Zhao and Liu *Advances in Difference Equations* (2016) 2016:20
DOI 10.1186/s13662-016-0743-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Multiple monotone positive solutions of integral BVPs for a higher-order fractional differential equation with monotone homomorphism

Kaihong Zhao[1*] and Juqing Liu[2]

*Correspondence:
zhaokaihongs@126.com
[1] Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650093, China
Full list of author information is available at the end of the article

Abstract

This paper is concerned with the integral boundary value problems of higher-order fractional differential equation with. In the sense of a monotone homomorphism, some sufficient criteria are established to guarantee the existence of at least two monotone positive solutions by employing the fixed point theorem of cone expansion and compression of functional type proposed by Avery, Henderson and O'Regan. As applications, some examples are provided to illustrate the validity of our main results.

MSC: 34B15; 34B18; 34B27

Keywords: integral boundary value problems; fractional differential equation; monotone positive solutions; monotone homomorphism; fixed point theorem

1 Introduction

In recent years, the fractional order differential equation has aroused great attention due to both the further development of fractional order calculus theory and the important applications for the theory of fractional order calculus in the fields of science and engineering such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Especially, the boundary value problems with Riemann-Stieltjes integral boundary conditions arise in a variety of different areas of applied mathematics and physics. For example, blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so on can be reduced to integral boundary problems. In a consequence, the subject of fractional differential equations is gaining much importance and attention. There have been many papers focused on boundary value problems of fractional ordinary differential equations (see [1–17]).

Recently, when ϕ is p-Laplacian operator, that is, $\phi(u) = \varphi_p(u) = |u|^{p-2}u$ ($p > 1$) and the nonlinear term does not depend on the first-order derivative, the existence problems of positive solutions of boundary value problems have attracted much attention. It is worth to notice that the oddness of p-Laplacian operator play a key role dealing with these problems. The existence of positive solutions for fractional differential equations with p-Laplacian operator have been studied by several authors (see [18–22] and the references therein). In fact, the oddness of some operators does not hold and is not necessary. As the improvement and generalization of p-Laplacian operator, some scholars have put forward the new operator ϕ satisfying the properties of homomorphism but being not the oddness. There have been many papers involving the integer order differential equations with a homomorphism ϕ (see [23–25] and references therein). However, to the best of our knowledge, there are no results concerning the higher-order fractional differential equations with a homomorphism ϕ. In this article, motivated by the above mentioned discussion, we study the existence of at least two monotone and concave positive solutions of integral boundary value problem for the nonlinear fractional differential equation with sign-changing nonlinearity and delayed or advanced arguments as follows (abbreviated by BVPs (1.1) throughout this paper):

$$\begin{cases} (\phi(D_{0^+}^{\nu}u(t)))' + h(t)f(t,u(\theta(t))) = 0, \quad t \in (0,1), \\ u(0) = au(1), \qquad u'(1) = bu'(0) + \lambda[u], \\ u^{(i)}(0) = 0, \quad i = 2,3,\ldots,n-1, \end{cases} \tag{1.1}$$

here $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and positive homomorphism with $\phi(0) = 0$, and $2 \le n-1 < \nu \le n$. $D_{0^+}^{\nu}$ is the Caputo fractional derivative and $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ is continuous function. $h \in C([0,1],(0,+\infty))$ and λ denotes linear functional on $C([0,1])$ given by $\lambda[u] = \int_0^1 u(t)\,dA(t)$ involving Stieltjes integrals with the suitable bounded variation $A(\cdot)$ on $[0,1]$.

Throughout this paper, it is worth to notice that $\lambda[u]$ is not assumed to be positive to all positive u because $dA(t)$ could be a sign-changing measure. In addition, a, b, and θ will be divided into two cases as follows:

Case I: $a,b \in (0,1)$ are two constants and $\theta \in C([0,1],[0,1])$ with $\theta(t) \ge t$ on $[0,1]$, it means $\theta(t)$ is an advanced argument.

Case II: $a,b \in (1,\infty)$ are two constants and $\theta \in C([0,1],[0,1])$ with $\theta(t) \le t$ on $[0,1]$, it means $\theta(t)$ is a delayed argument.

A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing and positive homomorphism if the following conditions are satisfied:

(1) if $x \le y$, then $\phi(x) \le \phi(y)$ for all $x,y \in \mathbb{R}$;
(2) ϕ is a continuous bijection and its inverse mapping ϕ^{-1} is also continuous;
(3) $\phi(xy) = \phi(x)\phi(y)$, for all $x,y \in \mathbb{R}^+$, where $\mathbb{R}^+ = [0,+\infty)$.

Remark 1.1 It is easy to see that the p-Laplacian operator $\varphi_p(x) = |x|^{p-2}x$ $(p > 1)$ satisfy the conditions (1)-(3), that is, φ_p is an increasing and positive homomorphism. Therefore, the operator ϕ of BVPs (1.1) is regarded as the improvement and generalization of the classical p-Laplacian operator $\varphi_p(x) = |x|^{p-2}x$ $(p > 1)$.

The remainder of this paper is organized as follows. In Section 2, we recall some useful definitions and properties and present the properties of the Green's functions. In Section 3 and Section 4, we give some sufficient conditions for the existence of at least two monotone positive solutions of BVPs (1.1) in Case I and Case II, respectively. Finally, some examples are also provided to illustrate our main results in Section 5.

2 Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the recent literature.

Definition 2.1 (see [26, 27]) The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $h : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\nu} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 (see [25, 26]) The Caputo fractional derivative of order $\nu > 0$ of a continuous function $h : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\nu} h(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} h^{(n)}(s)\, ds,$$

where $n-1 < \nu \leq n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 (see [26]) *Suppose that $\nu > 0$, $\nu \notin \mathbb{N}$. If $u \in C(0,1]$ and $D_{0^+}^{\nu} u \in L^1[0,1]$, then*

$$u(t) = I_{0^+}^{\nu} D_{0^+}^{\nu} u(t) + \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k, \quad \textit{for } t \in (0,1],$$

where n is the smallest integer greater than or equal to ν.

Now, we present the necessary definitions from the theory of cones in Banach spaces.

Definition 2.3 Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following two conditions:

(1) $u \in P$, $k \geq 0$ implies $ku \in P$;
(2) $u \in P$, $-u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $u \leq \nu$ if $\nu - u \in P$.

Definition 2.4 A mapping α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0,+\infty)$ is continuous and

$$\alpha\bigl(ku + (1-k)\nu\bigr) \geq k\alpha(u) + (1-k)\alpha(\nu), \quad u,\nu \in P, 0 \leq k \leq 1.$$

Similarly, a mapping β is called to be a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0,+\infty)$ is continuous and

$$\beta\bigl(ku + (1-k)\nu\bigr) \leq k\beta(u) + (1-k)\beta(\nu), \quad u,\nu \in P, 0 \leq k \leq 1.$$

A mapping γ is called a sublinear functional if and only if

$$\gamma(ku) \le k\gamma(u), \quad u \in P, 0 \le k \le 1.$$

Definition 2.5 Let E be a real Banach space. An operator $T : E \to E$ is said to be completely continuous if it is continuous and maps any bounded sets into the precompact sets.

Property A_1. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, +\infty)$ is said to satisfy Property A_1 if one of the following conditions holds:

(a) β is convex, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$ and $\inf_{u\in P\cap\partial\Omega} \beta(u) > 0$,
(b) β is sublinear, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$ and $\inf_{u\in P\cap\partial\Omega} \beta(u) > 0$,
(c) β is concave and unbounded.

