{9x^2}{2\lambda^2}-\frac{x^3}{\lambda^3}-\frac{9}{4} \end{pmatrix},$$

where

$$\hat{D}_{3,\xi}^{(0)}(x|\lambda) = \xi^3 x(x-1)(x-2),$$
$$\hat{D}_{3,\xi}^{(1)}(x|\lambda) = -\frac{\xi^3}{2}(\lambda+2x-2)\big(\lambda+2x-x^2-\lambda x\big),$$
$$\hat{D}_{3,\xi}^{(2)}(x|\lambda) = \frac{\xi^3}{2}(\lambda+x-1)\big(\lambda^2+4\lambda x-4\lambda+2x^2-4x\big),$$
$$\hat{D}_{3,\xi}^{(3)}(x|\lambda) = \frac{\xi^3}{4}(3\lambda+2x-2)\big(3\lambda^2+6\lambda x-6\lambda+2x^2-4x\big).$$

For more details and very recent results on these numbers, see [17].

5 Conclusions

We derived generalization of higher-order Daehee numbers and polynomials. Moreover, new matrix representations for these numbers and polynomials are obtained. This enabled us to obtain simple and short proofs of many previous results on higher-order Daehee numbers and polynomials. Furthermore, we investigated the relations between these numbers and polynomials and Stirling numbers, Nörlund numbers, and Bernoulli numbers of higher-order. Finally, some numerical results are given.

Appendix

Mathcad program for higher-order Daehee numbers using the recurrence relation (11):

$$
D(n,k) := \left|
\begin{array}{l}
D_{0,0} \leftarrow 1 \\
\text{for } i \in 1 \cdots n \\
\quad D_{i,0} \leftarrow 0 \\
\text{for } j \in 1 \cdots k \\
\quad D_{0,j} \leftarrow 1 \\
\text{for } i \in 0 \cdots n-1 \\
\quad \text{for } j \in 1 \cdots k \\
\qquad D_{i+1,j+1} \leftarrow \left(\frac{1}{i+j+1}\right) \cdot [j \cdot D_{i+1,j-1} - (i+1) \cdot (i+j) \cdot D_{i,j}] \\
D.
\end{array}
\right.
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors, BSE-D and AM with the consultation of each other carried out this work and drafted the manuscript together. Both authors read and approved the final manuscript.

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