Property A_2. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, +\infty)$ is said to satisfy Property A_2 if one of the following conditions holds:

(d) β is convex, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$,
(e) β is sublinear, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$,
(f) $\beta(u + v) \ge \beta(u) + \beta(v)$ for all $u, v \in P$, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$.

We will establish the existence of multiple monotone positive solutions to BVPs (1.1) by applying the following fixed point theorem of cone expansion and compression of functional type by Avery, Henderson and O'Regan.

Lemma 2.2 (see [28]) *Let E be a Banach space, $P \subset E$ be a cone, and Ω_1, Ω_2 be two bounded open balls of E centered at the origin with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator, α and γ are nonnegative continuous functional on P conditions such that either*

(B_1) *α satisfies Property A_1 with $\alpha(Tu) \ge \alpha(u)$, for all $u \in P \cap \partial\Omega_1$, and γ satisfies Property A_2 with $\gamma(Tu) \le \gamma(u)$, for all $u \in P \cap \partial\Omega_2$, or*

(B_2) *γ satisfies Property A_2 with $\gamma(Tu) \le \gamma(u)$, for all $u \in P \cap \partial\Omega_1$, and α satisfies Property A_1 with $\alpha(Tu) \ge \alpha(u)$, for all $u \in P \cap \partial\Omega_2$*

holds. Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Now we present the Green's functions for BVPs (1.1).

Lemma 2.3 *Denote $\rho(t) \triangleq \frac{a+(1-a)t}{(1-a)(1-b)}$. If $\lambda[\rho] = \int_0^1 \rho(t)\,dA(t) \neq 1$ and any $y \in C[0,1]$, then the unique solution of*

$$\begin{cases} D_{0^+}^{\nu} u(t) + y(t) = 0, \quad t \in (0,1), \\ u(0) = au(1), \qquad u'(1) = bu'(0) + \lambda[u], \\ u^{(i)}(0) = 0, \quad i = 2, 3, \dots, n-1, \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 H(t,s)y(s)\,ds,$$

where

$$H(t,s) = G(t,s) + K(t,s), \tag{2.2}$$

$$K(t,s) = \frac{\rho(t)}{1-\lambda[\rho]}\int_0^1 G(\tau,s)\,dA(\tau), \tag{2.3}$$

and

$$G(t,s) = \begin{cases} G_1(t,s), & 0 \le s \le t \le 1, \\ G_2(t,s), & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$G_1(t,s) = -\frac{(t-s)^{\nu-1}}{\Gamma(\nu)} - \frac{a}{1-a}\cdot\frac{(1-s)^{\nu-1}}{\Gamma(\nu)} + \frac{a+(1-a)t}{(1-a)(1-b)}\cdot\frac{(1-s)^{\nu-2}}{\Gamma(\nu-1)},$$

$$G_2(t,s) = -\frac{a}{1-a}\cdot\frac{(1-s)^{\nu-1}}{\Gamma(\nu)} + \frac{a+(1-a)t}{(1-a)(1-b)}\cdot\frac{(1-s)^{\nu-2}}{\Gamma(\nu-1)}.$$

Proof Applying Lemma 2.1 and noting that the boundary conditions $u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0$, BVPs (2.1) can be expressed as

$$\begin{aligned} u(t) &= -I_{0^+}^{\nu}y(t) + u(0) + u'(0)t + \frac{u''(0)}{2!}t^2 + \cdots + \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1} \\ &= -I_{0^+}^{\nu}y(t) + u(0) + u'(0)t. \end{aligned}$$

Thus, by the boundary value condition $u(0) = au(1)$ and $u'(1) = bu'(0) + \lambda[u]$, we get

$$u(0) = -aI_{0^+}^{\nu}y(1) + au(0) + au'(0), \qquad -I_{0^+}^{\nu-1}y(1) + u'(0) = bu'(0) + \lambda[u],$$

which imply that

$$\begin{aligned} u(0) &= \frac{1}{1-a}\left(-aI_{0^+}^{\nu}y(1) + \frac{a}{1-b}\left(I_{0^+}^{\nu-1}y(1) + \lambda[u]\right)\right), \\ u'(0) &= \frac{1}{1-b}\left(I_{0^+}^{\nu-1}y(1) + \lambda[u]\right). \end{aligned}$$

It follows from the definition of the Riemann-Liouville fractional integral that

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}y(s)\,ds - \frac{a}{(1-a)\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}y(s)\,ds \\ &\quad + \frac{a+(1-a)t}{(1-a)(1-b)}\left(\frac{1}{\Gamma(\nu-1)}\int_0^1 (1-s)^{\nu-2}y(s)\,ds + \lambda[u]\right) \\ &= \int_0^1 G(t,s)y(s)\,ds + \frac{a+(1-a)t}{(1-a)(1-b)}\lambda[u], \end{aligned} \tag{2.5}$$

where $G(t,s)$ is defined by (2.2). Integrating on both sides of (2.5) from 0 to 1, we have

$$\begin{aligned} \int_0^1 u(s)\,dA(s) &= \int_0^1 \left(\int_0^1 G(s,\tau)y(\tau)\,d\tau\right) dA(s) + \int_0^1 \frac{a+(1-a)s}{(1-a)(1-b)}\lambda[u]\,dA(s) \\ &= \int_0^1 \left(\int_0^1 G(\tau,s)\,dA(\tau)\right) y(s)\,ds + \lambda[\rho]\lambda[u], \end{aligned}$$

which implies

$$\lambda[u] = \int_0^1 u(s)\,dA(s) = \frac{1}{1-\lambda[\rho]}\int_0^1 \left(\int_0^1 G(\tau,s)\,dA(\tau)\right)y(s)\,ds.$$

Thus, the solutions of BVPs (2.1) are formulated by

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)\,ds + \frac{a+(1-a)t}{(1-a)(1-b)}\cdot\frac{1}{1-\lambda[\rho]}\int_0^1\left(\int_0^1 G(\tau,s)\,dA(\tau)\right)y(s)\,ds \\ &= \int_0^1 G(t,s)y(s)\,ds + \frac{\rho(t)}{1-\lambda[\rho]}\int_0^1\left(\int_0^1 G(\tau,s)\,dA(\tau)\right)y(s)\,ds \\ &= \int_0^1 G(t,s)y(s)\,ds + \int_0^1 K(t,s)y(s)\,ds \\ &= \int_0^1 H(t,s)y(s)\,ds, \end{aligned} \tag{2.6}$$

where $K(t,s)$ and $H(t,s)$ are defined by (2.3) and (2.2), respectively.

Now, we will prove the uniqueness of solution for BVPs (2.1). In fact, let $u_1(t)$ and $u_2(t)$ be any two solutions of (2.1). Denote $w(t) = u_1(t) - u_2(t)$, then (2.1) is changed into the following system:

$$\begin{cases} D_{0^+}^{\nu} w(t) = 0, \quad t\in(0,1), \\ w(0) = aw(1), \qquad w'(1) = bw'(0) + \lambda[w], \\ w^{(i)}(0) = 0, \quad i = 2,3,\ldots,n-1. \end{cases}$$

Similar to the above arguments, we get $w(t) = 0$, that is, $u_1(t) = u_2(t)$, which means that the solution for BVPs (2.1) is unique. The proof is complete. □

In the rest of this paper, we always assume that the conditions are fulfilled as follows:

(H_1) $0 \le \lambda[\rho] < 1$, $\kappa(s) \ge 0$ for $s\in[0,1]$, where $\kappa(s) \triangleq \int_0^1 G(t,s)\,dA(t)$.
(H_2) $f \in C([0,1]\times[0,+\infty),(-\infty,+\infty))$.

By Lemma 2.3, it is easy to obtain Lemma 2.4.

Lemma 2.4 *Assume that the condition* (H_1) *holds and any* $x\in C[0,1]$, *then the unique solution of*

$$\begin{cases} (\phi(D_{0^+}^{\nu} u(t)))' + x(t) = 0, \quad t\in(0,1), 2<\nu\le 3, \\ u(0) = au(1), \qquad u'(1) = bu'(0) + \lambda[u], \\ u^{(i)}(0) = 0, \quad i = 2,3,\ldots,n-1, \end{cases} \tag{2.7}$$

is given by

$$u(t) = -\int_0^1 H(t,s)\phi^{-1}\left(-\int_0^s x(\varsigma)\,d\varsigma\right)ds.$$

Lemma 2.5 *Assume that Case* I *and the condition* (H_1) *hold. Then the Green's function* $H(t,s)$ *defined by* (2.2) *has the following properties*:

(i) $H(t,s) \geq 0$ *for any* $t,s \in [0,1]$;
(ii) $\frac{\partial H(t,s)}{\partial t} \geq 0$ *for any* $t,s \in [0,1]$.

Proof When $0 \leq t \leq s \leq 1$, we have $\frac{\partial G_2(t,s)}{\partial t} = \frac{(1-s)^{\nu-2}}{(1-b)\Gamma(\nu-1)} \geq 0$. When $0 \leq s \leq t \leq 1$, we get

$$\begin{aligned}\frac{\partial G_1(t,s)}{\partial t} &= -\frac{(t-s)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(1-s)^{\nu-2}}{(1-b)\Gamma(\nu-1)} \\ &\geq -\frac{(1-s)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(1-s)^{\nu-2}}{(1-b)\Gamma(\nu-1)} = \frac{b(1-s)^{\nu-2}}{(1-b)\Gamma(\nu-1)} \geq 0.\end{aligned}$$

Therefore, $\frac{\partial G(t,s)}{\partial t} \geq 0$, that is, $G(t,s)$ is increasing with respect to t, which implies that

$$\begin{aligned}G(t,s) \geq G(0,s) &= -\frac{a}{1-a} \cdot \frac{(1-s)^{\nu-1}}{\Gamma(\nu)} + \frac{a}{(1-a)(1-b)} \cdot \frac{(1-s)^{\nu-2}}{\Gamma(\nu-1)} \\ &= \frac{a(\nu-1) - a(1-b)(1-s)}{(1-a)(1-b)} \cdot \frac{(1-s)^{\nu-2}}{\Gamma(\nu)} \\ &\geq \frac{a(\nu+b-2)}{(1-a)(1-b)} \cdot \frac{(1-s)^{\nu-2}}{\Gamma(\nu)} \geq 0,\end{aligned}$$

for $s \in [0,1]$. Hence, for $t,s \in [0,1]$,

$$\frac{\partial H(t,s)}{\partial t} = \frac{\partial G(t,s)}{\partial t} + \frac{\partial K(t,s)}{\partial t} = \frac{\partial G(t,s)}{\partial t} + \frac{\kappa(s)}{(1-b)(1-\lambda[\rho])} \geq 0.$$

Therefore, $H(t,s)$ is increasing with respect to t, which implies that

$$H(t,s) \geq H(0,s) = G(0,s) + \frac{a\kappa(s)}{(1-a)(1-b)(1-\lambda[\rho])} \geq 0,$$

for $s \in [0,1]$. The proof is complete. □

Let $E = C[0,1]$. Then E is a real Banach space with the norm $\|\cdot\|$ defined by $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Lemma 2.6 *Assume that Case* I *and the condition* (H_1) *hold. If* $x \in C([0,1],[0,+\infty))$, *then the unique solution* $u(t)$ *of BVPs* (2.7) *satisfies*
(1) $u(t) \geq 0$ *for any* $t \in [0,1]$;
(2) $u(t)$ *is increasing and concave on* $[0,1]$;
(3) $u(t) \geq a\|u\|$ *for any* $t \in [0,1]$.

Proof In view of $x \in C([0,1],[0,+\infty))$, we have $-\phi^{-1}(-\int_0^s x(\varsigma)\,d\varsigma) \geq 0$, $s \in [0,1]$. Together with $H(t,s) \geq 0$, $\frac{\partial H(t,s)}{\partial t} \geq 0$ and

$$\frac{\partial^2 H(t,s)}{\partial t^2} = \begin{cases} -\frac{(\nu-1)(\nu-2)(t-s)^{\nu-3}}{\Gamma(\nu)} \leq 0, & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases}$$

we have

$$u(t) = -\int_0^1 H(t,s)\phi^{-1}\left(-\int_0^s x(\varsigma)\,d\varsigma\right) ds \geq 0,$$

$$u'(t) = -\int_0^1 \frac{\partial H(t,s)}{\partial t}\phi^{-1}\left(-\int_0^s x(\varsigma)\,d\varsigma\right)ds \geq 0,$$
$$u''(t) = -\int_0^1 \frac{\partial^2 H(t,s)}{\partial t^2}\phi^{-1}\left(-\int_0^s x(\varsigma)\,d\varsigma\right)ds \leq 0.$$

Thus we know that $u(t)$ is increasing and concave on $[0,1]$, and $u(0) \leq u(t) \leq u(1) = \|u\|$. According to the boundary condition $u(0) = au(1)$, we obtain $u(t) \geq u(0) = a\|u\|$ for any $t \in [0,1]$. The proof is complete. □

Similar to the proof of Lemma 2.5 and Lemma 2.6, we have the following lemmas.

Lemma 2.7 *Assume that Case* II *and the condition* (H_1) *hold. Then the Green's function* $H(t,s)$ *defined by* (2.2) *has the following properties*:

(i) $H(t,s) \geq 0$ *for any* $t,s \in [0,1]$;
(ii) $\frac{\partial H(t,s)}{\partial t} \leq 0$ *for any* $t,s \in [0,1]$.

Lemma 2.8 *Assume that Case* II *and the condition* (H_1) *hold. If* $x \in C([0,1],[0,+\infty))$, *then the unique solution* $u(t)$ *of BVPs* (2.7) *satisfies*

(1) $u(t) \geq 0$ *for any* $t \in [0,1]$;
(2) $u(t)$ *is decreasing and concave on* $[0,1]$;
(3) $u(t) \geq \frac{1}{a}\|u\|$ *for any* $t \in [0,1]$.

For simplicity, we introduce some important notations as follows:

$$f^0 = \limsup_{u\to 0^+}\max_{t\in[0,1]}\frac{f(t,u)}{\phi(u)}, \qquad f_0 = \liminf_{u\to 0^+}\min_{t\in[0,1]}\frac{f(t,u)}{\phi(u)},$$
$$L_1 = \frac{-1}{\int_0^1 H(0,s)\phi^{-1}(-\int_0^s h(\varsigma)\,d\varsigma)\,ds}, \qquad L_2 = \frac{-1}{\int_0^1 H(1,s)\phi^{-1}(-\int_0^s h(\varsigma)\,d\varsigma)\,ds}.$$

It follows from $u(0) = au(1)$ that $H(0,s) = aH(1,s)$ and $L_2 = aL_1$.

From Lemma 2.4, we can obtain the following lemma.

Lemma 2.9 *Suppose that conditions* (H_1) *and* (H_2) *hold, then* $u \in E$ *is a solution of BVPs* (1.1) *if and only if* $u \in E$ *is a solution of the integral equation*

$$u(t) = -\int_0^1 H(t,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f\bigl(\varsigma,u\bigl(\theta(\varsigma)\bigr)\bigr)\,d\varsigma\right)ds, \quad t \in [0,1].$$

Define $F, T: E \to E$ to be the operators defined as

$$\begin{aligned}(Fu)(t) &= -\int_0^1 H(t,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f\bigl(\varsigma,u\bigl(\theta(\varsigma)\bigr)\bigr)\,d\varsigma\right)ds, \quad t \in [0,1],\\ (Tu)(t) &= -\int_0^1 H(t,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\bigl(\varsigma,u\bigl(\theta(\varsigma)\bigr)\bigr)\,d\varsigma\right)ds, \quad t \in [0,1],\end{aligned} \tag{2.8}$$

where $f^+(t,u(\theta(t))) = \max\{f(t,u(\theta(t))),0\}$. Then, by Lemma 2.9, the existence of solutions for BVPs (1.1) is translated into the existence of the fixed point for $u = Fu$, where F is given by (2.8). Thus, the fixed point of the operator F coincides with the solution of problem (1.1).

Lemma 2.10 *Assume that* (H_1) *and* (H_2) *hold, then* $F, T : E \to E$ *defined by* (2.8) *are completely continuous.*

Proof First, we shall show that $T : E \to E$ is completely continuous through three steps.

Step 1. Let $u \in E$, in view of the nonnegativity and continuity of functions $H(t,s)$, $h(t)$, $\theta(t)$, and $f^+(t, u(\theta(t)))$, we conclude that $T : E \to E$ is continuous.

Step 2. We will prove that T maps bounded sets into bounded sets. Indeed, it is enough to show that for any $\rho > 0$ there exists a positive constant M such that, for each $u \in \Omega_\rho = \{u \in E : \|u\| \le \rho\}$, $\|Tu\| \le M$ when $|h(t)| \le l_1$, $|f^+(t, u(\theta(t)))| \le l_2$, where l_i $(i = 1, 2)$ are some fixed positive constants. In fact, for each $t \in [0,1]$, $u \in \Omega_\rho$, by Lemma 2.5, we have

$$\bigl|(Tu)(t)\bigr| \le \phi^{-1}(l_1 l_2) \int_0^1 H(1,s)\,ds \triangleq M,$$

which imply that $\|Tu\| \le M$.

Step 3. T is equicontinuous. In fact, since $H(t,s)$ are continuous on $[0,1] \times [0,1]$, it is uniformly continuous on $[0,1] \times [0,1]$. Thus, for fixed $s \in [0,1]$ and for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that, for any $t_1, t_2 \in [0,1]$ with $|t_2 - t_1| < \delta$, we have $|H(t_2,s) - H(t_1,s)| < \frac{\epsilon}{\phi^{-1}(l_1 l_2)}$. Then

$$\begin{aligned}\bigl|(Tu)(t_2) - (Tu)(t_1)\bigr| &= \left|-\int_0^1 \bigl[H(t_2,s) - H(t_1,s)\bigr]\phi^{-1}\left(-\int_0^s h(\varsigma) f^+\bigl(\varsigma, u\bigl(\theta(\varsigma)\bigr)\bigr)\,d\varsigma\right)ds\right| \\ &< \frac{\epsilon}{\phi^{-1}(l_1 l_2)}\phi^{-1}(l_1 l_2) = \epsilon,\end{aligned}$$

which means that $T(\Omega_\rho)$ is equicontinuous on $[0,1]$. Thus, by means of the Arzela-Ascoli theorem, we see that $T : E \to E$ is completely continuous. Similarly, it is easy to show that $F : E \to E$ is also completely continuous. The proof is complete. □

3 Multiple monotone increasing positive solutions of BVPs (1.1) in Case I

In this section, we will discuss the existence of at least two monotone increasing positive solutions to BVPs (1.1) in Case I.

Let the cone $P \subset E$ be defined by

$$P = \bigl\{u \in E : u(t) \ge a\|u\|, u(t) \text{ is increasing and concave on } [0,1]\bigr\}. \tag{3.1}$$

The operators $F, T : E \to E$ are defined as (2.8). It follows from Lemma 2.10 that the operators $T, F : P \to P$ are all completely continuous. Define two continuous functionals α and γ on the cone P as follows:

$$\alpha(u) = \min_{t\in[0,1]}\bigl|u(t)\bigr| = u(0), \qquad \gamma(u) = \max_{t\in[0,1]}\bigl|u(t)\bigr| = u(1) = \|u\|.$$

It is clear that $\alpha(u) \le \gamma(u)$ for all $u \in P$.

Theorem 3.1 *Assume that* (H_1) *and* (H_2) *hold. If there exist constants* $r, R \in (0,\infty)$ *with* $r \le aR$ *such that the following conditions are satisfied*:

(C_1) $f(t,u) > \phi(L_1 r)$, *for all* $(t,u) \in [0,1] \times [r,R]$;

(C_2) $f(t,u) \leq \phi(L_2 R)$, *for all* $(t,u) \in [0,1] \times [aR,R]$;
(C_3) $0 \leq f^0 \leq \phi(L_2)$;
(C_4) $f(t,u) \geq 0$, *for all* $(t,u) \in [0,1] \times [0,R]$,

then BVPs (1.1) *have at least two increasing and concave positive solutions* $u_1(t)$, $u_2(t)$, *which satisfy* $0 \leq u_2(t) < r < u_1(t) \leq R$ *for* $t \in [0,1]$.

Proof Set $\Omega_1 = \{u \in E : \alpha(u) < r\}$. For any $u \in P \cap \partial\Omega_1$, we have $r = \alpha(u) = u(0) \leq u(t) \leq u(1) = \frac{u(0)}{a} = \frac{\alpha(u)}{a} = \frac{r}{a} \leq R$ for $t \in [0,1]$ and $r \leq u(\theta(t)) \leq R$ for $0 \leq t \leq \theta(t) \leq 1$. It follows from condition (C_1) that

$$-\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\big(\varsigma, u\big(\theta(\varsigma)\big)\big)\,d\varsigma\right) > -\phi^{-1}\left(-\int_0^s h(\varsigma)\phi(L_1 r)\,d\varsigma\right)$$
$$= -L_1 r\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right). \tag{3.2}$$

By (2.8), (3.2), and (3) of Lemma 2.6, we get

$$\begin{aligned}\alpha(Tu) = (Tu)(0) &= -\int_0^1 H(0,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\big(\varsigma, u\big(\theta(\varsigma)\big)\big)\,d\varsigma\right)ds \\ &> -L_1 r\int_0^1 H(0,s)\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right)ds = r = \alpha(u).\end{aligned} \tag{3.3}$$

Therefore,

$$\alpha(Tu) > \alpha(u), \quad u \in P \cap \partial\Omega_1. \tag{3.4}$$

On the other hand, taking $\Omega_2 = \{u \in E : \gamma(u) < R\}$ and $u \in \overline{\Omega}_1$, we have $r \geq \alpha(u) = u(0) = au(1) = a\gamma(u)$. Thus $R \geq \frac{r}{a} \geq \gamma(u)$, that is, $u \in \Omega_2$ implies $\overline{\Omega}_1 \subseteq \Omega_2$. For any $u \in P \cap \partial\Omega_2$, we obtain $aR = a\gamma(u) = au(1) = u(0) \leq u(t) \leq u(1) = \gamma(u) = R$ for $t \in [0,1]$ and $aR \leq u(\theta(t)) \leq R$ for $0 \leq t \leq \theta(t) \leq 1$. It follows from condition (C_2) that

$$-\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\big(\varsigma, u\big(\theta(\varsigma)\big)\big)\,d\varsigma\right) \leq -L_2 R\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right). \tag{3.5}$$

According to (2) of Lemma 2.6 and (3.5), we have

$$\begin{aligned}\gamma(Tu) = (Tu)(1) &= -\int_0^1 H(1,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\big(\varsigma, u\big(\theta(\varsigma)\big)\big)\,d\varsigma\right)ds \\ &\leq -L_2 R\int_0^1 H(1,s)\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right) = R = \gamma(u).\end{aligned} \tag{3.6}$$

So

$$\gamma(Tu) \leq \gamma(u), \quad u \in P \cap \partial\Omega_2. \tag{3.7}$$

Noting that we have the condition (C_3), namely, $f^0 \in [0, \phi(L_2)]$, for $\varepsilon = \phi(L_2) - f^0 \geq 0$, we know that there exists a sufficiently small constant $r_0 \in (0,r)$ such that

$$0 \leq f(t,u) \leq \big(f^0 + \varepsilon\big)\phi(u) = \phi(L_2 u) \leq \phi(L_2 r_0), \quad t \in [0,1], u \in [0,r_0]. \tag{3.8}$$

Set $\Omega_3 = \{u \in E : \gamma(u) < r_0\}$. It is clear that $\overline{\Omega}_3 \subseteq \Omega_1$. For any $u \in P \cap \partial\Omega_3$, we have $ar_0 = a\gamma(u) = au(1) = u(0) \leq u(t) \leq u(1) = \gamma(u) = r_0$ for $t \in [0,1]$ and $ar_0 \leq u(\theta(t)) \leq r_0$ for $0 \leq t \leq \theta(t) \leq 1$. It follows from (3.2) and (3.8) that

$$-\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\big(\varsigma, u\big(\theta(\varsigma)\big)\big)\,d\varsigma\right) \leq -L_2 r_0 \phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right). \tag{3.9}$$

Equation (3.9) and (2) of Lemma 2.6 give

$$\begin{aligned}\gamma(Tu) = (Tu)(1) &= -\int_0^1 H(1,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\big(\varsigma, u\big(\theta(\varsigma)\big)\big)\,d\varsigma\right)ds \\ &\leq -L_2 r_0 \int_0^1 H(1,s)\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right) = r_0 = \gamma(u),\end{aligned} \tag{3.10}$$

which implies that

$$\gamma(Tu) \leq \gamma(u), \quad u \in P \cap \partial\Omega_3. \tag{3.11}$$

Clearly, α satisfies condition (c) of Property A_1 and γ satisfies condition (d) of Property A_2. By (3.4), (3.7) and condition (B_1) of Lemma 2.2, we know that T has a fixed point $u_1 \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, that is, $r \leq u_1 \leq R$. Equations (3.4) and (3.11) together with condition (B_2) of Lemma 2.2 imply that T has a fixed point $u_2 \in P \cap (\overline{\Omega}_1 \setminus \Omega_3)$, namely, $r_0 \leq u_2 \leq r$. It is worth noting that (3.4) is a strict inequality, that is to say, the operator T has not the fixed point on the boundary $\partial\Omega_1$. Thus we obtain $0 \leq u_2(t) < r < u_1(t) \leq R$. By condition ($C_4$), we have $f(t,u_i) \geq 0$, for $t \in [0,1]$, that is, $f^+(t,u_i) = f(t,u_i)$ $(i = 1,2)$. Hence, $Fu_i = Tu_i$ $(i = 1,2)$. Consequently, BVPs (1.1) have at least two increasing and concave positive solutions with $0 \leq u_2(t) < r < u_1(t) \leq R$. The proof is complete. □

Similarly, we can get the following theorem.

Theorem 3.2 *Assume that* (H_1) *and* (H_2) *hold. If there exist constants* $r, R \in (0,\infty)$ *with* $r \leq R$ *such that the following conditions are satisfied*:

(C_5) $f(t,u) < \phi(L_2 r)$, *for all* $(t,u) \in [0,1] \times [ar, r]$;
(C_6) $f(t,u) \geq \phi(L_1 R)$, *for all* $(t,u) \in [0,1] \times [R, \frac{R}{a}]$;
(C_7) $\phi(L_1) \leq f_0 \leq +\infty$;
(C_8) $f(t,u) \geq 0$, *for all* $(t,u) \in [0,1] \times [0, \frac{R}{a}]$,

then BVPs (1.1) *have at least two increasing and concave positive solutions* $u_1(t)$, $u_2(t)$ *with* $0 \leq u_2(t) < r < u_1(t) \leq \frac{R}{a}$ *for* $t \in [0,1]$.

4 Multiple monotone decreasing positive solutions of BVPs (1.1) in Case II

In this section, we will discuss the existence of at least two monotone decreasing positive solutions to BVPs (1.1) in Case II.

Let the cone $P \subset E$ be defined by

$$P = \left\{u \in E : u(t) \geq \frac{1}{a}\|u\|, u(t) \text{ is decreasing and concave on } [0,1]\right\}. \tag{4.1}$$

The operators $F, T : P \to P$ are defined as (2.8). By Lemma 2.10, we know that the operators $T, F : P \to P$ are all completely continuous. Define two continuous functionals α and γ on the cone P as follows:

$$\alpha(u) = \min_{t\in[0,1]} \left|u(t)\right| = u(1), \qquad \gamma(u) = \max_{t\in[0,1]} \left|u(t)\right| = u(0) = \|u\|.$$

Obviously, $\alpha(u) \le \gamma(u)$ for all $u \in P$.

Theorem 4.1 *Assume that* (H_1) *and* (H_2) *hold. If there exist constants* $r, R \in (0, \infty)$ *with* $ar \le R$ *such that the following conditions are satisfied*:

(D_1) $f(t,u) > \phi(L_2 r)$, *for all* $(t,u) \in [0,1] \times [r,R]$;
(D_2) $f(t,u) \le \phi(L_1 R)$, *for all* $(t,u) \in [0,1] \times [\frac{R}{a}, R]$;
(D_3) $0 \le f^0 \le \phi(L_1)$;
(D_4) $f(t,u) \ge 0$, *for all* $(t,u) \in [0,1] \times [0,R]$,

then BVPs (1.1) *have at least two decreasing and concave positive solutions* $u_1(t)$, $u_2(t)$ *with* $0 \le u_2(t) < r < u_1(t) \le R$ *for* $t \in [0,1]$.

Proof Let $\Omega_1 = \{u \in E : \alpha(u) < r\}$. For any $u \in P \cap \partial\Omega_1$, we have $r = \alpha(u) = u(1) \le u(t) \le u(0) = au(1) = a\alpha(u) = ar \le R$ for $t \in [0,1]$ and $r \le u(\theta(t)) \le R$ for $0 \le \theta(t) \le t \le 1$. It follows from condition (D_1) that

$$\begin{aligned} -\phi^{-1}\left(-\int_0^s h(\varsigma) f^+\bigl(\varsigma, u\bigl(\theta(\varsigma)\bigr)\bigr)\,d\varsigma\right) &> -\phi^{-1}\left(-\int_0^s h(\varsigma)\phi(L_2 r)\,d\varsigma\right) \\ &= -L_2 r\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right). \end{aligned} \tag{4.2}$$

By (2.8), (4.2), and (3) of Lemma 2.8, we get

$$\begin{aligned} \alpha(Tu) = (Tu)(1) &= -\int_0^1 H(1,s)\phi^{-1}\left(-\int_0^s h(\varsigma) f^+\bigl(\varsigma, u\bigl(\theta(\varsigma)\bigr)\bigr)\,d\varsigma\right) ds \\ &> -L_2 r \int_0^1 H(1,s)\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right) ds = r = \alpha(u). \end{aligned} \tag{4.3}$$

Therefore,

$$\alpha(Tu) > \alpha(u), \quad u \in P \cap \partial\Omega_1. \tag{4.4}$$

Next, taking $\Omega_2 = \{u \in E : \gamma(u) < R\}$ and $u \in \overline{\Omega}_1$, we have $r \ge \alpha(u) = u(1) = \frac{u(0)}{a} = \frac{\gamma(u)}{a}$. Thus $R \ge ar \ge \gamma(u)$, that is, $u \in \Omega_2$ implies $\overline{\Omega}_1 \subseteq \Omega_2$. For any $u \in P \cap \partial\Omega_2$, we obtain $\frac{R}{a} = \frac{\gamma(u)}{a} = \frac{u(0)}{a} = u(1) \le u(t) \le u(0) = \gamma(u) = R$ for $t \in [0,1]$ and $\frac{R}{a} \le u(\theta(t)) \le R$ for $0 \le \theta(t) \le t \le 1$. It follows from condition (D_2) that

$$-\phi^{-1}\left(-\int_0^s h(\varsigma) f^+\bigl(\varsigma, u\bigl(\theta(\varsigma)\bigr)\bigr)\,d\varsigma\right) \le -L_1 R\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right). \tag{4.5}$$

In the light of (4.5) and (2) of Lemma 2.8, we have

$$\begin{aligned}\gamma(Tu) = (Tu)(0) &= -\int_0^1 H(0,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\left(\varsigma,u\left(\theta(\varsigma)\right)\right)d\varsigma\right)ds \\ &\leq -L_1R\int_0^1 H(0,s)\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right)ds = R = \gamma(u),\end{aligned} \tag{4.6}$$

which implies

$$\gamma(Tu) \leq \gamma(u), \quad u \in P\cap\partial\Omega_2. \tag{4.7}$$

Noting that the condition (D_3), namely, $f^0 \in [0,\phi(L_1)]$, for $\varepsilon = \phi(L_1) - f^0 \geq 0$, we know that there exists a sufficiently small constant $r_0 \in (0,r)$ such that

$$0 \leq f(t,u) \leq \left(f^0 + \varepsilon\right)\phi(u) = \phi(L_1u) \leq \phi(L_1r_0), \quad t \in [0,1], u \in [0,r_0]. \tag{4.8}$$

Set $\Omega_3 = \{u \in E : \gamma(u) < r_0\}$. Clearly, $\overline{\Omega}_3 \subseteq \Omega_1$. For any $u \in P\cap\partial\Omega_3$, we have $\frac{r_0}{a} = \frac{\gamma(u)}{a} = \frac{u(0)}{a} = u(1) \leq u(t) \leq u(0) = \gamma(u) = r_0$ for $t \in [0,1]$ and $\frac{r_0}{a} \leq u(\theta(t)) \leq r_0$ for $0 \leq \theta(t) \leq t \leq 1$. It follows from (4.2) and (4.8) that

$$-\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\left(\varsigma,u\left(\theta(\varsigma)\right)\right)d\varsigma\right) \leq -L_1r_0\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right). \tag{4.9}$$

Equation (4.9) together with (2) of Lemma 2.8 gives

$$\begin{aligned}\gamma(Tu) = (Tu)(0) &= -\int_0^1 H(0,s)\phi^{-1}\left(-\int_0^s h(\varsigma)f^+\left(\varsigma,u\left(\theta(\varsigma)\right)\right)d\varsigma\right)ds \\ &\leq -L_1r_0\int_0^1 H(0,s)\phi^{-1}\left(-\int_0^s h(\varsigma)\,d\varsigma\right)ds = r_0 = \gamma(u).\end{aligned} \tag{4.10}$$

So

$$\gamma(Tu) \leq \gamma(u), \quad u \in P\cap\partial\Omega_3. \tag{4.11}$$

Obviously, α satisfies condition (c) of Property A_1 and γ satisfies condition (d) of Property A_2. By (4.4), (4.7) and condition (B_1) of Lemma 2.2, we know that T has a fixed point $u_1 \in P\cap(\overline{\Omega}_2 \setminus \Omega_1)$, that is, $r \leq u_1 \leq R$. Equations (4.4) and (4.11) together with condition (B_2) of Lemma 2.2 imply that T has a fixed point $u_2 \in P\cap(\overline{\Omega}_1 \setminus \Omega_3)$, namely, $r_0 \leq u_2 \leq r$. It is worth noting that (4.4) is a strict inequality, that is to say, the operator T has not the fixed point on the boundary $\partial\Omega_1$. Thus we obtain $0 \leq u_2(t) < r < u_1(t) \leq R$. By condition ($\mathrm{D}_4$), we have $f(t,u_i) \geq 0$, for $t \in [0,1]$, that is, $f^+(t,u_i) = f(t,u_i)$ $(i = 1,2)$. Hence, $Fu_i = Tu_i$ $(i = 1,2)$. Consequently, BVPs (1.1) have at least two decreasing and concave positive solutions with $0 \leq u_2(t) < r < u_1(t) \leq R$. The proof is complete. □

Similar to the above arguments, we obtain the following theorem.

Theorem 4.2 *Assume that* (H_1) *and* (H_2) *hold. If there exist constants* $r,R \in (0,\infty)$ *with* $r \leq R$ *such that the following conditions are satisfied*:

(D_5) $f(t,u) < \phi(L_1 r)$, *for all* $(t,u) \in [0,1] \times [\frac{r}{a}, r]$;
(D_6) $f(t,u) \geq \phi(L_2 R)$, *for all* $(t,u) \in [0,1] \times [R, aR]$;
(D_7) $\phi(L_2) \leq f_0 \leq +\infty$;
(D_8) $f(t,u) \geq 0$, *for all* $(t,u) \in [0,1] \times [0, aR]$,

then BVPs (1.1) *have at least two decreasing and concave positive solutions* $u_1(t)$, $u_2(t)$ *with* $0 \leq u_2(t) < r < u_1(t) \leq aR$ *for* $t \in [0,1]$.

5 Illustrative examples

In this section, we give some examples to illustrate our main results.

Example 5.1 Consider the integral boundary value problem for the fractional differential equation

$$\begin{cases} (\phi(D_{0^+}^{5/2}u(t)))' + h(t)f(t,u(\theta(t))) = 0, \quad t \in (0,1), \\ u(0) = \frac{1}{2}u(1), \qquad u'(1) = \frac{1}{2}u'(0) + \int_0^1 (2t-1)u(t)\,dt, \qquad u''(0) = 0, \end{cases} \tag{5.1}$$

where $\nu = \frac{5}{2}$, $a = b = \frac{1}{2}$, $h(t) = 1$, $\theta(t) = \sqrt{t}$, $dA(t) = (2t-1)\,dt$, $\phi(u) = \begin{cases} u, & u < 0, \\ u^2, & u \geq 0, \end{cases}$ and

$$f(t,u) = \begin{cases} \frac{9(1+t^2)}{5}u^3 + \frac{1}{8}u^2, & (t,u) \in [0,1] \times [0,1], \\ \frac{9(1+t^2)}{5} + \frac{1}{8}u^2, & (t,u) \in [0,1] \times (1,4], \\ \frac{9(1+t^2)}{5} + \frac{1}{8}(u-4)^2, & (t,u) \in [0,1] \times (4,+\infty). \end{cases}$$

In view of $dA(t) = (2t-1)\,dt$, we have

$$\begin{aligned} &0 < \lambda[\rho] = 2\int_0^1 (t+1)(2t-1)\,dt = \frac{1}{3} < 1, \\ &\kappa(s) = \int_0^1 G(t,s)\,dA(t) = \frac{2\sqrt{1-s}}{105\sqrt{\pi}}\bigl[35 - 4(4s+3)(1-s)^2\bigr] \geq 0, \quad s \in [0,1]. \end{aligned}$$

Obviously, the conditions (H_1) and (H_2) hold. A simple calculation gives

$$\begin{aligned} &L_1 \approx 1.28162617, \\ &L_2 = \frac{L_1}{2} \approx 0.640813084. \end{aligned}$$

Choose $r = 1$, $R = 4$, obviously, $r < aR$. Thus we get

$$\begin{aligned} &f(t,u) \geq f(0,1) = \frac{77}{40} = 1.925 > \phi(L_1 r) \approx 1.64256564, \quad \forall (t,u) \in [0,1] \times [1,4], \\ &f(t,u) \leq f(1,4) = \frac{28}{5} = 5.6 < \phi(L_2 R) \approx 6.57026256, \quad \forall (t,u) \in [0,1] \times [2,4], \\ &0 \leq f^0 = \limsup_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t,u)}{u^2} = \frac{1}{8} < \phi(L_2) \approx 0.410641409, \end{aligned}$$

and $f(t,u) \geq 0$ for all $(t,u) \in [0,1] \times [0,4]$. Clearly, all the conditions of Theorem 3.1 are satisfied. Therefore, we know that BVP (5.1) has at least two increasing and concave positive solutions $u_1(t)$, $u_2(t)$ satisfying $0 \le u_2(t) < 1 < u_1(t) \le 4$ for $t \in [0,1]$.

Example 5.2 Consider the integral boundary value problem for the fractional differential equation

$$\begin{cases} (\phi(D_{0^+}^{7/2}u(t)))' + h(t)f(t,u(\theta(t))) = 0, \quad t \in (0,1), \\ u(0) = 2u(1), \qquad u'(1) = 2u'(0) + \frac{1}{5}u(\frac{1}{3}) + \frac{1}{4}u(\frac{2}{3}), \\ u''(0) = u'''(0) = 0, \end{cases} \tag{5.2}$$

where $\nu = \frac{7}{2}$, $a = b = 2$, $h(t) = e^t$, $\theta(t) = \sin t$, $\phi(u) = u$, $f(t,u) = \frac{1}{10}(u^4 + (1+t)\sqrt{2u} + t + 2)$, $(t,u) \in [0,1] \times [0,+\infty)$, and

$$A(t) = \begin{cases} 0, & 0 \le t < \frac{1}{3}, \\ \frac{1}{5}, & \frac{1}{3} \le t < \frac{2}{3}, \\ \frac{9}{20}, & \frac{2}{3} \le t \le 1. \end{cases}$$

By simple calculation, we get

$$0 < \lambda[\rho] = \frac{1}{5}\rho\left(\frac{1}{3}\right) + \frac{1}{4}\rho\left(\frac{2}{3}\right) = \frac{2}{3} < 1,$$
$$\kappa(s) = \frac{1}{5}G\left(\frac{1}{3},s\right) + \frac{1}{4}G\left(\frac{2}{3},s\right) \ge 0, \quad s \in [0,1],$$

$L_1 \approx 1.217724677$, and $L_2 = 2L_1 \approx 2.435449353$. Obviously, the conditions (H_1) and (H_2) hold.

Taking $r = 2$, $R = 3$, we obtain

$$f(t,u) \le f(1,2) = 2.3 < \phi(L_1 r) \approx 2.435449353, \quad \forall (t,u) \in [0,1] \times [2,3],$$
$$f(t,u) \ge f(0,3) = \frac{83 + \sqrt{6}}{10} \ge \phi(L_2 R) \approx 7.30634806, \quad \forall (t,u) \in [0,1] \times [1,2],$$
$$f_0 = \liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u} = +\infty \ge \phi(L_2),$$

and $f(t,u) \ge 0$ for all $(t,u) \in [0,1] \times [0,6]$. Thus, all the conditions of Theorem 4.2 are satisfied. So we conclude that BVP (5.2) has least two decreasing and concave positive solutions $u_1(t)$, $u_2(t)$ satisfying $0 \le u_2(t) < 2 < u_1(t) \le 6$ for $t \in [0,1]$.

Next, we provide an example when ϕ is p-Laplacian operator, that is, $\phi(u) = \varphi_p(u) = |u|^{p-2}u$ $(p > 1)$. Meanwhile, we compare with the previous well-known results of the literature [18, 19].

Example 5.3 Consider the integral boundary value problem for the fractional differential equation with the p-Laplacian

$$\begin{cases} (|D_{0^+}^{5/2}u(t)|D_{0^+}^{5/2}u(t))' + h(t)f(t,u(\theta(t))) = 0, \quad t \in (0,1), \\ u(0) = \frac{1}{2}u(1), \qquad u'(1) = \frac{1}{21}u'(0) + \frac{1}{5}\int_0^1 tu(t)\,dt, \qquad u''(0) = 0, \end{cases} \tag{5.3}$$

where $\nu = \frac{5}{2}$, $p = 3$, $q = \frac{3}{2}$, $a = \frac{1}{2}$, $b = \frac{1}{21}$, $h(t) = 1$, $\theta(t) = \sqrt{t}$, $dA(t) = \frac{1}{5}t\,dt$, and

$$f(t,u) = \begin{cases} \frac{7\sin\frac{t\pi}{2}}{10}u^2 + t^2u^3, & (t,u) \in [0,1] \times [0,1], \\ \frac{37+\cos\frac{t\pi}{2}}{10} + (u-1)^2, & (t,u) \in [0,1] \times (1,5], \\ \frac{7\tan\frac{t\pi}{4}}{10} + (u-5)^4, & (t,u) \in [0,1] \times (5,+\infty). \end{cases}$$

In view of $dA(t) = \frac{1}{5}t\,dt$, we have

$$\begin{aligned} &0 < \lambda[\rho] = \frac{1}{2}\int_0^1 (t+1)t\,dt = \frac{21}{120} < 1, \\ &\kappa(s) = \int_0^1 G(t,s)\,dA(t) = \frac{2\sqrt{1-s}}{5\Gamma(\frac{3}{2})}\left[\frac{7}{8} - \frac{1-s}{3} - \frac{2s(1-s)^2}{15} - \frac{2(1-s)^3}{21}\right] \geq 0, \quad s \in [0,1]. \end{aligned}$$

Obviously, the conditions (H_1) and (H_2) hold. By simple calculations, we have

$$\begin{aligned} L_1 &= \frac{-1}{\int_0^1 H(0,s)\phi^{-1}(-\int_0^s h(\varsigma)\,d\varsigma)\,ds} = \frac{-1}{\int_0^1 H(0,s)\varphi_{\frac{3}{2}}(-\int_0^s h(\varsigma)\,d\varsigma)\,ds} \\ &= \frac{-1}{\int_0^1 H(0,s)\varphi_{\frac{3}{2}}(-s)\,ds} = \frac{-1}{\int_0^1 H(0,s)(-\sqrt{s})\,ds} = \frac{1}{\int_0^1 H(0,s)\sqrt{s}\,ds} \\ &= \frac{1}{\int_0^1 [G(0,s) + \frac{\varrho(0)}{1-\lambda[\rho]}\kappa(s)]\sqrt{s}\,ds} = \frac{32{,}000}{9{,}493\sqrt{\pi}} \approx 1.9018294351 \end{aligned}$$

and

$$L_2 = \frac{-1}{\int_0^1 H(1,s)\phi^{-1}(-\int_0^s h(\varsigma)\,d\varsigma)\,ds} = \frac{L_1}{2} \approx 0.9509147176.$$

Choose $r = 1$, $R = 5$, obviously, $r < aR$. Thus we get

$$\begin{aligned} &f(t,u) \geq f(1,1) = 3.7 > \varphi_3(L_1r) = (L_1r)^2 \approx 3.6169543727, \quad \forall(t,u) \in [0,1] \times [1,5], \\ &f(t,u) \leq f(0,5) = 19.6 < \varphi_3(L_2R) = (L_2R)^2 \\ &\qquad \approx 22.605964621, \quad \forall(t,u) \in [0,1] \times [2.5,5], \\ &0 \leq f^0 = \limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{|u|u} = 0.7 < \varphi_3(L_2) = (L_2)^2 \approx 0.9042388001, \end{aligned}$$

and $f(t,u) \geq 0$ for all $(t,u) \in [0,1] \times [0,5]$. Clearly, all the conditions of Theorem 3.1 are satisfied. Therefore, we know that BVP (5.3) has at least two increasing and concave positive solutions $u_1(t)$, $u_2(t)$ satisfying $0 \leq u_2(t) < 1 < u_1(t) \leq 5$ for $t \in [0,1]$.

Remark 5.1 When $\theta(t) = t$, $h(t) = 1$, and $A(t) \equiv 0$, the system of Example 5.3 is changed into the equations of [18, 19]. In this paper, we consider the effect of time-delays and the integral boundary value conditions. Meanwhile, the operator ϕ includes the p-Laplacian operator. Therefore, our study improves and extends the previous well-known results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Author details
[1]Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650093, China.
[2]Department of Mathematics, Yuxi Normal University, Yuxi, Yunnan 653100, China.

Acknowledgements
The authors would like to thank the anonymous referees for their useful and valuable suggestions. This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant (No. 11161025; No. 11326101) and the Yunnan Province natural scientific research fund project (No. 2011FZ058).

Received: 30 April 2015 Accepted: 10 January 2016 Published online: 26 January 2016

References

1. Kilbas, AA, Trujillo, JJ: Differential equations of fractional order: methods, results and problems - I. Appl. Anal. **78**, 153-192 (2001)
2. Kilbas, AA, Trujillo, JJ: Differential equations of fractional order: methods, results and problems - II. Appl. Anal. **81**, 435-493 (2002)
3. Lakshmikantham, V, Leela, S: Nagumo-type uniqueness result for fractional differential equations. Nonlinear Anal. **71**, 2886-2889 (2009)
4. Ahmad, B, Nieto, JJ: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **58**(9), 1838-1843 (2009)
5. Ahmad, B, Ntouyas, S, Alsaedi, A: New existence results for nonlinear fractional differential equations with three-point integral boundary conditions. Adv. Differ. Equ. **2011**, Article ID 107384 (2011)
6. Henderson, J, Luca, R: Existence and multiplicity of positive solutions for a system of fractional boundary value problems. Bound. Value Probl. **2014**, 60 (2014)
7. Benchohra, M, Hamani, S, Ntouyas, SK: Boundary value problems for differential equations with fractional order and nonlocal conditions. Nonlinear Anal., Theory Methods Appl. **71**(7-8), 2391-2396 (2009)
8. Hao, ZC, Huang, YB: Existence of positive solutions to nonlinear fractional boundary value problem with changing sign nonlinearity and advanced arguments. Abstr. Appl. Anal. **2014**, Article ID 158436 (2014)
9. Jia, M, Liu, X: Three nonnegative solutions for fractional differential equations with integral boundary conditions. Comput. Math. Appl. **62**, 1405-1412 (2011)
10. Chen, T, Liu, W: An anti-periodic boundary value problem for the fractional differential equation with a *p*-Laplacian operator. Appl. Math. Lett. **25**(11), 1671-1675 (2012)
11. Zhao, KH, Gong, P: Positive solutions of Riemann-Stieltjes integral boundary problems for the nonlinear coupling system involving fractional-order differential. Adv. Differ. Equ. **2014**, 254 (2014)
12. Zhao, KH, Gong, P: Existence of positive solutions for a class of higher-order Caputo fractional differential equation. Qual. Theory Dyn. Syst. **14**(1), 157-171 (2015)
13. Zhao, KH, Gong, P: Positive solutions for impulsive fractional differential equations with generalized periodic boundary value conditions. Adv. Differ. Equ. **2014**, 255 (2014)
14. Zhao, KH: Multiple positive solutions of integral BVPs for high-order nonlinear fractional differential equations with impulses and distributed delays. Dyn. Syst. **30**(2), 208-223 (2015)
15. Zhao, KH, Gong, P: Positive solutions of *m*-point multi-term fractional integral BVP involving time-delay for fractional differential equations. Bound. Value Probl. **2015**, 19 (2015)
16. Zhao, KH: Impulsive boundary value problems for two classes of fractional differential equation with two different Caputo fractional derivatives. Mediterr. J. Math. (2015). doi:10.1007/s00009-015-0536-0
17. Zhao, KH: Triple positive solutions for two classes of delayed nonlinear fractional FDEs with nonlinear integral boundary value conditions. Bound. Value Probl. **2015**, 181 (2015)
18. Su, Y, Liu, Q, Liu, XL: Existence criteria for positive solutions of *p*-Laplacian fractional differential equations with derivative terms. Adv. Differ. Equ. **2013**, 119 (2013)
19. Liu, XP, Jia, M, Ge, WG: Multiple solutions of a *p*-Laplacian model involving a fractional derivative. Adv. Differ. Equ. **2013**, 126 (2013)
20. Liu, XP, Jia, M, Xiang, XF: On the solvability of a fractional differential equation model involving the *p*-Laplacian operator. Comput. Math. Appl. **62**, 1405-1412 (2011)
21. Chen, T, Liu, W: Solvability of some boundary value problems for fractional *p*-Laplacian equation. Abstr. Appl. Anal. **2013**, Article ID 432509 (2013)
22. Bereanu, C, Mawhin, J: Nonhomogeneous boundary value problems for some nonlinear equations with singular φ-Laplacian. J. Math. Anal. Appl. **352**, 218-233 (2009)
23. Sang, YB, Su, H: Positive solutions of nonlinear third-order *m*-point BVP for an increasing homeomorphism and homomorphism with sign-changing nonlinearity. J. Comput. Appl. Math. **225**, 288-300 (2009)
24. Wang, JH, Xiang, HJ, Zhao, YL: Monotone and concave positive solutions to a boundary value problem for higher-order fractional differential equation. Abstr. Appl. Anal. **2011**, Article ID 430457 (2011)
25. Cui, Y, Zou, Y: Existence results and the monotone iterative technique for nonlinear fractional differential systems with coupled four-point boundary value problems. Abstr. Appl. Anal. **2014**, Article ID 242591 (2014)
26. Kilbas, A, Srivastava, H, Trujillo, J: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
27. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1993)
28. Avery, R, Henderson, J, O'Regan, D: Functional compression-expansion fixed point theorem. Electron. J. Differ. Equ. **2008**, 22 (2008)